Answers to specimen paper questions

Most of the answers below go into rather more detail than is really needed. Please let me know of any mistakes.

Question 1.

(a) The *degree* of a vertex x is the number of edges to which it belongs. If G has vertices of degree d_1, d_2, $\ldots$, d_n, where $d_1 \geq d_2, \ldots, \geq d_n$ then the *degree sequence* of G is

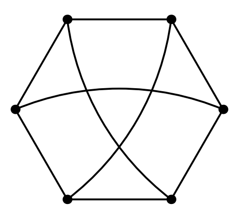
$$(d_1, d_2, \ldots, d_n).$$

We say that G is *simple* if there is at most one edge between any two of its vertices.

(b) **Theorem** (Handshaking Theorem): If G is a graph with degree sequence $(d_1, \ldots, d_n)$ then G has exactly $2(d_1 + \ldots + d_n)$ edges.

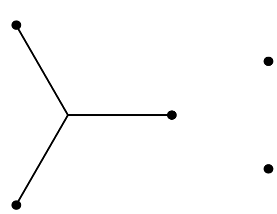
Proof: let $x_1, \ldots, x_n$ be the vertices of G, ordered so that $\deg x_i = d_i$. In the sum $d_1 + \ldots + d_n$ we count each edge twice: an edge $\{x_i, x_j\}$ is counted once in d_i as an edge leaving x_i, and once in d_j as an edge leaving x_j. So the total number of edges of G is $2(d_1 + \ldots + d_n)$.

(c) (i) Yes. The complete bipartite graph $K_{3,3}$ has $(3, 3, 3, 3, 3, 3)$ as its degree sequence.



(ii) No. The sum of the degrees is 15. But the Handshaking Theorem implies that the sum of the degrees of a graph is even.

(iii) Yes. The simple graph shown below has degree sequence $(3, 1, 1, 1, 0, 0)$.



(iv) No. Suppose G is a simple graph with degree sequence $(4, 2, 1, 1, 0)$. Let x be the vertex of degree 4. As G is simple and has only 5 vertices, x must be adjacent to all the other four vertices of G. In particular x is adjacent to the vertex of degree 0, a contradiction.

Question 2. *Correction: in the graph in part (c), the edge running southwest from vertex y should be deleted.*

(a) A *closed path* is a sequence of vertices $(x_0, x_1, \ldots, x_m, x_0)$ such that
(i) $x_i \neq x_j$ if $i \neq j$;
(ii) x_{i-1} is adjacent to x_i for $1 \leq i \leq m$;
(iii) x_m is adjacent to x_0.
A closed path is *Hamiltonian* if it visits every vertex of G.

The graph G is *bipartite* if there is a partition of its vertices into subsets A and B such that if $\{x, y\}$ is an edge of G then either $x \in A$ and $y \in B$, or $x \in B$ and $y \in A$.

(b) Suppose that G has a closed Hamiltonian path. We may order the vertices in this path so that its initial vertex lies in A. Then, as G is bipartite, the next vertex must lie in B, the next in A, and so on. The path therefore has the form

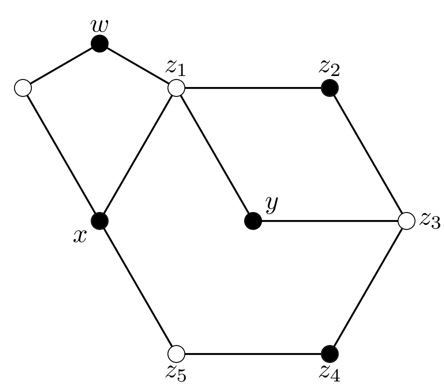
$$(a_1, b_1, a_2, b_2, \ldots, a_n, b_n, a_1)$$

where $a_i \in A$, $b_i \in B$ for $1 \leq i \leq n$. Every vertex in G appears in this list, so we must have

$$A = \{a_1, \ldots, a_n\} \quad \text{and} \quad B = \{b_1, \ldots, b_n\}.$$

In particular, $n = |A| = |B|$.

