A FIELD-INDEPENDENT FILTRATION OF PLETHYSTIC MODULES FOR $\mathrm{SL}_2(\mathbb{F})$ THAT CATEGORIFIES A PRODUCT RULE FOR THE CARTAN SUBALGEBRA OF $U_q(\mathfrak{sl}_2)$

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Abstract. We lift a product rule in the Cartan subalgebra of quantum $\mathfrak{sl}_2$ to a filtration of the plethystic representation $\Delta^{(n,m)}\mathrm{Sym}^d E$ of the affine group scheme of the algebraic group SL_2, where E is the natural representation and $\Delta^{(n,m)}$ the Weyl functor. This is a significant step towards a categorification of quantum $\mathfrak{sl}_2$. Our filtration is an addition to a growing family of field-independent isomorphisms of SL_2 representations that include Hermite reciprocity and the Wronskian isomorphism. It is the first such field-independent result requiring multiple filtration layers. It is proved by combinatorial techniques using the authors' symmetric functions model for Weyl modules.

1. Introduction

Let $\mathbb{F}$ be an arbitrary field, let E be the natural representation of $\mathrm{SL}_2(\mathbb{F})$, and let Δ^λ be the Weyl functor (see §2.6) whose formal character is the Schur function s_λ. The main result of this paper is a field-independent filtration of the $\mathrm{SL}_2(\mathbb{F})$-representations $\Delta^{(n,m)}\,\mathrm{Sym}^d E$.

Theorem 1.1. *Let $m, n, d \in \mathbb{N}$ with $m \leqslant n, d$. The $\mathrm{SL}_2(\mathbb{F})$-plethysm $\Delta^{(n,m)}\,\mathrm{Sym}^d E$ has the filtration*

$$\begin{gathered}
\mathrm{Sym}_{n+m}\,\mathrm{Sym}^{d-m} E \otimes \Delta^{(n-m)}\,\mathrm{Sym}^m E\\
\mathrm{Sym}_{n+m}\,\mathrm{Sym}^{d-m+1} E \otimes \Delta^{(n-m+1,1)}\,\mathrm{Sym}^{m-1} E\\
\vdots\\
\mathrm{Sym}_{n+m}\,\mathrm{Sym}^{d-2} E \otimes \Delta^{(n-2,m-2)}\,\mathrm{Sym}^2 E\\
\mathrm{Sym}_{n+m}\,\mathrm{Sym}^{d-1} E \otimes \Delta^{(n-1,m-1)}\,\mathrm{Sym}^1 E
\end{gathered}$$

from top layer to bottom layer.

The primary motivation for this theorem is that it categorifies a product rule in the Cartan subalgebra of the quantum group $\mathcal{U}_q(\mathfrak{sl}_2)$. We explain this now. Let $\mathcal{A} = \mathbb{Z}[q, q^{-1}]$. Lusztig's integral form $\mathcal{U}_\mathcal{A}(\mathfrak{sl}_2)$ for the Cartan subalgebra of $\mathcal{U}_q(\mathfrak{sl}_2)$ is spanned by the *Lusztig elements*

$$\begin{bmatrix} K; c \\ n \end{bmatrix} = \frac{[K;c][K;c-1]\cdots[K;c-n+1]}{[n]_q[n-1]_q\cdots[1]_q},$$

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where $c \in \mathbb{N}_0$, $n \in \mathbb{N}_0$, $[K;a] = (q^a K - q^{-a}K^{-1})/(q-q^{-1})$ and $[b]_q = (q^b - q^{-b})/(q-q^{-1})$ is the usual quantum integer. By Theorem 6.2 in the authors' paper [GMSW26a], the multiplication rule for the Lusztig elements is

$$\begin{bmatrix} K;c \\ n \end{bmatrix} \begin{bmatrix} K;b \\ m \end{bmatrix} = \sum_{i \geqslant 0} \begin{bmatrix} n-c+b \\ i-c \end{bmatrix}_q \begin{bmatrix} m-b+c \\ i-b \end{bmatrix}_q \begin{bmatrix} K;i \\ n+m \end{bmatrix}, \tag{1.1}$$

where $\left[\begin{smallmatrix} n-c+b \\ i-c \end{smallmatrix}\right]_q$ and $\left[\begin{smallmatrix} m-b+c \\ i-b \end{smallmatrix}\right]_q$ are the quantum binomial coefficients defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q \cdots [n-k+1]_q}{[k]_q \dots [1]_q}.$$

Formally substituting q^d for K, specializing by $b = m$ and $c = n$ and using the basic identity $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_q = \left[\begin{smallmatrix} n \\ n-k \end{smallmatrix}\right]_q$ four times we obtain the q-binomial identity

$$\begin{bmatrix} d+n \\ d \end{bmatrix}_q \begin{bmatrix} d+m \\ d \end{bmatrix}_q = \sum_{i \geqslant 0} \begin{bmatrix} n \\ n+m-i \end{bmatrix}_q \begin{bmatrix} m \\ n+m-i \end{bmatrix}_q \begin{bmatrix} d+i \\ n+m \end{bmatrix}_q,$$

which we interpreted as a form of the q-Pfaff–Saalscütz identity and proved combinatorially in [GMSW26a]. Taking the difference of two instances of this identity we obtain

$$\begin{aligned} &\begin{bmatrix} d+n \\ d \end{bmatrix}_q \begin{bmatrix} d+m \\ d \end{bmatrix}_q - \begin{bmatrix} d+n+1 \\ d \end{bmatrix}_q \begin{bmatrix} d+m-1 \\ d \end{bmatrix}_q \\ &\quad = \sum_{i \geqslant 0} \left(\begin{bmatrix} n \\ n+m-i \end{bmatrix}_q \begin{bmatrix} m \\ n+m-i \end{bmatrix}_q - \begin{bmatrix} n+1 \\ n+m-i \end{bmatrix}_q \begin{bmatrix} m-1 \\ n+m-i \end{bmatrix}_q \right) \begin{bmatrix} d+i \\ n+m \end{bmatrix}_q \\ &\quad = \sum_{k \geqslant 0} \left(\begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} m \\ k \end{bmatrix}_q - \begin{bmatrix} n+1 \\ k \end{bmatrix}_q \begin{bmatrix} m-1 \\ k \end{bmatrix}_q \right) \begin{bmatrix} d+n+m-k \\ n+m \end{bmatrix}_q . \end{aligned}$$

By the basic corollary of the Jacobi–Trudi identity in (2.4), when $n \geqslant m$, the left-hand side is equal to the plethystic product $s_{(n,m)} \circ s_d(q, q^{-1})$. Similarly the expression in parentheses in the final line is $s_{(n-k,m-k)} \circ s_k(q, q^{-1})$. Thus the multiplication rule (1.1) implies the identity

$$s_{(n,m)} \circ s_d(q, q^{-1}) = \sum_{k=1}^{m} \begin{bmatrix} d+n+m-k \\ n+m \end{bmatrix}_q s_{(n-k,m-k)} \circ s_k(q, q^{-1}). \tag{1.2}$$

Interpreting $s_{(n,m)} \circ s_d(q, q^{-1})$ as the character of $\Delta^{(n,m)} \operatorname{Sym}^d E$, $\left[\begin{smallmatrix} d+n+m-k \\ n+m \end{smallmatrix}\right]_q$ as the character of $\operatorname{Sym}_{n+m} \operatorname{Sym}^{d-k} E$ and $s_{(n-k,m-k)} \circ s_k(q, q^{-1})$ as the character of $\Delta^{(n-k,m-k)} \operatorname{Sym}^k E$ (as explained in §2.7) we see that Theorem 1.1 is a modular lift of (1.2) that categorifies (1.1).

A novel categorification. At the turn of the century, tensor 2-categories were constructed whose Grothendieck ring is isomorphic to the representation ring of $\mathcal{U}_q(\mathfrak{sl}_2)$ [BFK99, Str05, FKS07]. This may be regarded as a first approximation to the problem of categorifying the quantum group itself. In this setting, the weight spaces of a module are replaced by abelian categories, and the Chevalley generators E and F act as exact functors between them. Subsequently, an idempotented form of the algebra $\mathcal{U}_q(\mathfrak{sl}_2)$ was categorified by Lauda [Lau10], extending the earlier categorification of the positive part $\mathcal{U}_q^+(\mathfrak{sl}_2)$ by Crane and Frenkel [CF94] at a root of unity. This lead to the categorification of (idempotented forms of) all quantum groups by Rouquier [Rou08] and by Khovanov and Lauda [KL09]. In all of these categorifications, each generator K of the Cartan is replaced with a system of mutually orthogonal idempotents (one for each weight space). None of these earlier categorifications can express (1.1). Indeed, Theorem 1.1 is the first categorification of an instance of the multiplication rule (1.1) requiring a filtration of $\mathrm{SL}_2(\mathbb{F})$-modules for $\mathbb{F}$ an arbitrary field. As such we believe it is a significant step towards a categorification of $\mathcal{U}_q(\mathfrak{sl}_2)$.

Modular plethystic isomorphisms. We emphasise that Theorem 1.1 holds over an arbitrary field $\mathbb{F}$. As such, it provides a filtration of rational representations for the affine group scheme of the algebraic group SL_2. It is notable as an addition to a small but growing family of field-independent isomorphisms of plethysms, and the *first* that requires a filtration with multiple layers.

The earliest results in this area are *Hermite reciprocity*

$$\mathrm{Sym}^k \mathrm{Sym}_n E \cong \mathrm{Sym}_n \mathrm{Sym}^k E, \tag{1.3}$$

and the *Wronskian isomorphism*

$$\mathrm{Sym}_n \mathrm{Sym}^k E \cong \textstyle\bigwedge^n \mathrm{Sym}^{n+k-1} E, \tag{1.4}$$

each proved independently in [AFP$^+$19, McDW22]. The latter may be regarded as a modular lift of the well-known fact that the number of n-multisubsets of a set of size $k+1$ is $\binom{n+k}{n}$. Next in [MW25], the authors proved Theorem 1.1 when $m = 1$; this is the very special case when the filtration has only one layer. Later [GMSW25] lifted the trinomial revision identity $\binom{M+N}{N}\binom{M+N+d}{M+N} = \binom{N+d}{N}\binom{M+N+d}{M}$ to an isomorphism

$$\mathrm{Sym}_M \mathrm{Sym}^N E \otimes \mathrm{Sym}_{M+N} \mathrm{Sym}^d E \cong \mathrm{Sym}_N \mathrm{Sym}^d E \otimes \mathrm{Sym}_M \mathrm{Sym}^{d+N} E \tag{1.5}$$

and Stanley's hook-content formula (see [Sta99, Theorem 7.21.2]) for hooks to

$$\mathrm{Sym}_M \mathrm{Sym}^{N-1} E \otimes \mathrm{Sym}_{M+N} \mathrm{Sym}^{d-N+1} E \cong \Delta^{(M+1,1^{N-1})} \mathrm{Sym}^d E\,. \tag{1.6}$$

A very interesting new direction was begun in [IOT25], where the authors show that

$$(\mathrm{Sym}^{\ell m} \mathbb{F}^{2\ell^2})^{\mathrm{SL}_\ell(\bar{\mathbb{F}}) \times \mathrm{SL}_\ell(\bar{\mathbb{F}})} \cong \mathrm{Sym}_\ell \mathrm{Sym}^m \mathbb{F}^2; \tag{1.7}$$

this is an isomorphism between an invariant ring (related to the Kronecker product) and an $\mathrm{SL}_2(\mathbb{F})$-plethysm.

