Combinatorics	Name
MAP363 Sheet 1	Number
Hand in by 19th February	Year
	Mark: /10
	date marked: / /2007

Please attach your working, with this sheet at the front.

Guidance on notation: graphs may have multiple edges, but may not have loops. A graph is simple if it has no multiple edges. ($\star$) indicates an optional question, included for interest only.

1. (a) Show that if two graphs have the same degree sequence then they have the same number of vertices and the same number of edges. Find two non-isomorphic graphs with the degree sequence $(2, 2, 2, 1, 1)$.

(b) Find all simple graphs on 4 vertices, up to isomorphism.

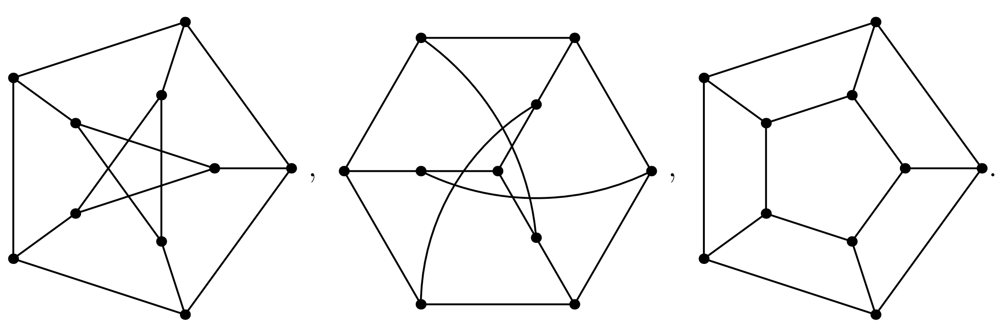
2. For each of the sequences below, decide whether or not it is the degree sequence of a graph. (If it is, give an explicit example, if not, say why not.)

$$\text{(i) } (9, 7, 5, 3, 1), \text{ (ii) } (3, 2, 2, 1), \text{ (iii) } (10, 3, 3, 2), \text{ (iv) } (3, 3, 3, 1).$$

Do any of your answers change if the graph has to be simple?

3. Let G and G' be graphs. Suppose that $f : V(G) \to V(G')$ is an isomorphism. Let $x, y \in V(G)$. Use induction to show that that the distance between x and y in G is equal to the distance between $f(x)$ and $f(y)$ in G'.

4. (a) Show that the graphs below all have the same degree sequence.



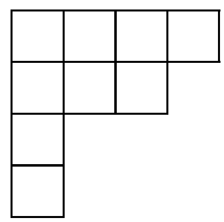
(b) Show that the first two graphs are isomorphic. *Hint:* start by finding a closed path of length 6 in the first.

(c) Show that neither is isomorphic to the third.

Turn over

5. Let G be a connected graph with k vertices of odd degree, where $k > 0$. Show that the minimum number of trails with mutually distinct edges needed to cover every edge of G is $k/2$.

How many continuous pen-strokes are needed to draw the diagram below if it is forbidden to draw any line segment twice?



6. (a) Show that the complete graph on n vertices has $n(n-1)/2$ edges.

(b) Show that if G is a simple graph on 6 vertices which is not connected then G has at most 10 edges. Can equality occur?

7. ($\star$) Assume that any two people are either friends or enemies. Show that in a room of 6 people that are either 3 mutual friends, or 3 mutual enemies. Show also that there are two people who have an equal number of enemies present in the room.

8. ($\star$) Here are two further results on derangements.

(a) Let $a_k(n)$ be the number of permutations of $\{1, 2, \ldots, n\}$ with exactly k fixed points. Use results from lectures to prove that

$$a_k(n) = \frac{n!}{k!}\left(1 - \frac{1}{1!} + \frac{1}{2!} - \ldots + \frac{(-1)^{n-k}}{(n-k)!}\right).$$

Hence, or otherwise, give a simple expression for $a_0(n) - a_1(n)$.

(b) Let $e(n)$ be the number of derangements of $\{1, 2, \ldots, n\}$ which are even permutations, and $o(n)$ the number which are odd permutations. By evaluating the determinant of the matrix

$$\begin{pmatrix} 0 & 1 & 1 & \ldots & 1 \\ 1 & 0 & 1 & \ldots & 1 \\ 1 & 1 & 0 & \ldots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \ldots & 0 \end{pmatrix}$$