(c) (i) A bipartition is indicated by the colouring below.



(ii) Suppose that G has a closed Hamiltonian path. If this path doesn't use the edge $\{x, y\}$ then it is a closed Hamiltonian path in the bipartite graph above. This contradicts (b) as the sets in the bipartition have different sizes.

Hence the path must use the edge $\{x, y\}$. By ordering the vertices we may assume it starts at x and then goes to y. There are two such paths which visit vertices z_4 and z_5, namely

$$(x, y, z_1, z_2, z_3, z_4, z_5, x) \quad \text{and} \quad (x, y, z_3, z_4, z_5).$$

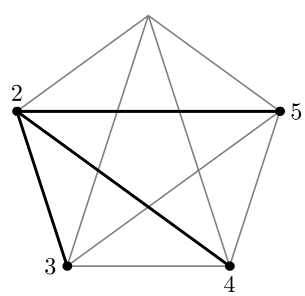
Neither of these paths visit vertex w, so neither is Hamiltonian. Hence G does not have a closed Hamiltonian path.

Question 3.

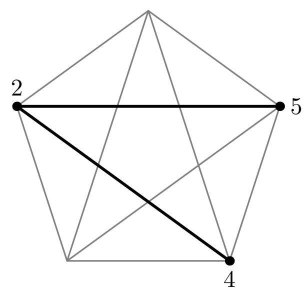
(a) A subgraph T of G is a *spanning tree* if (i) it contains all the vertices of G, and (ii) it is a tree, i.e. it is a connected graph with no closed paths.

(b) Suppose that b is the leaf of T with the smallest label, and that b is adjacent to vertex s. The first entry in the Prüfer code of T is s. To find the remaining entries, delete b and the edge $\{b, s\}$ from T, and then repeat with the new tree. Stop when only 2 vertices are left.

In the given tree, the smallest numbered leaf is 1, which is attached to 5. So we start by writing down 5. We then delete 1 and the edge $\{1, 5\}$, leaving the tree below.



Now the smallest numbered leaf is 3 which is adjacent to 2. So we write down 2. We then delete 3 and the edge $\{2, 3\}$ leaving the tree below.

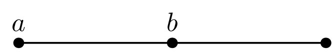


Now the smallest numbered leaf is 4, so we write down 2 and delete $\{2, 4\}$. The remaining tree has just two vertices so we stop.

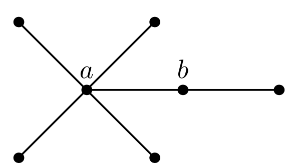
The Prüfer code is therefore $(5, 2, 2)$.

[*There are many ways to describe the algorithm; any reasonably clear explanation along with the right Prüfer code would get full marks.*]

(c) If a spanning tree has Prüfer code (a, a, a, a, b) then it must have 5 leaves (since a vertex is a leaf if and only if it doesn't appear in the code). At least four of these leaves are attached to a, and they are deleted in the first 4 steps of the algorithm. This leaves a tree on 3 vertices. This tree must have the form below, since b is one of its vertices, and b is not a leaf.



The original tree is therefore of the form shown below.



We can choose any number in $\{1, 2, \ldots, 7\}$ for a, and any of the remaining numbers for b. Hence there are 42 trees with Prüfer code of the form (a, a, a, a, b) where $a \neq b$.

Question 4.

Let E be the edge set of N and let $c(x, y)$ be the capacity of the edge (x, y).

(a) A *flow* in N is an assignment of a real number $f(x, y)$ to each edge (x, y) of N such that

(i) $f(x, y) < c(x, y)$ for all edges x, y;

(ii) if x is a vertex of N other than s and t then

$$\sum_{y:(x,y)\in E} f(x,y) = \sum_{z:(z,x)\in E} f(z,x).$$

The *value* of a flow f is the total flow in the edges leaving s, i.e.

$$\operatorname{val} f = \sum_{y:(s,y)\in E} f(s,y).$$

A flow is *maximal* if its value is as large as possible.