We emphasise the jump in complexity that Theorem 1.1 represents over these earlier results. The techniques developed to prove (1.5) and (1.6) are extended in the proof of Theorem 1.1; they are very different from the algebraic geometry approach in [AFP$^+$19] or the basis dependent methods used in [MW25]. We expect that the new and mainly combinatorial techniques presented in §3 and §4 will be very useful in further proofs of characteristic-free isomorphisms for SL_2.

2. Preliminaries

2.1. **Partitions.** We follow [Sta99]. A *partition* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is a weakly decreasing sequence of positive integers. We say $\ell(\lambda) = k$ is its *length* and $|\lambda| = \lambda_1 + \cdots + \lambda_k$ its *size*. By convention, we let $\lambda_L = 0$ if $L > \ell(\lambda)$. We write exponents for repeated parts: for instance $(7, 4^2, 3, 1^3) = (7, 4, 4, 3, 1, 1, 1)$. In this paper, partitions of two parts are often the indices of maps, in which case we sometimes abbreviate $f_{(a,b)}$ by f_{ab}.

The *sum* of partitions $\lambda + \mu$ is entry-wise, for instance $(3,2,1) + (4,3,3,2) = (7,5,4,2)$. The *union* of partitions $\lambda \sqcup \mu$ is the sorted list of the multiset union of their parts, for instance $(3,2,1) \sqcup (4,3,3,2) = (4,3^3,2^2,1)$.

The *Young diagram* of a partition λ is the set

$$[\lambda] = \{(i,j) \in \mathbb{Z}^2 \mid 1 \leqslant i \leqslant \ell(\lambda),\ 1 \leqslant j \leqslant \lambda_i\}$$

which we represent in the English convention as a top-left justified array of boxes,

$$[(7,4,4,2)] = \ydiagram{7,4,4,2}\ .$$

The *transpose* λ' of a partition λ is the partition whose Young diagram is $\{(i,j) \mid (j,i) \in [\lambda]\}$. We write $\mu \subseteq \lambda$ if $[\mu] \subseteq [\lambda]$. In that case, the *skew partition* λ/μ is the formal pair (λ,μ), and the Young diagram of the skew partition is the set $[\lambda/\mu] = [\lambda] \setminus [\mu]$. Given $n, d \in \mathbb{N}_0$ let $L(n,d) = \{\lambda \in \mathrm{Par} \mid \lambda_1 \leqslant d,\ \ell(\lambda) \leqslant n\}$ be the set of partitions fitting in an $n \times d$ box.

A *tableau* is a map $T : Y \to [n]$, where Y is the Young diagram of a partition λ or a skew partition λ/μ. We say λ or λ/μ is the *shape* of the tableau and $[n]$ is the *alphabet*. The *weight* of a tableau T is the tuple $w(T) = (w_1, w_2, \ldots, w_n)$ where w_i counts the number of boxes in $[\lambda]$ or $[\lambda/\mu]$ sent to i by T. A tableau is *semistandard* if

$$T(i,j) \leqslant T(i,j+1) \quad \text{and} \quad T(i,j) < T(i+1,j)$$

wherever the function is defined. We represent a tableau by placing $T(i,j)$ inside the box (i,j) of the Young diagram; then a tableau is semistandard if it is weakly increasing when rows are read left-to-right and strictly increasing when columns are read top-to-bottom.

2.2. **Alternating and symmetric polynomials.** Our general reference for symmetric polynomials and symmetric functions is [Sta99, Ch. 7]. Let $\mathbf{x}_n = (x_1, \ldots, x_n)$. Let the symmetric group $\mathfrak{S}_n$ act on the polynomial ring $\mathbb{F}[\mathbf{x}_n]$ by permuting the variables. A polynomial $p(\mathbf{x}_n)$ is *alternating* if $\sigma \cdot p(\mathbf{x}_n) = -p(\mathbf{x}_n)$ for all transpositions $\sigma \in \mathfrak{S}_n$ and $p(\ldots, x, x, \ldots) = 0$ whenever two entries coincide. (Note the second condition is needed if $\mathbb{F}$ has characteristic two.) A polynomial is *symmetric* if it is invariant under the $\mathfrak{S}_n$ action.

Let $\rho_n = (n-1, n-2, \ldots, 1, 0)$ be the staircase partition. As λ ranges over all partitions of length at most n, the determinants

$$a_{\lambda+\rho_n}(\mathbf{x}_n) = \det(x_j^{\lambda_i + (\rho_n)_i}) = \det(x_j^{\lambda_i + n - i})$$

form a basis of the $\mathbb{F}$-vector space $A[\mathbf{x}_n]$ of alternating polynomials. When $\lambda = \varnothing$, then $a_{\rho_n}(\mathbf{x}_n) = \prod_{i<j}(x_i - x_j)$ is the Vandermonde determinant. The $\mathbb{F}$-vector space $\Lambda[\mathbf{x}_n]$ of symmetric polynomials satisfies

$$a_{\rho_n}(\mathbf{x}_n)\, \Lambda[\mathbf{x}_n] = A[\mathbf{x}_n].$$

2.3. **Schur polynomials and Littlewood–Richardson coefficients.** The corresponding basis of $\Lambda[\mathbf{x}_n]$ is that of *Schur polynomials*,

$$s_\lambda(\mathbf{x}_n) = \frac{a_{\lambda+\rho_n}(\mathbf{x}_n)}{a_{\rho_n}(\mathbf{x}_n)}.$$

A classic result is that

$$s_\lambda(\mathbf{x}_n) = \sum_{T \in \mathrm{SSYT}_n(\lambda)} \mathbf{x}_n^{w(T)},$$

where $\mathrm{SSYT}_n(\lambda)$ is the set of semistandard Young tableaux of shape λ in the alphabet $[n]$, and where $\mathbf{x}_n^{(\alpha_1,\ldots,\alpha_n)} = x_1^{\alpha_1}\cdots x_n^{\alpha_n}$. More generally, the *skew Schur polynomials* are defined as

$$s_{\lambda/\mu}(\mathbf{x}_n) = \sum_{T \in \mathrm{SSYT}_n(\lambda/\mu)} \mathbf{x}_n^{w(T)}$$

for skew partitions λ/μ.

The *Littlewood–Richardson coefficients* are the structure constants $c^\lambda_{\mu\nu}$ of the product of Schur polynomials,

$$s_\mu(\mathbf{x}_n)s_\nu(\mathbf{x}_n) = \sum_\lambda c^\lambda_{\mu\nu} s_\lambda(\mathbf{x}_n).$$

They are also the constants expressing skew Schur polynomials in the Schur basis,

$$s_{\lambda/\mu}(\mathbf{x}_n) = \sum_\nu c^\lambda_{\mu\nu} s_\nu(\mathbf{x}_n).$$

The *reverse reading word* $\mathrm{word}(T)$ of a tableau T is the sequence of its entries obtained by reading rows right-to-left, beginning with the topmost row and finishing with the bottom-most. A sequence $(t_1,\ldots,t_L)$ is said to be a *ballot sequence* if at every prefix $(t_1,\ldots,t_k)$ there are at least as many instances of i as of $i+1$, for every $i=1,\ldots,n-1$. Defining

$$\mathrm{LR}^\lambda_{\mu,\nu}(n) = \{T \in \mathrm{SSYT}_n(\lambda/\mu) \mid w(T) = \nu \text{ and } \mathrm{word}(T) \text{ is a ballot sequence}\},$$

we have $c^\lambda_{\mu\nu} = \#\mathrm{LR}^\lambda_{\mu,\nu}(n)$. The elements of $\mathrm{LR}^\lambda_{\mu,\nu}(n)$ are called *Littlewood–Richardson tableaux*.

2.4. **Tensor spaces.** Given an $\mathbb{F}$-vector space V let

$$\textstyle\bigwedge^n V = V^{\otimes n} \big/ (\cdots \otimes w \otimes w \otimes \cdots)$$

be its *exterior power*. Note this definition is correct even when $\mathbb{F}$ has characteristic 2, and that setting $w = u+v$ gives the relation $\cdots u \otimes v \cdots + \cdots v \otimes u \cdots$, as expected for the exterior power. There are two symmetric power functors, the *upper* and *lower symmetric powers*

$$\mathrm{Sym}^n V = V^{\otimes n} \big/ (\cdots u \otimes v \cdots - \cdots v \otimes u \cdots) \quad \text{and} \quad \mathrm{Sym}_n V = (V^{\otimes n})^{\mathfrak{S}_n},$$

where the symmetric group $\mathfrak{S}_n$ acts by permuting tensor factors. Typically $\mathrm{Sym}^n V$ and $\mathrm{Sym}_n V$ are not isomorphic over $\mathbb{F}$ as representations of $\mathrm{GL}(V)$, but they are always isomorphic working over $\mathbb{C}$.

Remark 2.1. *Analogously, one could consider the* lower exterior power *defined as the invariants under the* signed *action of* $\mathfrak{S}_n$. *Here, in contrast to the symmetric powers, a routine computation shows that it is* $\mathrm{GL}(V)$*-isomorphic to the exterior power defined above, for any field* $\mathbb{F}$. *Throughout the paper, we write* $\bigwedge^n V$ *for both cases, as the construction will be usually clear from the context.*

Throughout this paper we work with many-fold tensor products of representations. We introduce the following shorthand notation for convenience.