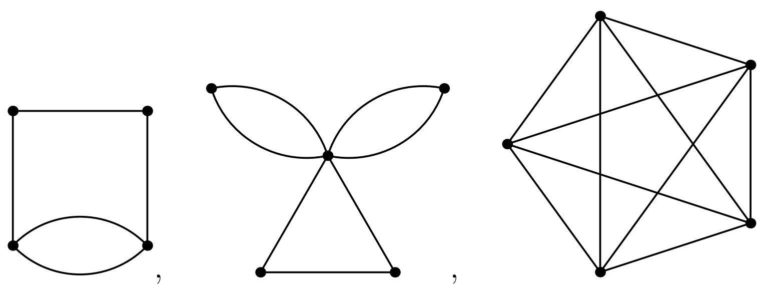
in two different ways, show that $e(n) - o(n) = (-1)^{n-1}(n-1)$. *Hint:* the eigenvectors and hence eigenvalues may be found directly; the determinant is the product of the eigenvalues.

Combinatorics	Name
MAP363 Sheet 2	Number
Hand in by 6th March	Year
	Mark: /10
	date marked: / /2007

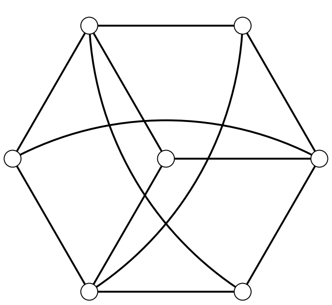
Please attach your working, with this sheet at the front.

Guidance on notation: graphs may have multiple edges, but may not have loops. A graph is simple if it has no multiple edges. ($\star$) indicates an optional question, included for interest only.

1. Which of the graphs shown below have (i) a closed Eulerian trail? (ii) a closed Hamiltonian path?



2. Show that the graph below is isomorphic to the complete bipartite graph $K_{4,3}$.



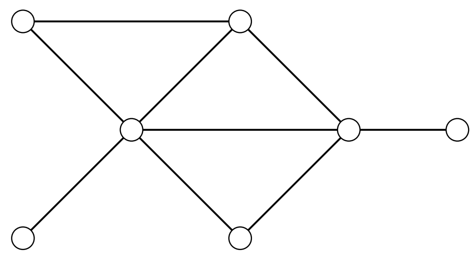
For which $n \in \mathbf{N}_0$ does $K_{4,3}$ have (i) a closed path of length n; (ii) a closed trail of length n?

3. Use Euler's formula to give an alternative proof that $K_{3,3}$ is not planar. *Hint:* adapt the proof given in lectures for K_5, noting that each face in a planar drawing of $K_{3,3}$ must be bounded by at least 4 edges.

Turn over

4. Professors Beeblebrox, Catalan, Descartes, Euler, Frobenius, Gauss and Hamilton go punting. Gauss considers it beneath his dignity to be in the same punt as anyone except Euler or Frobenius. Euler will tolerate Gauss, but dislikes Descartes. No-one except Hamilton is willing to share a punt with Beeblebrox.

(a) Explain the relevance of the graph shown below.



(b) By colouring this graph, or otherwise, determine the minimum number of punts required. How many possible seating plans are there using this number of punts?

5. Let G be a planar graph.

(a) Show that it follows from Euler's formula that G has a vertex of degree 5 or less. *Hint:* suppose G has n vertices, e edges, and f faces, and that all vertices of G have degree 6 or more. Show that $e \geq 3n$ and $f \leq 2e/3$, and then use Euler's formula to get a contradiction.

(b) Show by induction on the number of vertices of G that G may be coloured using 6 colours. *Hint:* in the inductive step delete from G the vertex given by (a), along with all the edges to which it belongs.

6. (a) Show that if G is a bipartite graph then every closed path in G has even length.

(b) Now suppose that G is a connected graph and that every closed path in G has even length. Fix $x \in G$. Set

$$A = \{y \in V(G) : \text{there is a path from } x \text{ to } y \text{ of even length}\},$$
$$B = \{y \in V(G) : \text{there is a path from } x \text{ to } y \text{ of odd length}\}.$$

Show that $A \cup B = V(G)$, $A \cap B = \varnothing$ and that if $\{u, v\}$ is an edge of G then either $u \in A$ and $v \in B$ or $u \in B$ and $v \in A$. Deduce that G is bipartite.

(c) ($\star$) Show that a connected bipartite graph has a unique bipartition.

7. ($\star$) Let $a, b, c \in \mathbf{N}_0$ with $a \geq b \geq c$.

(a) Show that (a, b) is the degree sequence of a graph if and only if $a = b$.

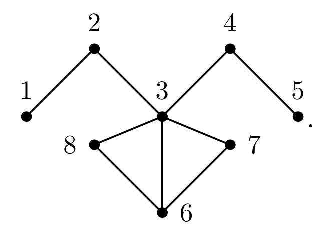
(b) Show that (a, b, c) is the degree sequence of a graph if and only if $a \leq b + c$ and $a + b + c$ is even.

