MT5454 Combinatorics: MSc Mini-project

Attempt Questions 1, 2 and 3 and *one of* Questions 4, 5, 6.

To be submitted to the Mathematics Office McCrea 243 by 12 noon, Monday 11th January 2016. Put your candidate number *but not your name or student number* on the top sheet.

Hand-written answers are acceptable but use of LaTeX is encouraged. Any general results from lectures or Wilf's book *generatingfunctionology* may be used without proof provided that they are clearly stated.

Each question is worth 25 marks: 4 marks per question will be given for the clarity and readability of your answers.

A *set partition of* $\{1,\ldots,n\}$ *into* m *sets* is a set $\{A_1,\ldots,A_m\}$ of m disjoint subsets of $\{1,\ldots,n\}$ such that $A_1 \cup \ldots \cup A_m = \{1,\ldots,n\}$.

For $n, m \in \mathbf{N}_0$, define the *Stirling Number of the Second Kind* $\left\{ {n \atop m} \right\}$ to be the number of set partitions of $\{1,\ldots,n\}$ into m sets.

For example, $\left\{ {3 \atop 2} \right\} = 3$; the relevant set partitions are $\big\{\{1\},\{2,3\}\big\}$, $\big\{\{2\},\{1,3\}\big\}$, $\big\{\{3\},\{2,3\}\big\}$, and $\left\{ {4 \atop 2} \right\} = 7$. To avoid overcounting, note that $\big\{\{2\},\{1,3\}\big\} = \big\{\{1,3\},\{2\}\big\} = \big\{\{3,1\},\{2\}$, and so on.

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1. (a) Make a table showing $\left\{ {n \atop m} \right\}$ for $0 \le m \le n \le 5$. [*Hint: check your answer against sequence A008277 in the Online Encyclopedia of Integer Sequences.*]

(b) Find $\left\{ {n \atop 0} \right\}$, $\left\{ {n \atop 1} \right\}$ and $\left\{ {n \atop n} \right\}$ for each $n \in \mathbf{N}_0$ and $\left\{ {n \atop n-1} \right\}$ for each $n \in \mathbf{N}$. Justify your answers.

(c) Prove that $\left\{ {n \atop 2} \right\} = 2^{n-1} - 1$ for each $n \in \mathbf{N}$.

(d) Let $m, n \in \mathbf{N}$. Show that $m!\left\{ {n \atop m} \right\}$ is the number of surjective functions $\{1,2,\ldots,n\} \to \{1,2,\ldots,m\}$. [*Hint: what two surjective functions* $\{1,2,3,4\} \to \{1,2\}$ *can be defined in a natural way given the set partition* $\big\{\{1,3\},\{2,4\}\big\}$*?*]

(e) Using the Principle of Inclusion and Exclusion, or otherwise, prove that

$$m!\left\{ {n \atop m} \right\} = \sum_{r=0}^{m} (-1)^r \binom{m}{r} (m-r)^n$$

for all $m, n \in \mathbf{N}$. [*Hint: see Question 9 on Sheet 2 for a step-by-step solution. Bear in mind that the notation is slightly different.*]

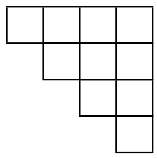
2. (a) Prove that $\left\{ {n \atop m} \right\} = \left\{ {n-1 \atop m-1} \right\} + m\left\{ {n-1 \atop m} \right\}$ for all $n, m \in \mathbf{N}$.

(b) Let $f_m(x) = \sum_{n=0}^{\infty} \left\{ {n \atop m} \right\} x^n$. Prove that

$$f_m(x) = \frac{x^m}{(1-x)(1-2x)\ldots(1-mx)}$$

for all $m \in \mathbf{N}$.

3. For $n \in \mathbf{N}_0$ let T_n denote the right-justified triangular board with k squares in row k for each $k \in \{1, 2, \ldots, n\}$. For example T_4 is shown below.



(a) Prove that $r_k(T_n) = r_k(T_{n-1}) + \big(n-(k-1)\big) r_{k-1}(T_{n-1})$ for all n, $k \in \mathbf{N}$. [*Hint: interpret the second summand as the number of ways to place $k-1$ non-attacking rooks on the subboard T_{n-1} of T_n obtained by deleting the rightmost column, and then to put one further rook somewhere in this column.*]

(b) Use Q2(a) to show that $f_{T_n}(x) = \sum_{k=0}^{n} \genfrac\{\}{0pt}{}{n+1}{n+1-k} x^k$ for each $n \in \mathbf{N}_0$.

(c) Show that $\sum_{r=0}^{n} (-1)^r r! \genfrac\{\}{0pt}{}{n+1}{r+1} = 0$ for all $n \in \mathbf{N}$. [*Hint: use Theorem 6.10.*]

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4. Give a bijective proof of the result in Q3(b).

5. For $n \in \mathbf{N}_0$, let $B_n = \sum_{m=0}^{n} \genfrac\{\}{0pt}{}{n}{m}$ be the number of set partitions of $\{1, \ldots, n\}$ into any number of sets. (These numbers are called *Bell Numbers*.)

(a) Write down the values of B_0, B_1, B_2, B_3, B_4.

(b) Using Q1(e), or otherwise, show that $\sum_{n=0}^{\infty} \frac{1}{n!} \genfrac\{\}{0pt}{}{n}{m} x^n = (\mathrm{e}^x - 1)^m / m!$ for each $m \in \mathbf{N}_0$.

(c) Hence show that $\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{n!} \genfrac\{\}{0pt}{}{n}{m} x^n y^m = \exp\big((\mathrm{e}^x - 1)y\big)$.

(d) Using (c), or otherwise, find, with proof, a closed form for $\sum_{n=0}^{\infty} B_n x^n / n!$.

(e) Prove that $\sum_{m=0}^{n} (-1)^m \binom{n}{m} B_{n+1-m} = B_n$ for each $n \in \mathbf{N}_0$.

[*Hint: for an 'otherwise' solution to this question you could use the theory in Chapter 3 of generatingfunctionology.*]

6. Let $\genfrac\{\}{0pt}{}{n}{m}^\star$ be the number of set partitions of $\{1, 2, \ldots, n\}$ into m sets such that no set contains both k and $k+1$ for any $k \in \{1, 2, \ldots, n-1\}$. For example, $\genfrac\{\}{0pt}{}{4}{3}^\star = 3$ counts the set partitions $\big\{\{1,3\},\{2\},\{4\}\big\}$, $\big\{\{1,4\},\{2\},\{3\}\big\}$, $\big\{\{1\},\{2,4\},\{3\}\big\}$.

(a) Find $\genfrac\{\}{0pt}{}{n}{2}^\star$ and $\genfrac\{\}{0pt}{}{n}{n-1}^\star$ for each $n \in \mathbf{N}$. Justify your answers.

(b) Find with proof a recurrence analogous to the one in Q2(a) for $\genfrac\{\}{0pt}{}{n}{m}^\star$.

(c) For each $m \in \mathbf{N}_0$, find, with proof, a closed form for $\sum_{n=0}^{\infty} \genfrac\{\}{0pt}{}{n}{m}^\star x^n$.