Definition 2.2. Let V be an $\mathbb{F}$-vector space. Given a partition λ, let

$$V^{\otimes\lambda} = \bigotimes_{(i,j)\in[\lambda]} V(i,j),$$

where $V(i,j)$ is an isomorphic copy of V. Let

$$\mathrm{Sym}_\lambda V = \mathrm{Sym}_{\lambda_1} V \otimes \cdots \otimes \mathrm{Sym}_{\lambda_{\ell(\lambda)}} V$$

and

$$\textstyle\bigwedge^\lambda V = \bigwedge^{\lambda'_1} V \otimes \cdots \otimes \bigwedge^{\lambda'_{\lambda_1}} V\,.$$

We introduce a pictorial way of representing the three objects above: we represent $V^{\otimes\lambda}$ by the Young diagram of λ, the space $\mathrm{Sym}_\lambda V$ by a row tabloid of shape λ, and $\bigwedge^\lambda V$ by a column tabloid, in each case filled with the coordinates of the boxes. For instance, for $\lambda = (5,3,3)$:

$$V^{\otimes\lambda} = \begin{array}{|c|c|c|c|c|}\hline 11 & 12 & 13 & 14 & 15 \\ \hline 21 & 22 & 23 \\ \cline{1-3} 31 & 32 & 33 \\ \cline{1-3}\end{array}\,, \quad \mathrm{Sym}_\lambda V = \begin{array}{|ccccc|}\hline 11 & 12 & 13 & 14 & 15 \\ \hline 21 & 22 & 23 & \multicolumn{2}{|c}{} \\ \cline{1-3} 31 & 32 & 33 & \multicolumn{2}{|c}{} \\ \cline{1-3}\end{array}\,, \quad \text{and} \quad \textstyle\bigwedge^\lambda V = \begin{array}{|c|c|c|c|c|}\hline 11 & 12 & 13 & 14 & 15 \\ \cline{4-5} 21 & 22 & 23 \\ 31 & 32 & 33 \\ \cline{1-3}\end{array}\,.$$

2.5. **Product and coproduct.** Following [Wey03, §1.1.1], there is a product $\heartsuit$ and coproduct $\spadesuit$ for symmetric powers

$$\mathrm{Sym}_\lambda V \underset{\spadesuit_\lambda}{\overset{\heartsuit_\lambda}{\rightleftarrows}} \mathrm{Sym}_{|\lambda|} V,$$

as well as a product $\diamondsuit$ and coproduct $\clubsuit$ for exterior spaces,

$$\textstyle\bigwedge^\lambda V \underset{\clubsuit_\lambda}{\overset{\diamondsuit_\lambda}{\rightleftarrows}} \bigwedge^{|\lambda|} V\,.$$

We drop the subscripts of the maps when they are understood from context.

Remark 2.3. *Let $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ be a partition of length ℓ and size M. The products are best understood as elements of the group algebra of the symmetric group $\mathfrak{S}_M$ which acts on $V^{\otimes M}$ by permuting tensor factors. The action extends linearly to $\mathbb{F}\mathfrak{S}_M$. The product $\heartsuit$ can be identified with the group algebra element*

$$\heartsuit_\lambda = \sum_{\sigma\in\mathrm{Shuff}(\lambda)} \sigma$$

and $\diamondsuit$, in light of Remark 2.1, with the group algebra element

$$\diamondsuit_{\lambda'} = \sum_{\sigma\in\mathrm{Shuff}(\lambda)} \mathrm{sgn}(\sigma)\sigma,$$

where $\mathrm{Shuff}(\lambda)$ *is a set of minimal length coset representatives of the quotient by the Young subgroup, $\mathfrak{S}_M/(\mathfrak{S}_{\lambda_1} \times \cdots \times \mathfrak{S}_{\lambda_\ell})$. These representatives are called* shuffles, *and in the one-line notation they are the permutations of M such that each of the subsets $\{1,\ldots,\lambda_1\}, \{\lambda_1+1,\ldots,\lambda_1+\lambda_2\}, \ldots$ appears in linear order* [BB05, Ch. 2, Exercise 4]*. As elements of the group algebra, $\spadesuit$ and (by Remark 2.1) $\clubsuit$ are simply the identity.*

2.6. **Weyl modules as images.** For the construction of Weyl modules we follow [Wey03, p. 43] or [McD25]; the latter construction is equivalent but more elementary and avoids divided symmetric powers. Again let V be an $\mathbb{F}$-vector space. The coproducts

$$\spadesuit_{(1^{\lambda_j})} : \mathrm{Sym}_{\lambda_j} V \hookrightarrow V(j,1) \otimes \cdots \otimes V(j,\lambda_j)$$

induce a map $\spadesuit^\lambda := \spadesuit_{(1^{\lambda_1})} \otimes \cdots \otimes \spadesuit_{(1^{\lambda_{\ell(\lambda)}})} : \mathrm{Sym}_\lambda V \hookrightarrow V^{\otimes\lambda}$. Similarly, the products

$$\begin{aligned} \diamondsuit_{(\lambda'_i)} : V(1,i) \otimes \cdots \otimes V(\lambda'_i, i) &\twoheadrightarrow \textstyle\bigwedge^{\lambda'_i} V \\ v_1 \otimes \cdots \otimes v_{\lambda'_i} &\mapsto v_1 \wedge \cdots \wedge v_{\lambda'_i} \end{aligned}$$

induce a map $\diamondsuit^\lambda = \diamondsuit_{(\lambda'_1)} \otimes \cdots \otimes \diamondsuit_{(\lambda'_{\lambda_1})} : V^{\otimes\lambda} \twoheadrightarrow \bigwedge^\lambda V$.

Definition 2.4. The *Weyl module* $\Delta^\lambda V$ is defined to be the image of the composition $\psi_\lambda = \diamondsuit^\lambda \circ \spadesuit^\lambda$:

$$\Delta^\lambda V = \mathrm{im}(\diamondsuit^\lambda \circ \spadesuit^\lambda : \mathrm{Sym}_\lambda V \xrightarrow{\psi_\lambda} \textstyle\bigwedge^\lambda V). \tag{2.1}$$

Pictorially, ψ_λ is the composite

$$\psi_\lambda : \begin{array}{|lllll|} \hline 11 & 12 & 13 & 14 & 15 \\ \hline 21 & 22 & 23 & \multicolumn{2}{|l}{} \\ \cline{1-3} 31 & 32 & 33 & \multicolumn{2}{|l}{} \\ \cline{1-3} \end{array} \hookrightarrow \begin{array}{|c|c|c|c|c|} \hline 11 & 12 & 13 & 14 & 15 \\ \hline 21 & 22 & 23 & \multicolumn{2}{|l}{} \\ \cline{1-3} 31 & 32 & 33 & \multicolumn{2}{|l}{} \\ \cline{1-3} \end{array} \twoheadrightarrow \begin{array}{|c|c|c|c|c|} \hline 11 & 12 & 13 & 14 & 15 \\ \cline{4-5} 21 & 22 & 23 & \multicolumn{2}{|l}{} \\ 31 & 32 & 33 & \multicolumn{2}{|l}{} \\ \cline{1-3} \end{array} .$$

Since the construction is functorial, this defines $\Delta^\lambda V$ as a representation of $\mathrm{GL}(V)$, and if V is a representation of $\mathrm{SL}_2(\mathbb{F})$, by restriction we see that $\Delta^\lambda V$ is a representation of $\mathrm{SL}_2(\mathbb{F})$.

2.7. **The characters of Weyl modules and plethysm.** The formal character of the Weyl module $\Delta^\lambda V$ as a representation of $\mathrm{GL}(V)$ is the Schur polynomial $s_\lambda(\mathbf{x}_{d+1}) = s_\lambda(x_1, \ldots, x_{d+1})$, where $d+1 = \dim V$. Recall that E is the natural 2-dimensional representation of $\mathrm{SL}_2(\mathbb{F})$. If V is the representation $\mathrm{Sym}^d E$ of $\mathrm{SL}_2(\mathbb{F})$ then the eigenvalues of the diagonal matrix $\left(\begin{smallmatrix} q & 0 \\ 0 & q^{-1} \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{F})$ acting on V are $q^d, q^{d-2}, \ldots, q^{-d}$. Thus, the *$q$-character* of $\mathrm{Sym}^d E$ is

$$q^d + q^{d-2} + \cdots + q^{-d} = [d+1]_q\,. \tag{2.2}$$

Moreover, the character of $\Delta^\lambda \mathrm{Sym}^d E$ on this matrix is obtained by substituting these eigenvalues for the formal variables $x_1, x_2, \ldots$, and so the q-character of $\Delta^\lambda \mathrm{Sym}^d E$ is

$$s_\lambda(q^d, q^{d-2}, \ldots, q^{-d}) = s_\lambda \circ s_d(q, q^{-1}) \tag{2.3}$$

where the equality follows from $s_d(q,q^{-1}) = q^d + q^{d-2} + \cdots + q^{-d}$ and the definition of the plethysm product by monomial substitution (see [LR11, Theorem 7]).

Remark 2.5. *More generally, composition of Weyl functors Δ^λ corresponds to the plethysm product on Schur functions: see* [Sta99, page 448]*, or* [Mac95, 160, (7.3)] *for an even more general result on polynomial functors. The simplified treatment in this subsection suffices for our purposes.*

As a corollary of (2.3) we prove the result relating q-binomial coefficients to characters of two-row Weyl modules used in the derivation of (1.1). Let $d, m, n \in \mathbb{N}_0$ with $n \geqslant m$. It follows from the specialism $h_n(1, q, \ldots, q^d)$ of the complete homogeneous symmetric function of degree n in [Mac95, Ch. 1, §2, Example 3] that $\left[{d+n \atop d}\right]_q = h_n(q^d, \ldots, q^{-d})$. Hence

$$\begin{aligned}\begin{bmatrix} d+n \\ d \end{bmatrix}_q \begin{bmatrix} d+m \\ d \end{bmatrix}_q - \begin{bmatrix} d+n+1 \\ d \end{bmatrix}_q \begin{bmatrix} d+m-1 \\ d \end{bmatrix}_q & \\ &= (h_n h_m - h_{n+1} h_{m-1})(q^d, q^{d-2}, \ldots, q^{-d}) \\ &= s_{(n,m)}(q^d, q^{d-2}, \ldots, q^{-d}) \\ &= s_{(n,m)} \circ s_d(q, q^{-1}) \end{aligned} \tag{2.4}$$

where we used the Jacobi–Trudi identity and then (2.3). See [GK25, §6, (3)] for other applications of this identity.