(c) State and prove a generalisation to an arbitrary number of vertices.

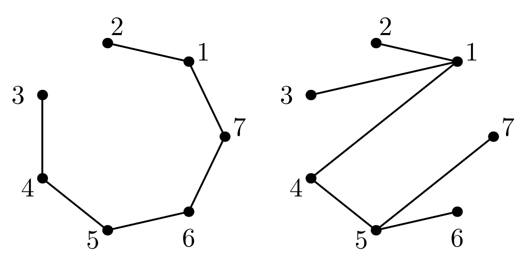
Combinatorics	Name
MAP363 Sheet 3	Number
Hand in by 21st March	Year
	Mark: /10
	date marked: / /2007

Please attach your working, with this sheet at the front.

Guidance on notation: graphs may have multiple edges, but may not have loops. A graph is simple if it has no multiple edges. A *tree* is a connected simple graph with no closed paths. ($\star$) indicates an optional question, included for interest only.

1. Find all spanning trees in the following graph: 

2. Find the Prüfer codes of the spanning trees in K_7 shown below.



Which spanning trees in K_7 have Prüfer codes $(1, 1, 1, 1, 1)$ and $(3, 3, 4, 4, 6)$? Which spanning trees in K_7 have Prüfer codes of the form (i, j, k, l, m) where i, j, k, l, m are mutually distinct?

3. The purpose of this question is to give an alternative proof of Theorem 4.2, that the number of edges in a tree on n vertices is $n - 1$.

(a) Let T be a tree. Show that if $(x_0, x_1, \ldots, x_{m-1}, x_m)$ is a path in T of greatest possible length then x_0 and x_m are leaves.

(b) Deduce that every non-empty tree has a leaf.

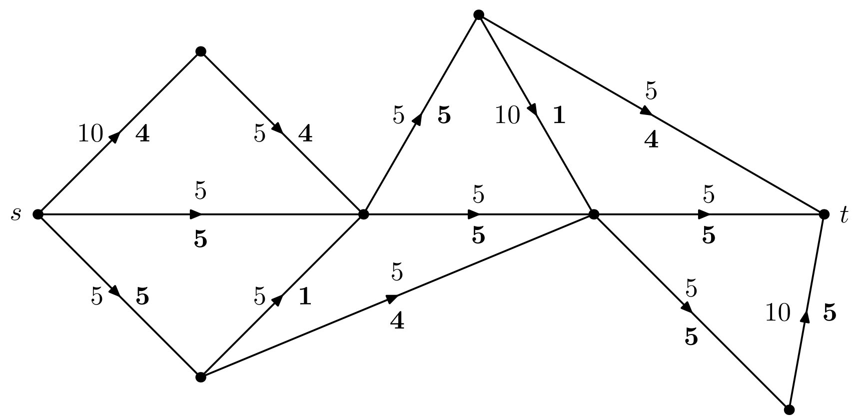
(c) Use the previous part and induction to prove that a tree on n vertices has exactly $n - 1$ edges.

4. Show that a connected simple graph is a tree if and only if there is a unique path between any two of its vertices.

Turn over

5. Let $(s_1, \ldots, s_{n-2})$ be the Prüfer code of a tree on $\{1, 2, \ldots, n\}$. Show that if vertex j has degree d then j appears exactly $d - 1$ times in the sequence $(s_1, \ldots, s_{n-2})$. Deduce that the tree has leaves $\{1, 2, \ldots, n\} \backslash \{s_1, \ldots, s_{n-2}\}$.

6. The diagram below shows a network with source s and sink t. Plain numbers give the capacities, bold numbers one possible flow.



(a) Check that the bold numbers give a valid flow from s to t.

(b) Apply one iteration of the Ford–Fulkerson algorithm to this flow.

(c) Find, with proof, a maximal flow in this network.

7. Let f be a flow in a network with source s, target t and edge list E. By summing $f(x, y)$ over all edges $(x, y) \in E$ and using conservation of flow show that

$$\sum_{y:(s,y)\in E} f(s, y) = \sum_{z:(z,t)\in E} f(z, t).$$

8. ($\star$) Families with 5, 4, 4 and 2 members plan to go on holiday. They have at their disposal three cars, each able to seat 4 people, and one motorbike, which is able to seat 2 people. It is essential that no two members of the same family travel in the same vehicle. Show that the problem of maximising the number of people who go on holiday can be formulated as a max-flow problem in a suitable network.

9. ($\star$) Let u_n be the average number of leaves in a spanning tree in the complete graph on $\{1, 2, \ldots, n\}$.