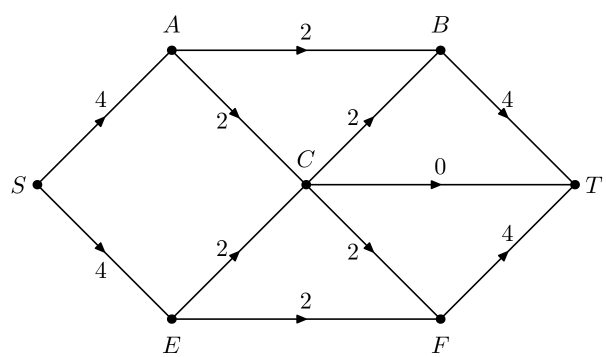
(b) (S,T) is a cut of N if $\{S,T\}$ is a partition of the vertices of N such that $s \in S$ and $t \in T$. The *capacity* of the cut (S,T) is

$$\operatorname{cap}(S,T) = \sum c(x,y)$$

where the sum is over all edges (x,y) of N such that $x \in S$ and $y \in T$.

(c) The value of any flow is bounded above by the total capacity of all the edges from S to T. Hence if there is a flow whose value is equal to this total capacity, it must be maximal.

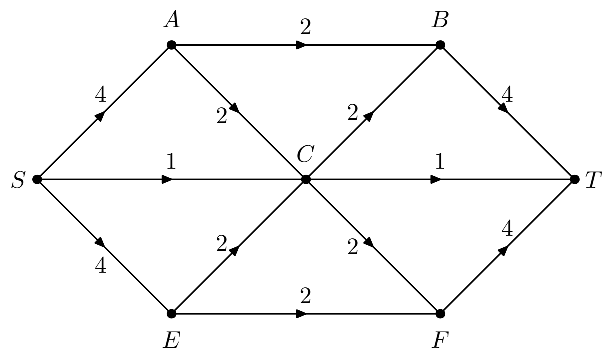
(d) Since only 4 units can leave A and E, it seems reasonable to guess that the flow shown below will be close to maximal. (The numbers on the edges show the flow values.)



In fact this flow *is* maximal. To prove this we use part (c). If $S = \{S,A,E\}$ and $T = \{B,C,F,T\}$ then $f(x,y) = c(x,y)$ whenever $x \in S$ and $y \in T$.) There are no edges from T to S. Hence the hypothesis in (c) is satisfied, and so the flow is maximal.

[*It would also be fine to say that the total capacity of the pipes leaving* A *and* E *(the two neighbours of* S*) is* 8*, and so at most* 8 *units of flow can leave* S.]

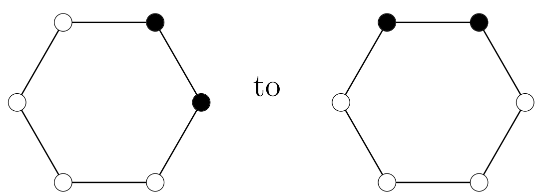
Using the new pipe we can increase the flow value by 1.



To prove that this new flow is maximal we again use (c), this time with the sets $S' = \{S, A, E, C\}$ and $T' = \{B, F, T\}$.

Question 5

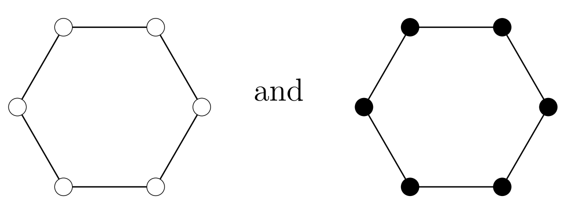
Rotating the hexagon permutes the set X of vertex colourings. For example, k sends



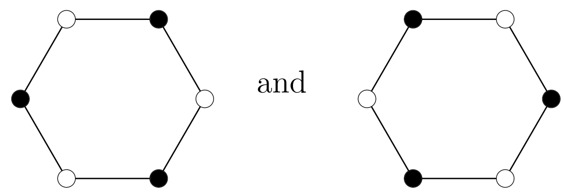
and so on. By identifying k with this permutation of X we may think of $\langle k \rangle$ as a subgroup of $\mathrm{Sym}(X)$.