2.8. **Weyl modules as vector spaces of polynomials.**

One-row and one-column Weyl modules. In [GMSW25] we constructed the isomorphism of $\mathrm{SL}_2(\mathbb{F})$-modules

$$\mathrm{Sym}_n \mathrm{Sym}^d E \cong \Lambda_{\leqslant d}[\mathbf{x}_n],$$

where the right-hand side is the vector space of symmetric polynomials in $\mathbf{x}_n = (x_1, \ldots, x_n)$ with coefficients in $\mathbb{F}$ and of degree at most d *in each variable separately.* An $\mathbb{F}$-basis of $\Lambda_{\leqslant d}[\mathbf{x}_n]$ is given by

$$\{s_\lambda(\mathbf{x}_n) \mid \lambda \in L(n,d)\},$$

where s_λ is a Schur polynomial and, as defined in §2.1, $L(n,d) = \{\lambda \in \mathrm{Par} \mid \lambda_1 \leqslant d,\ \ell(\lambda) \leqslant n\}$ is the set of partitions fitting in an $n \times d$ box [GMSW26b]. The group $\mathrm{SL}_2(\mathbb{F})$ acts on $\Lambda_{\leqslant d}[\mathbf{x}_n]$ via

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} f(x_1, \ldots, x_n) = \prod_{i=1}^{n} (\beta x_i + \delta)^d \cdot f\left(\frac{\alpha x_1 + \gamma}{\beta x_1 + \delta}, \ldots, \frac{\alpha x_n + \gamma}{\beta x_n + \delta}\right). \tag{2.5}$$

To keep track of the names of the variables and their maximum degree, we have the following pictorial representation:

$$\mathrm{Sym}_n \mathrm{Sym}^d E = \Lambda_{\leqslant d}[x_1, \ldots, x_n] = \boxed{x_1\ x_2 \cdots x_n}_{\,d}\,.$$

Remark 2.6. *The symmetric coproduct* $\spadesuit$ *becomes the usual coproduct inherited from the (plethystic) Hopf algebra structure of* $\Lambda[\mathbf{x}]$ *(see* [ALRS13, §5.1]*). On the Schur basis,*

$$\spadesuit_{ab}\, s_\lambda(\mathbf{x}_a, \mathbf{y}_b) = \sum_{\mu \subseteq \lambda} s_\mu(\mathbf{x}_a) \otimes s_{\lambda/\mu}(\mathbf{y}_b)\,. \tag{2.6}$$

Suppose $\lambda \in L(a+b, k)$*. Note that* $s_\mu(\mathbf{x}_a) \neq 0$ *requires* $\ell(\mu) \leqslant a$*, so* $\mu \in L(a,k)$*. Similarly, for* $s_{\lambda/\mu}(\mathbf{y}_b) \neq 0$*, each column of the skew partition* λ/μ *has to be of height at most* b*, equivalently* $\mu \subseteq \lambda \subseteq \mu \sqcup (k^b)$*, thus we may rewrite the above as*

$$\spadesuit_{ab}\, s_\lambda = \sum_{\substack{\mu \in L(a,k) \\ \mu \subseteq \lambda \subseteq \mu \sqcup (k^b)}} s_\mu \otimes s_{\lambda/\mu}\,. \tag{2.7}$$

A single-column Weyl module $\bigwedge^n \mathrm{Sym}^d E$ is identified with the $\mathbb{F}$-vector space $A_{\leqslant d}[\mathbf{x}_n]$ of alternating polynomials in n variables, each of degree at most d. We have a similar pictorial representation:

$$\bigwedge^n \mathrm{Sym}^d E = A_{\leqslant d}[x_1, \dots, x_n] = \boxed{\begin{matrix} x_1 \\ \vdots \\ x_n \end{matrix}}_{\,d} .$$

A basis of $\bigwedge^n \mathrm{Sym}^d E$ is therefore $\{a_{\lambda+\rho_n}(\mathbf{x}_n) \mid \lambda \in L(n, d-n+1)\}$.

Remark 2.7. *The exterior product $\diamondsuit_{(k,\ell)'}$ also admits a convenient description on the basis above. For $\lambda \in L(k, d-k+1)$ and $\mu \in L(\ell, d-\ell+1)$,*

$$\diamondsuit_{(k,\ell)'}(a_{\lambda+\rho_k}(\mathbf{x}_k) \otimes a_{\mu+\rho_\ell}(\mathbf{y}_\ell)) = \mathrm{sgn}_{k\ell}(\lambda,\mu) a_{(\lambda+\rho_k)\sqcup(\mu+\rho_\ell)}(\mathbf{x}_k, \mathbf{y}_\ell),$$

with

$$\mathrm{sgn}_{k\ell}(\lambda,\mu) = \begin{cases} \mathrm{sgn}(\tau) & \text{if the parts of } (\lambda+\rho_k) \sqcup (\mu+\rho_\ell) \text{ are distinct,} \\ 0 & \text{otherwise,} \end{cases}$$

where τ is the permutation sorting the concatenated sequence of parts into decreasing order. Explicitly,

$$\mathrm{sgn}(\tau) = (-1)^{\#\mathrm{inv}(\lambda_1+k-1,\, \lambda_2+k-2,\, \dots,\, \lambda_k,\, \mu_1+\ell-1,\, \mu_2+\ell-2,\, \dots,\, \mu_\ell)}.$$

Wronskian and the (co)products. The interpretation of Weyl modules as $\mathbb{F}$-vector spaces of polynomials becomes a useful setting to study the field-independent morphisms (1.4)–(1.6). For instance the Wronskian isomorphism (1.4) is realised as

$$\begin{aligned} \mathrm{wr}_n : \Lambda_{\leqslant k}[\mathbf{x}_n] &\to A_{\leqslant n+k-1}[\mathbf{x}_n] \\ p(\mathbf{x}_n) &\mapsto a_{\rho_n}(\mathbf{x}_n)\, p(\mathbf{x}_n). \end{aligned} \tag{2.8}$$

See [McDW22] for the original map and [Gri23, GMSW25] for the interpretation above.

The following computation in the group algebra is useful later. Write $\mathbf{x}_n^\alpha$ for the operator that multiplies a polynomial by $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. Recall from §2.5 that $\diamondsuit$ is the product for exterior powers, $\spadesuit$ is the coproduct for symmetric powers, and ψ is the composition from Definition 2.4.

Lemma 2.8. $\diamondsuit_{(2^k)}\mathrm{wr}_k^{\otimes 2} = \diamondsuit_{kk}\mathbf{x}_k^{\rho_k}\mathbf{y}_k^{\rho_k}\psi_{kk}$.

Proof. Begin by noting $a_{\rho_k}(\mathbf{x}_k) = \sum_{\tau \in \mathfrak{S}_k} \mathrm{sgn}(\tau)\tau \cdot \mathbf{x}_k^{\rho_k}$. By Remark 2.3 the left-hand side becomes the group algebra element

$$\sum_{\sigma \in \mathrm{Shuff}(k,k)} \sum_{\tau_1 \in \mathfrak{S}_k} \sum_{\tau_2 \in \mathfrak{S}_k} \mathrm{sgn}(\sigma)\mathrm{sgn}(\tau_1)\mathrm{sgn}(\tau_2)\ \sigma(\tau_1 \cdot \mathbf{x}_k^{\rho_k} \otimes \tau_2 \cdot \mathbf{y}_k^{\rho_k})$$

where we have used two alphabets $\mathbf{x}$ and $\mathbf{y}$ to distinguish tensor factors. Now using the fact that $\mathrm{Shuff}(k,k)$ is a set of minimal length coset representatives of $\mathfrak{S}_{2k}/(\mathfrak{S}_k \times \mathfrak{S}_k)$ this becomes $\sum_{\pi \in \mathfrak{S}_{2k}} \mathrm{sgn}(\pi)\ \pi \cdot \mathbf{x}_k^{\rho_k}\mathbf{y}_k^{\rho_k}$.

To write the right-hand side we follow the action of the different symmetric groups. The operator is

$$\begin{aligned} &\diamondsuit_{kk}\mathbf{x}_k^{\rho_k}\mathbf{y}_k^{\rho_k}(\diamondsuit_2 \otimes \overset{k}{\cdots} \otimes \diamondsuit_2)(\spadesuit_{(1^k)} \otimes \spadesuit_{(1^k)}) \\ &\qquad = \sum_{\sigma \in \mathrm{Shuff}(2^k)} \sum_{\substack{\tau_1 \in \mathfrak{S}_2 \\ \cdots \\ \tau_k \in \mathfrak{S}_2}} \mathrm{sgn}(\sigma)\mathrm{sgn}(\tau_1)\cdots\mathrm{sgn}(\tau_k)\sigma \cdot \mathbf{x}_k^{\rho_k}\mathbf{y}_k^{\rho_k}\ \tau_1 \cdots \tau_k, \end{aligned}$$

where the different copies of the symmetric group $\mathfrak{S}_2$ act on the sets $\{x_i, y_i\}$. Note that $\tau_i(\mathbf{x}_k^{\rho_k}\mathbf{y}_k^{\rho_k}) = \mathbf{x}_k^{\rho_k}\mathbf{y}_k^{\rho_k}$, so the τ_i's commute with $\mathbf{x}_k^{\rho_k}\mathbf{y}_k^{\rho_k}$. The expression becomes

$$\sum_{\pi \in \mathfrak{S}_{2k}} \operatorname{sgn}(\pi)\, \pi \cdot \mathbf{x}_k^{\rho_k}\mathbf{y}_k^{\rho_k}. \qquad \square$$

The evaluation map. The $\mathrm{SL}_2(\mathbb{F})$-isomorphisms in [GMSW25] were stated and proved using the evaluation map studied here. A generalisation and restriction of this map play an important role in the proof of Theorem 1.1. We begin by introducing the most general version of the evaluation map required.

Definition 2.9. Let λ be a partition and let $N, a, b \in \mathbb{N}_0$. The *evaluation map* is defined by

$$\begin{aligned} \pi_{\lambda,N,a,b} : \mathrm{Sym}_{|\lambda|+N} \mathrm{Sym}^b E \otimes (\mathrm{Sym}^a E)^{\otimes \lambda} &\to \mathrm{Sym}_N \mathrm{Sym}^b E \otimes (\mathrm{Sym}^{a+b} E)^{\otimes \lambda} \\ P(\mathbf{x}_{|\lambda|}, \mathbf{y}_N) Q(\mathbf{z}_{|\lambda|}) &\mapsto P(\mathbf{z}_{|\lambda|}, \mathbf{y}_N) Q(\mathbf{z}_{|\lambda|}). \end{aligned} \tag{2.9}$$

The parameters λ, N, a, b are usually clear from the context, in which case we write just π instead of $\pi_{\lambda,N,a,b}$.

Proposition 2.10. *The map $\pi_{\lambda,N,a,b}$ is well defined and $\mathrm{SL}_2(\mathbb{F})$-equivariant. If $N \geqslant a$ then $\pi_{\lambda,N,a,b}$ is injective for any partition λ and $b \in \mathbb{N}_0$.*

Proof. All these properties follow immediately from the proofs of Proposition 4.9 and Proposition 4.10 in [GMSW25]. In the proof of Proposition 4.9 the polynomial $Q(\mathbf{z})$ in [GMSW25] is assumed symmetric, but this property is never used in the calculation completing the proof. A similar remark applies to Proposition 4.10. $\square$

Proposition 2.11. *By restriction, $\pi_{\lambda,N,a,b}$ induces an $\mathrm{SL}_2(\mathbb{F})$-homomorphism*

$$\mathrm{Sym}_{|\lambda|+N} \mathrm{Sym}^b E \otimes \mathrm{Sym}_\lambda \mathrm{Sym}^a E \to \mathrm{Sym}_N \mathrm{Sym}^b E \otimes \mathrm{Sym}_\lambda \mathrm{Sym}^{a+b} E \tag{2.10}$$

which is injective when $N \geqslant a$.