(a) Find u_2 and u_3 and check that $u_4 = 36/16$ and $u_5 = 320/125$.

(b) Use Prüfer codes to show that

$$u_n = n\left(1 - \frac{1}{n}\right)^{n-2}$$

Hint: start by finding the probability that vertex 1 is a leaf.

(c) Hence show that

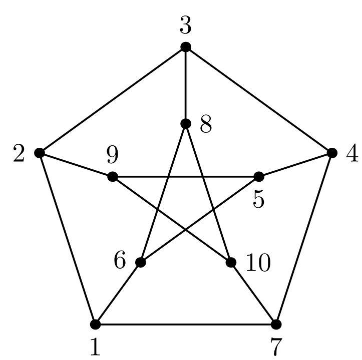
$$\frac{u_n}{n} \to \frac{1}{\mathrm{e}} \quad \text{as } n \to \infty.$$

Combinatorics	Name
MAP363 Sheet 4	Number
Hand in by 10th April	Year
	Mark: /10
	date marked: / /2007

Please attach your working, with this sheet at the front.

Guidance on notation: A partition of a natural number n is a sequence of natural numbers $(\lambda_1, \ldots, \lambda_k)$ such that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k$ and $\lambda_1 + \ldots + \lambda_k = n$. ($\star$) indicates an optional question, included for interest only.

1. Let G be the Petersen graph (shown below).



Let H be the subgroup of $\mathrm{Sym}\{1, 2, \ldots, 10\}$ consisting of the permutations h such that

$$\{i, j\} \text{ is an edge of } G \iff \{h(i), h(j)\} \text{ is an edge of } G.$$

(a) Find an element of H of order 5.

(b) Find an element of H of order 6. (*Hint:* The labelling above is consistent with Example 6.3 from lectures.)

(c) ($\star$) Find an element of H of order 4. Hence, or otherwise, draw the Petersen graph in a way that exhibits 4-fold symmetry.

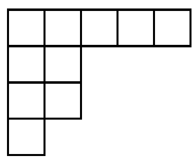
2. A 5-bead necklace is made using red and blue beads. Two necklaces should be regarded as the same if one is a rotation of the other. Show that 8 different necklaces can be made. How does this answer change if reflections are also considered?

3. An 8-bead necklace is made using c different colours of beads. Two necklaces should be regarded as the same if one is a rotation of the other. (Do not consider reflections.)

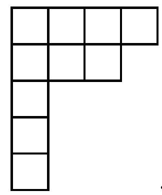
(a) Find the number of different necklaces as a polynomial in c.

(b) When $c = 2$, how many necklaces have exactly 4 beads of each colour?

4. The *conjugate* of a partition is obtained by reflecting its Young diagram in its major diagonal. For example $(5, 2, 2, 1)$ has conjugate $(4, 3, 1, 1, 1)$ since



reflects to



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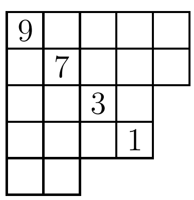
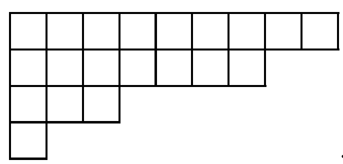
It is usual to write λ' for the conjugate of λ.

(a) Show that λ has exactly k parts if and only if k is the the largest part of λ'.

(b) Show that the number of self-conjugate partitions of n is equal to the number of partitions of n into distinct odd parts. [*Hint:* There is a bijective proof based on straightening 'hooks':



$\longleftrightarrow$



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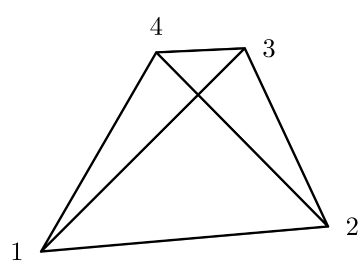
(c) Find a closed form for the generating function of self-conjugate partitions.

5. ($\star$) A paperweight manufacturer wishes to sell regular tetrahedra with faces painted red, blue and green. (It is not required that all three different colours are used.) Counting two paperweights as the same if one is a rotation of the other, how many different models can be sold?



Hint: Let $X = \{1, 2, 3, 4\}$ and let $H \leq \mathrm{Sym}(X)$ be the subgroup of permutations induced by rotations of the tetrahedron shown above. Start by proving that H consists of all even permutations of $\{1, 2, 3, 4\}$, so

$$H = \{1, (12)(34), (13)(24), (14)(23), (123), \dots, (234)\}$$

where there are in total 8 elements of order 3.