PRIFYSGOL CYMRU ABERTAWE
UNIVERSITY OF WALES SWANSEA

DEGREE EXAMINATIONS 2007

MODULE MAP363

Combinatorics

Time Allowed — 2 hours

There are SIX questions on the paper.
A candidate's best THREE questions will be used for assessment.

No calculators are permitted.

Each question has equal weight. The maximum possible mark is 75/75.

turn over

1. Let G be a graph with vertex set V and edge list E.

Define the *degree* of a vertex $x \in V$. What does it mean to say that $(x_0, x_1, \ldots, x_m)$ is a *trail* in G? What is the *length* of this trail? When is this trail *closed*? What does it mean to say that G is *connected*? [6 Marks]

Suppose that G is connected and that all vertices of G have even degree. Show that G has a closed trail of non-zero length. Hence, or otherwise, prove that G has a closed trail passing through all the edges in E. [12 Marks]

For $n \in \mathbb{N}$, let K_n denote the complete graph with vertex set $\{1, 2, \ldots, n\}$. When does K_n have a closed trail passing through all its edges? [3 Marks]

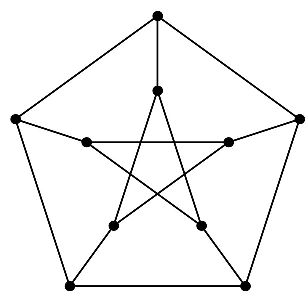
For each $n \in \mathbb{N}$ determine the minimum number of continuous pen-strokes needed to draw K_n. [4 Marks]

2. (a) Let G be a simple graph. What does it mean to say that G is (i) *planar*? (ii) a *tree*? [4 Marks]

(b) State and prove Euler's formula relating the number of vertices, edges and faces of a simple planar graph. [10 Marks]

[*You may assume that a tree on n vertices has precisely $n - 1$ edges.*]

(c) What is the minimum length of a closed path in the Petersen graph (shown below)?



By counting edges, show that if there is a planar drawing of the Petersen graph with f faces, then $5f \leq 30$. Hence show that the Petersen graph is not planar. [7 Marks]

Find with proof the smallest number of edges that can be removed from the Petersen graph in order to leave a planar graph.
[4 Marks]

3. Let X be a finite subset of $\mathbb{N}$ of size $n \geq 2$.

(a) Let T be a spanning tree in the complete graph on X. What does it mean to say that a vertex of T is a *leaf*? Define the *Prüfer code* of T. [6 Marks]

(b) Find the spanning tree in the complete graph on the set $\{1, 2, 3, 4, 5, 6\}$ with Prüfer code $(3, 5, 4, 3)$. [5 Marks]

(c) Use Prüfer codes to prove that the number of spanning trees in the complete graph on X is n^{n-2}. [10 Marks]

How many of these trees have exactly 2 leaves? [4 Marks]

4. Let N be a network with vertex set V and edge set E. Write $c(x, y)$ for the capacity of the edge $(x, y) \in E$. Let $s \in V$ be the source vertex and let $t \in V$ be the target vertex.

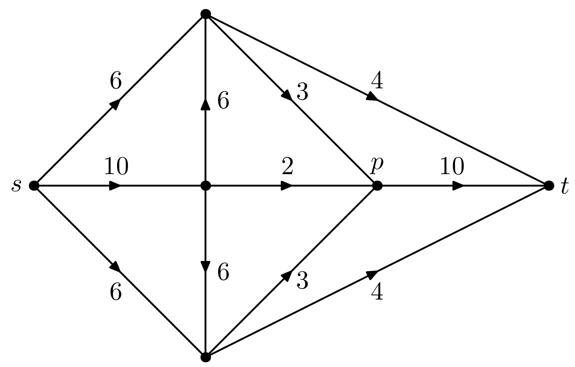
(a) What does it mean to say that f is a *flow* in N? Define the *value* of f, $\operatorname{val} f$. What does it mean to say that f is *maximal*?

What does it mean to say that (S, T) is a *cut* of N? Define the *capacity* of (S, T), $\operatorname{cap}(S, T)$. [7 Marks]

(b) Prove that if f is any flow in N and (S, T) is any cut then $\operatorname{val} f \leq \operatorname{cap}(S, T)$. Show that if equality holds, then the flow f is maximal. [9 Marks]

(c) Find, with proof, a maximal flow in the network below. [6 Marks]

(The numbers show the capacity of the edges.)



Now suppose that at most m units may pass through vertex p. Find a formula for the value of the maximal flow in terms of m. [3 Marks]

5. (a) Let X be a set and let $G \leq \mathrm{Sym}(X)$ be a permutation group. For $g \in G$, let $\mathrm{Fix}\, g = \{x \in X : g(x) = x\}$. Prove that

$$\frac{1}{|G|} \sum_{g \in G} |\,\mathrm{Fix}\, g|$$

is the number of orbits of G on X. [12 Marks]

[*You may assume the orbit-stabiliser theorem, provided it is clearly stated.*]

(b) A 9-bead necklace is made using c different colours of beads. Two necklaces are regarded as the same if one is a rotation of the other. (Reflections should *not* be considered.) Find the number of different necklaces as a polynomial in c. [9 Marks]

If there are two colours, red and blue, find the number of different necklaces which have 6 red beads and 3 blue beads. [4 Marks]

6. Let $n \in \mathbb{N}_0$. Define a *partition* of n. [2 Marks]

Prove that if $p(n)$ is the number of partitions of n, then

$$\sum_{n=0}^{\infty} p(n)x^n = \prod_{r=1}^{\infty} \frac{1}{1-x^r}.$$ [8 Marks]

Show that the number of partitions of n into odd parts is equal to the number of partitions of n with distinct parts. [8 Marks]

What is meant by the *Young diagram* of a partition? [2 Marks]

Let $m \in \mathbb{N}$. Show that the number of partitions of n with largest part $\leq m$ is equal to the number of partitions of n with at most m parts. [5 Marks]