Proof. By linearity, it suffices to consider a generic pure tensor of the domain:

$$P(\mathbf{x}_\lambda, \mathbf{y}_N) Q_{\lambda_1}(z_1, \ldots, z_{\lambda_1}) \cdots Q_{\lambda_t}(z_{|\lambda|-\lambda_t+1}, \ldots, z_{|\lambda|})$$

for some symmetric polynomials $P, Q_{\lambda_1}, \ldots, Q_{\lambda_t}$, where $t = \ell(\lambda)$. By Definition 2.9, this element is mapped to

$$P(\mathbf{z}_{|\lambda|}, \mathbf{y}_N) Q_{\lambda_1}(z_1, \ldots, z_{\lambda_1}) \cdots Q_{\lambda_t}(z_{|\lambda|-\lambda_t+1}, \ldots, z_{|\lambda|}),$$

which is symmetric in each of the alphabets $\mathbf{y}, \{z_1, \ldots, z_{\lambda_1}\}, \ldots, \{z_{|\lambda|-\lambda_t+1}, \ldots, z_{|\lambda|}\}$ separately. Moreover, each of the variables in $\mathbf{y}$ appears in each monomial with degree at most b, and each of the variables in $\mathbf{z}$ with degree at most $a + b$. Thus the restriction described above is well defined. Injectivity and $\mathrm{SL}_2(\mathbb{F})$-equivariance are both inherited from the properties of π proved in Proposition 2.10. $\square$

We may abuse notation by also writing the restricted map as $\pi_{\lambda,N,a,b}$ or, when the context makes the parameters clear, just as π.

Remark 2.12. *When $\lambda = (M)$, the restricted map is precisely π from* [GMSW25].

In the proof of Theorem 1.1 we will make use of yet another restriction of π which we postpone until §3.1.

Two-row Weyl modules as vector spaces of polynomials. In order to interpret Weyl modules in this context, let $\lambda = (n, m)$ to obtain

$$\mathrm{Sym}_{(n,m)} \mathrm{Sym}^d E \cong \Lambda_{\leqslant d}[x_{11}, \ldots, x_{1n}] \otimes \Lambda_{\leqslant d}[x_{21}, \ldots, x_{2m}],$$

which is spanned by pure tensors of symmetric polynomials

$$f \otimes g = f(x_{11}, \ldots, x_{1n}) g(x_{21}, \ldots, x_{2m}).$$

By Remark 2.3, the map $\psi_{(n,m)}$ from (2.1) corresponds to multiplication by the element $\prod_{i=1}^{m} \big(1-(x_{1i}, x_{2i})\big)$ of the group algebra $\mathbb{F}\mathfrak{S}(x_{11}, \ldots, x_{2m})$ of the symmetric group in the elements of $\mathbf{x}$. Thus, in light of Remark 2.1, $\Delta^{(n,m)} \mathrm{Sym}^d E$ is the submodule of $\bigwedge^{(n,m)} \mathrm{Sym}^d E \leqslant (\mathrm{Sym}^d E)^{\otimes (n+m)}$ spanned by the elements

$$\prod_{i=1}^{m} (1 - (x_{1i}, x_{2i})) f(x_{11}, \ldots, x_{1n}) g(x_{21}, \ldots, x_{2m})$$

for symmetric f and g.

2.9. **Weyl modules as quotients.** By the first isomorphism theorem, we can describe $\mathrm{im}(\psi_\lambda)$ as a quotient of $\mathrm{Sym}_\lambda V$ by a certain submodule $U_\lambda = \ker \psi_\lambda$. This is done in [Wey03, 2.1.15(a)], in a construction that we present here only for two-row partitions. (For an equivalent presentation see [McD25, Theorem 1.2].) In the two-row case, U_λ is defined as the submodule of $\mathrm{Sym}_\lambda V$ spanned by the images $\mathrm{im}(\theta_{uv})$ of the following maps, for all choices of $0 \leqslant u \leqslant \lambda_1$ and $0 \leqslant v \leqslant u+v < \lambda_2$. The map θ_{uv} is defined as the composition

$$\mathrm{Sym}_{(u, \lambda_1 - u + \lambda_2 - v, v)} V \xrightarrow{1 \otimes \spadesuit \otimes 1} \mathrm{Sym}_{(u, \lambda_1 - u, \lambda_2 - v, v)} V \xrightarrow{\heartsuit \otimes \heartsuit} \mathrm{Sym}_{(\lambda_1, \lambda_2)} V$$

where, as defined in §2.5, $\spadesuit$ is the coproduct for symmetric powers and $\heartsuit$ is the product for symmetric powers.

The following property of U_{nm} is a consequence of Lemma 2.8, and will be useful in the proof of Theorem 1.1. Recall that wr is the isomorphism from (1.4), the maps $\diamondsuit$ and $\spadesuit$ are the product for exterior powers and the coproduct for symmetric powers defined in §2.5, and the map ψ is the composition from Definition 2.4.

Proposition 2.13. *For $0 \leqslant i \leqslant m-1$, $k = m - i$, and $\tilde{n} = n - m + i = n - k$, we have* $(\mathrm{wr}_k \otimes 1)^{\otimes 2} (\spadesuit_{k\tilde{n}} \otimes \spadesuit_{ki}) U_{nm} \subseteq \ker(\diamondsuit_{(2^k)} \otimes \psi_{\tilde{n}i})$.

Proof. The claim is that U_{nm}, that is the kernel of ψ_{nm}, is annihilated by $(\diamondsuit_{(2^k)} \otimes \psi_{\tilde{n}i})(\mathrm{wr}_k \otimes 1)^{\otimes 2} (\spadesuit_{k\tilde{n}} \otimes \spadesuit_{ki})$. We use Lemma 2.8 to write

$$\begin{aligned}(\diamondsuit_{(2^k)} \otimes \psi_{\tilde{n}i})(\mathrm{wr}_k \otimes 1)^{\otimes 2} (\spadesuit_{k\tilde{n}} \otimes \spadesuit_{ki}) &= (\diamondsuit_{kk} \mathbf{x}_{1,*}^{\rho_k} \mathbf{x}_{2,*}^{\rho_k} \psi_{kk} \otimes \psi_{\tilde{n}i})(\spadesuit_{k\tilde{n}} \otimes \spadesuit_{ki}) \\ &= (\diamondsuit_{kk} \mathbf{x}_{1,*}^{\rho_k} \mathbf{x}_{2,*}^{\rho_k} \otimes 1)(\psi_{kk} \otimes \psi_{\tilde{n}i})(\spadesuit_{k\tilde{n}} \otimes \spadesuit_{ki}) \\ &= (\diamondsuit_{kk} \mathbf{x}_{1,*}^{\rho_k} \mathbf{x}_{2,*}^{\rho_k} \otimes 1) \psi_{nm} \,.\end{aligned}$$

The claim is then clear. □

3. Two auxiliary maps

In this section we define two $\mathrm{SL}_2(\mathbb{F})$-homomorphisms which we will use in the proof of Theorem 1.1, namely

$$\tilde{\pi} : \mathrm{Sym}_{n+m} \mathrm{Sym}^{d-k} E \otimes \Delta^{(\tilde{n},i)} \mathrm{Sym}^k E \hookrightarrow \mathrm{Sym}_{2k} \mathrm{Sym}^{d-k} E \otimes \Delta^{(\tilde{n},i)} \mathrm{Sym}^d E$$

and

$$\varphi : \mathrm{Sym}_{(n,m)} \mathrm{Sym}^k E \twoheadrightarrow \mathrm{Sym}_{(\tilde{n},i)} \mathrm{Sym}^k E,$$

where $i \in \{0, 1, \ldots, m-1\}$, $k = m - i$, and $\tilde{n} = n - m + i$. We will show that they are an injection and a surjection, respectively. Both maps are of independent interest as plethystic lifts of certain q-binomial identities stated in [GMSW26b].

3.1. **The $\tilde{\pi}$ map.** Let λ be a partition of size M. In this subsection we prove, by means of an explicit field-independent $\mathrm{SL}_2(\mathbb{F})$-homomorphism, that whenever $N \geqslant a$

$$\begin{bmatrix} N+b \\ b \end{bmatrix}_q \cdot s_\lambda \circ s_{a+b}(q, q^{-1}) - \begin{bmatrix} M+N+b \\ b \end{bmatrix}_q \cdot s_\lambda \circ s_a(q, q^{-1}) \tag{3.1}$$

is Schur positive, meaning that it can be expressed as a non-negative linear combination of quantum integers. When $\lambda = (M)$ and $N = a$, the difference becomes 0 — this is an instance of trinomial revision. Thus our proof follows and generalises the proof of (1.5) in [GMSW25].

Towards this goal, we show that if $N \geqslant a$ then $\pi_{\lambda,N,a,b}$ restricts to an $\mathrm{SL}_2(\mathbb{F})$-equivariant injection $\tilde{\pi}$ as follows (note that $\Delta^\lambda V \leqslant V^{\otimes\lambda}$ by Remark 2.1 and Definition 2.4):

$$\begin{tikzcd}
\mathrm{Sym}_{M+N} \mathrm{Sym}^b E \otimes \Delta^\lambda \mathrm{Sym}^a E \arrow[r, dashed, "\tilde{\pi}"] \arrow[d, hook] & \mathrm{Sym}_N \mathrm{Sym}^b E \otimes \Delta^\lambda \mathrm{Sym}^{a+b} E \arrow[d, hook] \\
\mathrm{Sym}_{M+N} \mathrm{Sym}^b E \otimes (\mathrm{Sym}^a E)^{\otimes\lambda} \arrow[r, "\pi_{\lambda,N,a,b}"'] & \mathrm{Sym}_N \mathrm{Sym}^b E \otimes (\mathrm{Sym}^{a+b} E)^{\otimes\lambda}
\end{tikzcd}$$

As in Remark 2.3, consider the action of $\mathbb{F}\mathfrak{S}_M$ on $(\mathrm{Sym}^\bullet E)^{\otimes\lambda}$.

Lemma 3.1. *The action of $\mathbb{F}\mathfrak{S}_M$ on the second tensor factors commutes with π.*

Proof. By linearity, it suffices to verify the statement for $\sigma \in \mathfrak{S}_M$ acting on variables $z_1, z_2, \ldots, z_M$. We compute directly:

$$\begin{aligned}
\sigma\pi(P(\mathbf{x}_M,\, \mathbf{y}_N)Q(\mathbf{z}_M)) &= \sigma\left(P(\mathbf{z}_M, \mathbf{y}_N)Q(\mathbf{z}_M)\right) = P(\mathbf{z}_M, \mathbf{y}_N) \cdot \sigma Q(\mathbf{z}_M) \\
&= \pi\left(P(\mathbf{x}_M, \mathbf{y}_N) \cdot \sigma Q(\mathbf{z}_M)\right) = \pi\sigma\left(P(\mathbf{x}_M, \mathbf{y}_N)Q(\mathbf{z}_M)\right).
\end{aligned}$$

where the second and last equalities hold because P (unlike Q) is a symmetric polynomial. □

Recall from 2.5 that $\diamondsuit$ is the product on exterior powers and $\spadesuit$ is the coproduct on symmetric powers.