There are 2^6 colourings in all (choose one colour for each of the six vertices) and all of them are fixed by the identity e.

If a colouring is fixed by k then all its vertices must have the same colour. So there are 2 such colourings:



There are 4 colourings fixed by k^2 (rotation by 120°): the two above and the two below.



A similar argument shows that there are $2^3 = 8$ vertex colourings fixed by k^3 (rotation by 180°).

Since k^4 is the inverse of k^2, it fixes the same colourings as k^2. Hence k^4 fixes 4 colourings. Similarly we deduce that k^5 fixes just 2 colourings.

Using the given formula we find that K has

$$\frac{2^6 + 2 + 2^2 + 2^3 + 2^2 + 2}{6} = \frac{84}{6} = 14$$

orbits on $\mathrm{Sym}(X)$. Orbits correspond to distinct colourings, so there are 14 ways to colour the hexagon (when we regard two colourings as the same if one can be rotated into the other).

There are 6 reflections to consider. Three of them, through the mid-points of opposite edges, fix $2^3 = 8$ colourings, and the other three, through pairs of opposite vertices, fix $2^4 = 16$ colourings. The number of orbits is

$$\frac{2^6 + 2 + 2^2 + 2^3 + 2^2 + 2 + 3.2^3 + 3.2^4}{12} = \frac{84}{12} + \frac{3.24}{12} = 13.$$

There are therefore 13 distinct colourings if we also allow reflections.

Question 6

(a) The generating function of f is the power series

$$f(0) + f(1)x + f(2)x^2 + \dots.$$

(b) Suppose that $15 = 3a + 5b$. Subtracting off multiples of 3 from 15 gives $12, 9, 6, 3$, none of which is divisible by 5. So if $a \geq 1$ then $a = 5$ and $b = 0$. Similarly if $b \geq 1$ then $b = 3$ and $a = 0$. So $f(15) = 2$.

Suppose that $14 = 3a + 5b$. Subtracting off multiples of 3 leaves $11, 8, 5, 2$, of which only 5 is divisible by 5. So $a = 3$ and $b = 1$ is the only possibility, and $f(14) = 1$. Similarly $f(16) = 1$.

Let $n \in \mathbb{N}$. The coefficient of x^n in

$$(1 + x^3 + x^6 + x^9 + \ldots)(1 + x^5 + x^{10} + x^{15} + \ldots)$$

is equal to the number of ways to choose a term x^{3a} on the left-hand-side and a term x^{5b} on the right-hand-side such that $3a + 5b = n$. Hence

$$\begin{aligned}(1 + x^3 + x^6 + \ldots)(1 + x^5 + x^{10} + \ldots) &= f(0) + f(1)x + f(2)x^2 + \ldots \\ &= F(x).\end{aligned}$$

Now sum the geometric series to get

$$F(x) = \frac{1}{(1 - x^3)(1 - x^5)}$$

as required.

(c) Multiplying through by $(1 - x^3)(1 - x^5)$ gives

$$F(x) - x^3F(x) - x^5F(x) + x^8F(x) = 1.$$

For $n \geq 8$, the coefficient of x^n on the left-hand-side is

$$f(n) - f(n - 3) - f(n - 5) + f(n - 8).$$

The coefficient of x^n on the right-hand-side is 0 for all $n \geq 1$. Hence

$$f(n) - f(n - 3) - f(n - 5) + f(n - 8) = 0 \quad \text{for } n \geq 8.$$