Corollary 3.2. *Let $|\lambda| = M$ and $N \geqslant a$. Then, there is an injective* $\mathrm{SL}_2(\mathbb{F})$-*homomorphism*

$$\tilde{\pi} : \mathrm{Sym}_{M+N} \mathrm{Sym}^b E \otimes \Delta^\lambda \mathrm{Sym}^a E \hookrightarrow \mathrm{Sym}_N \mathrm{Sym}^b E \otimes \Delta^\lambda \mathrm{Sym}^{a+b} E\,.$$

Proof. We will first show that this restriction of π is well-defined. Recall from §2.6 that

$$\Delta^\lambda \mathrm{Sym}^a E = \diamondsuit^\lambda \spadesuit^\lambda \Big(\mathrm{Sym}_\lambda \mathrm{Sym}^a E\Big).$$

Since $\spadesuit^\lambda$ is a canonical inclusion, we view $\spadesuit^\lambda \mathrm{Sym}_\lambda \mathrm{Sym}^a E$ as an isomorphic copy of $\mathrm{Sym}_\lambda \mathrm{Sym}^a E$ inside the space $(\mathrm{Sym}^a E)^{\otimes\lambda}$ and so by Proposition 2.11,

$$\begin{aligned}
&\pi\Big(\mathrm{Sym}_{M+N} \mathrm{Sym}^b E \otimes \spadesuit^\lambda \mathrm{Sym}_\lambda \mathrm{Sym}^a E\Big) \\
&\qquad\qquad \leqslant \mathrm{Sym}_N \mathrm{Sym}^b E \otimes \spadesuit^\lambda \mathrm{Sym}_\lambda \mathrm{Sym}^{a+b} E\,.
\end{aligned}$$

View $\diamondsuit^\lambda$ as an element of the group algebra $\mathbb{F}\mathfrak{S}_M$ as in Remark 2.3. By the observation above and Proposition 2.11,

$$\begin{aligned}
\pi_{\lambda,N,a,b}&\Big(\operatorname{Sym}_{M+N}\operatorname{Sym}^b E\otimes\Delta^\lambda\operatorname{Sym}^a E\Big)\\
&= \pi(1\otimes\diamondsuit^\lambda)\left(\operatorname{Sym}_{M+N}\operatorname{Sym}^b E\otimes\spadesuit^\lambda\operatorname{Sym}_\lambda\operatorname{Sym}^a E\right)\\
&= (1\otimes\diamondsuit^\lambda)\pi\left(\operatorname{Sym}_{M+N}\operatorname{Sym}^b E\otimes\spadesuit^\lambda\operatorname{Sym}_\lambda\operatorname{Sym}^a E\right)\\
&\leqslant (1\otimes\diamondsuit^\lambda)\left(\operatorname{Sym}_{N}\operatorname{Sym}^b E\otimes\spadesuit^\lambda\operatorname{Sym}_\lambda\operatorname{Sym}^{a+b} E\right)\\
&= \operatorname{Sym}_N\operatorname{Sym}^b E\otimes\Delta^\lambda\operatorname{Sym}^{a+b}E\,.
\end{aligned}$$

The $\mathrm{SL}_2(\mathbb{F})$-equivariance and injectivity of $\tilde{\pi}$ is inherited from π. $\square$

Now taking the characters on either side using (2.3) we find that (3.1) is the character of the quotient of the codomain of $\tilde{\pi}$ by its domain, and so, by (2.2), it is a linear combination of quantum integers, as we claimed at the start of this subsection.

For ease of reference, we state the following special case.

Corollary 3.3. *Suppose $\tilde{n}+k=n\geqslant m=i+k$. Then, there is an injective $\mathrm{SL}_2(\mathbb{F})$-homomorphism*

$$\tilde{\pi}:\operatorname{Sym}_{n+m}\operatorname{Sym}^{d-k}E\otimes\Delta^{(\tilde{n},i)}\operatorname{Sym}^k E\hookrightarrow\operatorname{Sym}_{2k}\operatorname{Sym}^{d-k}E\otimes\Delta^{(\tilde{n},i)}\operatorname{Sym}^d E.$$

3.2. **The φ map.** We now prove by an explicit field-independent homomorphism of $\mathrm{SL}_2(\mathbb{F})$-plethysms that

$$\begin{bmatrix}n+k\\k\end{bmatrix}_q\begin{bmatrix}m+k\\k\end{bmatrix}_q-\begin{bmatrix}n\\k\end{bmatrix}_q\begin{bmatrix}m\\k\end{bmatrix}_q$$

is Schur positive. By introducing two other variables i and $\tilde{n}$ such that $k=m-i$ and $\tilde{n}=n-k$, the expression above becomes

$$\begin{bmatrix}n+k\\k\end{bmatrix}_q\begin{bmatrix}m+k\\k\end{bmatrix}_q-\begin{bmatrix}\tilde{n}+k\\k\end{bmatrix}_q\begin{bmatrix}i+k\\k\end{bmatrix}_q.$$

By a similar argument to the end of the previous subsection, it suffices to construct a surjective $\mathrm{SL}_2(\mathbb{F})$-homomorphism

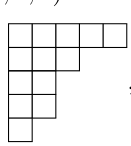
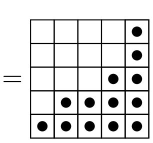
$$\varphi:\operatorname{Sym}_{(n,m)}\operatorname{Sym}^k E\to\operatorname{Sym}_{(\tilde{n},i)}\operatorname{Sym}^k E\,.$$

The construction requires some combinatorial preliminaries.

The flip of a partition. Recall from 2.1 that $L(k,k)$ is the set of partitions fitting into a $k\times k$ box. Given a partition $\theta\in L(k,k)$, let $\mathsf{flip}_k\theta\subseteq[(k^k)]$ be the set of boxes obtained by reflecting the Young diagram $[\theta]$ by the main antidiagonal of $[(k^k)]$. Explicitly,

$$(r,c)\in\mathsf{flip}_k\theta\iff(k+1-c,k+1-r)\in[\theta].$$

For instance, if $\theta=(5,3,2,2,1)$ then we have

$\theta =$  , $\mathsf{flip}_5\theta =$  .

The complementary boxes $[(k^k)] \setminus \mathsf{flip}_k\theta$ can be read as the Young diagram of a partition; in the example above this partition is $(4,4,3,1)$.

The next combinatorial result appears also in [FGGK26].

Lemma 3.4. *Given $\nu, \theta \in L(k,k)$, we have $(\nu+\rho_k) \sqcup (\theta+\rho_k) = \rho_{2k}$ if and only if*

$$[\nu] \sqcup \mathsf{flip}_k\theta = [(k^k)],$$

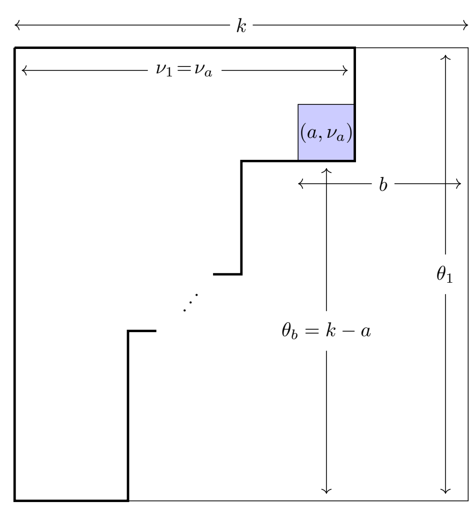
where $\sqcup$ denotes disjoint union.

Proof. We work by induction on $|\nu|$. When $\nu = \varnothing$ it is clear that $(\nu+\rho_k)\sqcup(\theta+\rho_k) = \rho_{2k}$ if and only if $\theta = (k^k)$, and this is the case if and only if $[\nu]\sqcup\mathsf{flip}_k\theta = [(k^k)]$. For the inductive step, let $(a, \nu_a) \in [\nu]$ be the position of the northern-most removable box. More formally, a is maximal such that $\nu_1 = \cdots = \nu_a$. Let

$$\nu^- = (\nu_1, \ldots, \nu_a - 1, \ldots, \nu_k)$$

be the partition obtained from ν by removing this box from its Young diagram. Let $b = k - \nu_a + 1$. The diagram below, in which the boundary of $[\nu]$ is indicated by the thicker line, shows that $\nu \sqcup \mathsf{flip}_k\theta = [(k^k)]$ if and only if $\mathsf{flip}_k\theta \cup \{(a, \nu_a)\} = \mathsf{flip}_k\theta^+$, where

$$\theta^+ = (\theta_1, \ldots, \theta_b + 1, \ldots, \theta_k).$$



We now have a chain of double implications. The second double implication is the inductive hypothesis for ν^-, while all the rest are simple rewritings, using

$\nu_a = k-b+1$ and $\theta_b = k-a$ and that $\rho_k = (k-1,\dots,1,0)$:

$$\begin{aligned}
&[\nu] \sqcup \mathsf{flip}_k\theta = [(k^k)] \\
&\quad\iff [\nu^-] \sqcup \mathsf{flip}_k\theta^+ = [(k^k)] \\
&\quad\iff (\nu^- + \rho_k) \sqcup (\theta^+ + \rho_k) = \rho_{2k} \\
&\quad\iff (\nu_1 + k-1, \dots, \nu_a - 1 + k - a, \dots, \nu_k) \\
&\qquad\quad \sqcup (\theta_1 + k - 1, \dots, \theta_b + 1 + k - b, \dots, \theta_k) = \rho_{2k} \\
&\quad\iff (\nu_1 + k-1, \dots, 2k-a-b, \dots, \nu_k) \\
&\qquad\quad \sqcup (\theta_1 + k - 1, \dots, 2k-a-b+1, \dots, \theta_k) = \rho_{2k} \\
&\quad\iff (\nu_1 + k-1, \dots, 2k-a-b+1, \dots, \nu_k) \\
&\qquad\quad \sqcup (\theta_1 + k - 1, \dots, 2k-a-b, \dots, \theta_k) = \rho_{2k} \\
&\quad\iff (\nu_1 + k-1, \dots, \nu_a + k - a, \dots, \nu_k) \\
&\qquad\quad \sqcup (\theta_1 + k - 1, \dots, \theta_b + k - b, \dots, \theta_k) = \rho_{2k} \\
&\quad\iff (\nu + \rho_k) \sqcup (\theta + \rho_k) = \rho_{2k}.
\end{aligned}$$

This completes the inductive step. $\square$

Corollary 3.5. *Let $\nu, \theta \in L(k,k)$. Then*

$$a_{(\nu+\rho_k)\sqcup(\theta+\rho_k)} = \begin{cases} a_{\rho_{2k}} & \text{when } [\nu] \sqcup \mathsf{flip}_k\theta = [(k^k)] \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Note that $(\nu+\rho_k)\sqcup(\theta+\rho_k) \in L(2k, 2k-1)$. The alternating polynomial $a_{(\nu+\rho_k)\sqcup(\theta+\rho_k)}$ is non-zero if and only if $(\nu+\rho_k)\sqcup(\theta+\rho_k)$ has all parts distinct, which is the case only when it equals ρ_{2k}. By Lemma 3.4, this is equivalent to $[\nu] \sqcup \mathsf{flip}_k\theta = [(k^k)]$. $\square$

Construction and surjectivity of φ. By the polynomial interpretation of one-row Weyl modules, we consider the map φ constructed as

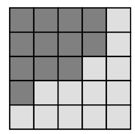
$$\begin{aligned}
\varphi:\quad & \mathrm{Sym}_{(n,m)}\mathrm{Sym}^k E \to \mathrm{Sym}_{(\tilde{n},i)}\mathrm{Sym}^k E \\
& \Lambda_{\leqslant k}[\mathbf{x}_n] \otimes \Lambda_{\leqslant k}[\mathbf{z}_m] \to \Lambda_{\leqslant k}[\mathbf{y}_{\tilde{n}}] \otimes \Lambda_{\leqslant k}[\mathbf{w}_i] \\
& \qquad s_\lambda \otimes s_\mu \mapsto \sum_{\substack{\nu,\theta:\ [\nu]\sqcup\mathsf{flip}_k\theta = [(k^k)] \\ \nu\subseteq\lambda\subseteq\nu\sqcup(k^{\tilde{n}}) \\ \theta\subseteq\mu\subseteq\theta\sqcup(k^i)}} \mathrm{sgn}_{kk}(\nu,\theta) s_{\lambda/\nu} \otimes s_{\mu/\theta},
\end{aligned} \tag{3.2}$$

where $\mathrm{sgn}_{kk}(\nu,\theta)$ is defined in Remark 2.7. As we shall see, by following the computations in the proof of Lemma 4.2, the map φ arises as the composition

$$(\mathrm{wr}_{2k}^{-1} \otimes 1)(\diamondsuit_{(2^k)} \otimes 1)(\mathrm{wr}_k \otimes 1)^{\otimes 2}(\spadesuit_{k\tilde{n}} \otimes \spadesuit_{ki}),$$

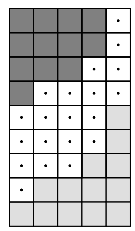
thus it is a well-defined $\mathrm{SL}_2(\mathbb{F})$-homomorphism. (Here $\diamondsuit$ and $\spadesuit$ are the product on exterior powers and coproduct on symmetric powers defined in §2.5.) To show surjectivity we establish the stronger property that the restriction to $\mathrm{Sym}_{(n,i)}\mathrm{Sym}^k E$ is surjective.

Remark 3.6. *There are many partitions involved in the definition of the image of φ. They fit nicely in the following puzzle, which we illustrate with an example* *for $k = 5$, $i = 2$, and $\tilde{n} = 4$. We begin with two partitions $\nu = (4, 4, 3, 1)$ and $\theta = (5, 3, 2, 2, 1)$ such that $\nu \sqcup \mathsf{flip}_5\theta = [(5^5)]$:*



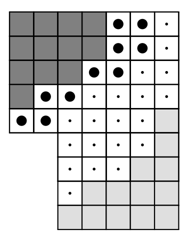
.

Next, we consider a partition $\lambda \in L(n, k)$ such that $\nu \subseteq \lambda \subseteq \nu \sqcup (k^{\tilde{n}}) = \nu \sqcup (5^4)$, which means that $[\lambda/\nu]$ fits in this smaller area marked with $\boxdot$s below, in which every column has height $\tilde{n} = 4$:



.

On the flip side, we have $\theta \subseteq \mu \subseteq \theta \sqcup (k^i) = \theta \sqcup (5^2)$. And so we conclude that the reflection of $[\mu/\theta]$ along the main antidiagonal of $[(k^k)]$ (which we denote $\mathsf{flip}_k(\mu/\theta)$ for consistency) fits into the following area marked with $\boxed{\bullet}$s, in which every row has width $i = 2$:



.

In summary: the $\blacksquare$ area is (a translation of) $[\nu]$, the $\square$ area is (a translation of) $\mathsf{flip}_k\theta$, the $\boxdot$ area is a super-set of $[\lambda/\nu]$ and the $\boxed{\bullet}$ area is a super-set of $\mathsf{flip}_k(\mu/\theta)$. This diagram depends only on $n, i, \tilde{n}$ and ν.

In the following lemma we see $\Lambda_{\leqslant k}[\mathbf{z}_i] = \langle s_\tau \mid \tau \in L(i, k)\rangle$ as a subspace of $\Lambda_{\leqslant k}[\mathbf{z}_m] = \langle s_\tau \mid \tau \in L(m, k)\rangle$.

Lemma 3.7. *The restriction of φ to*

$$\begin{aligned} \Lambda_{\leqslant k}[\mathbf{x}_n] \otimes \Lambda_{\leqslant k}[\mathbf{z}_i] &\to \Lambda_{\leqslant k}[\mathbf{y}_{\tilde{n}}] \otimes \Lambda_{\leqslant k}[\mathbf{w}_i] \\ s_\lambda \otimes s_\mu &\mapsto \sum_{\substack{\nu,\theta\colon [\nu]\sqcup \mathsf{flip}_k\theta = [(k^k)] \\ \nu\subseteq\lambda\subseteq\nu\sqcup(k^{\tilde{n}}) \\ \theta\subseteq\mu\subseteq\theta\sqcup(k^i)}} \mathrm{sgn}_{kk}(\nu,\theta) s_{\lambda/\nu} \otimes s_{\mu/\theta} \end{aligned}$$

is a surjective linear map.

Proof. We proceed by induction on $\mu \in L(i, k)^{\preccurlyeq}$, where the ordering $\preccurlyeq$ is lexicographic, to show that the restriction of φ to

$$\Lambda_{\leqslant k}[\mathbf{x}_n] \otimes \langle s_\tau \mid \tau \in L(i, k),\ \tau \preccurlyeq \mu\rangle \twoheadrightarrow \Lambda_{\leqslant k}[\mathbf{y}_{\tilde{n}}] \otimes \langle s_\tau \mid \tau \in L(i, k),\ \tau \preccurlyeq \mu\rangle,$$

is surjective. For $\mu = (k^i)$ this gives the desired conclusion.

When $\mu = \varnothing$, we have only one pair (ν, θ) in the sum, namely $\nu = (k^k)$ and $\theta = \varnothing$ with $\mathrm{sgn}_{kk}((k^k), \varnothing) = 1$. Thus,

$$\varphi(s_\lambda \otimes 1) = s_{\lambda/(k^k)} \otimes 1.$$

Let λ range over $\{\lambda \in L(n,k) \mid (k^k) \subseteq \lambda\}$. Each such partition can be written as $\lambda = \pi \sqcup (k^k)$ for some $\pi \in L(\tilde{n}, k)$. We have $s_{\lambda/(k^k)} = s_\pi$. Therefore, the image of $\Lambda_{\leqslant k}[\mathbf{x}_n] \otimes \mathbb{F}$ is $\Lambda_{\leqslant k}[\mathbf{y}_{\tilde{n}}] \otimes \mathbb{F}$.

For the induction step, suppose that

$$\Lambda_{\leqslant k}[\mathbf{x}_n] \otimes \langle s_\tau \mid \tau \in L(i,k),\ \tau \prec \mu \rangle \twoheadrightarrow \Lambda_{\leqslant k}[\mathbf{y}_{\tilde{n}}] \otimes \langle s_\tau \mid \tau \in L(i,k),\ \tau \prec \mu \rangle$$

and we want to show that the same holds if the symbol $\prec$ is changed for $\preccurlyeq$ in both the domain and codomain. For this, consider the image of an element $s_\lambda \otimes s_\mu$ for a partition λ of the form $\pi \sqcup (k^k)$ as above:

$$s_{\pi \sqcup (k^k)} \otimes s_\mu \mapsto s_\pi \otimes s_\mu + \sum_{(\nu,\theta) \neq ((k^k), \varnothing)} \mathrm{sgn}_{kk}(\nu, \theta) s_{(\pi \sqcup (k^k))/\nu} \otimes s_{\mu/\theta}, \tag{3.3}$$

where, again, the coefficient at the leading term is $1 = \mathrm{sgn}_{kk}((k^k), \varnothing)$. We shall now show that $\sum_{(\nu,\theta) \neq ((k^k), \varnothing)} \mathrm{sgn}_{kk}(\nu, \theta) s_{\lambda/\nu} \otimes s_{\mu/\theta}$ is in

$$\Lambda_{\leqslant k}[\mathbf{y}_{\tilde{n}}] \otimes \langle s_\tau \mid \tau \in L(i,k),\ \tau \prec \mu \rangle.$$

Since $(\pi \sqcup (k^k))/\nu \subseteq (\nu \sqcup (k^{\tilde{n}}))/\nu$ we have that $s_{(\pi \sqcup (k^k))/\nu}$ is a symmetric polynomial in $\tilde{n}$ variables, each of degree at most k. Therefore, $s_{(\pi \sqcup (k^k))/\nu} \in \Lambda_{\leqslant k}[\mathbf{y}_{\tilde{n}}]$. On the other hand, write $s_{\mu/\theta} = \sum_\tau c^\mu_{\theta\tau} s_\tau$. If the coefficient $c^\mu_{\theta\tau}$ is nonzero, then τ is contained in μ and therefore $\tau \preccurlyeq \mu$. Moreover, since $\theta \neq \varnothing$, we have $|\tau| = |\mu| - |\theta| < |\mu|$, and hence $\tau \prec \mu$. Hence $s_{\mu/\theta} \in \langle s_\tau \mid \tau \in L(i,k),\ \tau \prec \mu \rangle$. Thus, by induction, the sum in (3.3) is in the image of φ, and consequently so is $s_\pi \otimes s_\mu$. Altogether we have shown that for any fixed μ

$$\langle s_\lambda \otimes s_\mu \mid \lambda \in L(n,k),\ (k^k) \subseteq \lambda \rangle \to \Lambda_{\leqslant k}[\mathbf{y}_{\tilde{n}}] \otimes \langle s_\tau \mid \tau \in L(i,k),\ \tau \preccurlyeq \mu \rangle.$$

We conclude

$$\Lambda_{\leqslant k}[\mathbf{x}_n] \otimes \langle s_\tau \mid \tau \in L(i,k),\ \tau \preccurlyeq \mu \rangle \twoheadrightarrow \Lambda_{\leqslant k}[\mathbf{y}_{\tilde{n}}] \otimes \langle s_\tau \mid \tau \in L(i,k),\ \tau \preccurlyeq \mu \rangle,$$

as claimed. □

Since the restriction of φ above is surjective, and has the same codomain as φ, we immediately deduce the following corollary.

Corollary 3.8. *The map φ defined in* (3.2) *is surjective.*

Proof. Since $\mathrm{Sym}_{(n,i)} \mathrm{Sym}^k E \subseteq \mathrm{Sym}_{(n,m)} \mathrm{Sym}^k E$ (as vector spaces, but not necessarily as modules), by Lemma 3.7 we have

$$\mathrm{Sym}_{(\tilde{n},i)} \mathrm{Sym}^k E = \varphi(\mathrm{Sym}_{(n,i)} \mathrm{Sym}^k E) \subseteq \operatorname{im} \varphi \subseteq \mathrm{Sym}_{(\tilde{n},i)} \mathrm{Sym}^k E,$$

and so $\operatorname{im} \varphi$ equals the codomain of φ as vector spaces, thus φ is surjective. □

4. The proof of Theorem 1.1

4.1. **Set-up and notation.** In this section we assume the following notation: for any fixed $i = 0, 1, \ldots, m-1$, let $k = m - i$ and $\tilde{n} = n - m + i = n - k$.

Theorem 1.1 is equivalent to the existence of nested $\mathrm{SL}_2(\mathbb{F})$-modules

$$0 = M_m \leqslant M_{m-1} \leqslant \ldots \leqslant M_0 = \Delta^{(n,m)} \operatorname{Sym}^d E \tag{4.1}$$

which satisfy the following diagram of short exact sequences:

$$\begin{tikzcd}
 & M_m \arrow[r, hook] & M_{m-1} \arrow[r, two heads] \arrow[dl, hook] & L_{m-1} \\
 & M_{m-2} \arrow[r, hook] \arrow[dl, two heads] & \cdots \arrow[dl] & \\
L_{m-2} & M_2 \arrow[r, hook] \arrow[dl, two heads] & M_1 \arrow[r, two heads] \arrow[dl, hook] & L_1 \\
L_2 & M_0 \arrow[dl, two heads] & & \\
L_0 & & &
\end{tikzcd}$$

where $L_i = \operatorname{Sym}_{n+m} \operatorname{Sym}^{d-k} E \otimes \Delta^{(\tilde{n},i)} \operatorname{Sym}^k E$ are the filtration layers from Theorem 1.1.

In order to define the modules M_i, we first construct maps

$$\mathcal{M}_i : M_i \to \tilde{L}_i$$

where for $0 \leqslant i \leqslant m-1$ we define

$$\tilde{L}_i = \operatorname{Sym}_{2k} \operatorname{Sym}^{d-k} E \otimes \Delta^{(\tilde{n},i)} \operatorname{Sym}^d E.$$

In a later section, we show that the image of M_i lands inside an isomorphic copy of L_i inside $\tilde{L}_i$ (see §4.3). These maps $\mathcal{M}_i$ are given by the diagram in Figure 1; we now proceed to construct them. Consider the map $\widehat{\mathcal{M}}_i : \operatorname{Sym}_{(n,m)} \operatorname{Sym}^d E \to \tilde{L}_i$ given by a composition

$$\widehat{\mathcal{M}}_i = (\mathrm{wr}_{2k}^{-1} \otimes 1)(\diamondsuit_{(2^k)} \otimes \psi_{\tilde{n}i})(\mathrm{wr}_k \otimes 1)^{\otimes 2}(\spadesuit_{k\tilde{n}} \otimes \spadesuit_{ki}).$$

Here the maps $\diamondsuit$ and $\spadesuit$ are the product for exterior powers and the coproduct for symmetric powers defined in §2.4, the map ψ in §2.6 and the map wr is the isomorphism from (2.8). All of the above are $\mathrm{SL}_2(\mathbb{F})$-equivariant, hence so is $\widehat{\mathcal{M}}_i$. By Proposition 2.13, $\widehat{\mathcal{M}}_i(U_{nm}) = 0$, so by definition of Weyl modules from §2.9, $\widehat{\mathcal{M}}_i$ descends to the homomorphism

$$\overline{\mathcal{M}}_i : \Delta^{(n,m)} \operatorname{Sym}^d E \to \tilde{L}_i.$$

The modules M_i and homomorphisms $\mathcal{M}_i$ are defined inductively, by setting $M_0 = \Delta^{(n,m)} \operatorname{Sym}^d E$, and for $0 \leqslant i \leqslant m-1$:

$$\mathcal{M}_i = \overline{\mathcal{M}}_i|_{M_i} \qquad \text{and} \qquad M_{i+1} = \ker(\mathcal{M}_i).$$

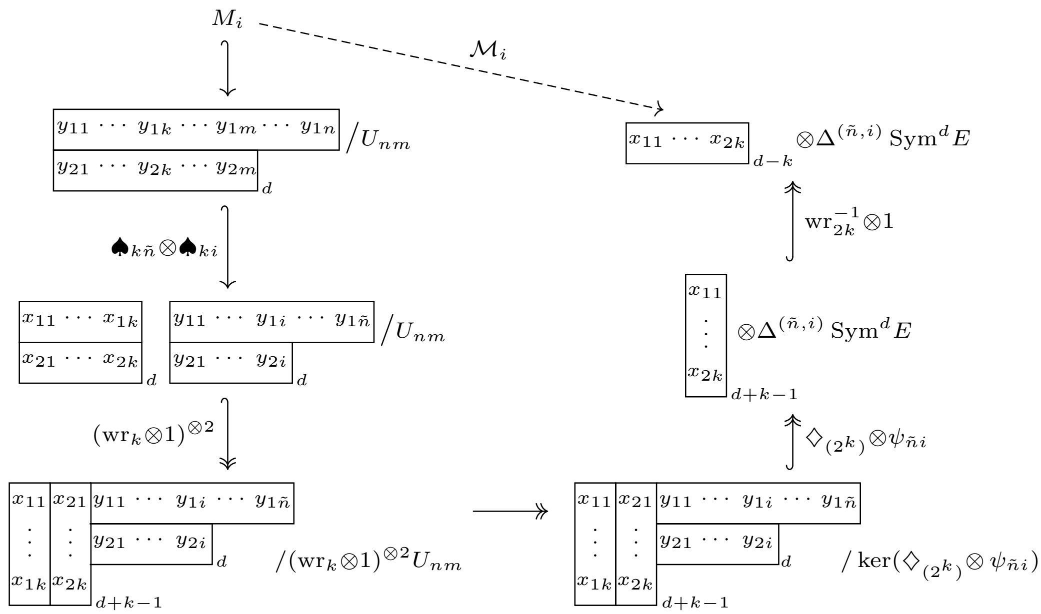


FIGURE 1. The restriction $\mathcal{M}_i$ of the composition $\widehat{\mathcal{M}_i}$ to M_i. The rectangular shapes introduced in §2.4 stand for symmetric and exterior powers. The maps $\spadesuit$ and $\diamondsuit$ are the coproduct of symmetric powers and the product of exterior powers defined in §2.5, the map ψ is the composition from Definition 2.4. We write simply U_{nm} for $(\spadesuit_{k\tilde{n}} \otimes \spadesuit_{ki})U_{nm}$ (see Remark 2.3).

4.2. **Proof outline.** We pause here to assess what remains to complete the proof. In the previous section we defined a nested sequence of $\mathrm{SL}_2(\mathbb{F})$-modules

$$M_m \leqslant M_{m-1} \leqslant \ldots \leqslant M_0 = \Delta^{(n,m)}\,\mathrm{Sym}^d E.$$

We will show that $M_m = 0$ in the concluding dimension counting argument in §4.4.

As we pointed out, we need to show that for $i = 0, 1, \ldots, m-1$ there exists a short exact sequence

$$0 \to M_{i+1} \to M_i \to L_i \to 0.$$

Note that $\tilde{\pi}_{(\tilde{n},i),2k,k,d-k}$ has domain L_i and codomain $\tilde{L}_i$, so by Corollary 3.3,

$$L_i \cong \tilde{\pi}(L_i) \leqslant \tilde{L}_i,$$

and, by definition, $M_{i+1} = \ker(\mathcal{M}_i)$. To conclude the proof of Theorem 1.1 it therefore remains to show that $\mathrm{im}(\mathcal{M}_i) = \tilde{\pi}(L_i) \cong L_i$. Indeed, this will show that the short exact sequence

$$0 \to \ker(\mathcal{M}_i) \to M_i \to \mathrm{im}(\mathcal{M}_i) \to 0$$

is precisely

$$0 \to M_{i+1} \to M_i \to L_i \to 0.$$

We will show that $\mathrm{im}\,\mathcal{M}_i = \tilde{\pi}(L_i)$ in two steps, first arguing that $\tilde{\pi}(L_i)$ is contained in the image of $\mathcal{M}_i$ (viewed just as a linear map of $\mathbb{F}$-vector spaces) and then concluding by dimension counting.

4.3. **$\tilde{\pi}(L_i)$ is a subspace of $\operatorname{im}\mathcal{M}_i$.** In this subsection we turn to consider $\mathcal{M}_i$ primarily as a linear map to deduce that $\mathcal{M}_i(M_i)$ *contains* $\tilde{\pi}(L_i)$. Recall the π map from Definition 2.9 and consider the $\mathrm{SL}_2(\mathbb{F})$-module

$$S_i = (\pi_{(n,m),0,k,d-k}(A_i \otimes B_i) + U_{nm})/U_{nm} \leqslant \Delta^{(n,m)} \operatorname{Sym}^d E,$$

where

$$A_i = \operatorname{Sym}_{n+m} \operatorname{Sym}^{d-k} E \quad \text{and} \quad B_i = \operatorname{Sym}_{(n,m)} \operatorname{Sym}^k E\,.$$

Lemma 4.1. *S_i is a submodule of M_i.*

Proof. By definition, S_i is a submodule of $M_0 = \Delta^{(n,m)} \operatorname{Sym}^d E$. Suppose that for $j < i$, S_i is a submodule of M_j and we will show that S_i is a submodule of M_{j+1}. The claim then follows by induction. Let $k' = m - j$ and $\tilde{n}' = n - m + j = n - k'$. Note that $k < k'$ because $j < i$.

Recall that $M_{j+1} = \ker(\mathcal{M}_j)$ and $\mathcal{M}_j$ is obtained by reducing $\widehat{\mathcal{M}}_j$ to $\overline{\mathcal{M}}_j$ and then restricting to M_j. It suffices then to show that $\pi(A_i \otimes B_i)$ is annihilated by

$$\widehat{\mathcal{M}}_j = (\diamondsuit_{(2^{k'})} \otimes \psi_{(\tilde{n}',j)})(\mathrm{wr}_{k'} \otimes 1)^{\otimes 2}(\spadesuit_{(k',\tilde{n}')} \otimes \spadesuit_{(k',j)})\,,$$

where, as defined in §2.5, $\diamondsuit$ is the product on exterior powers and $\spadesuit$ is the coproduct on symmetric powers. The Wronskian wr is given by multiplication by an alternating polynomial in the variables of the second tensor factor; since π acts as an evaluation homomorphism, this multiplication commutes with π. Together with Lemma 3.1 we obtain that the whole composition commutes with π:

$$\widehat{\mathcal{M}}_j \pi(A_i \otimes B_i) = \pi(A_i \otimes \widehat{\mathcal{M}}_j(B_i)).$$

We encourage the reader to recall Definition 2.9 and the proof of Lemma 3.1 to see that in the second line above, the composition indeed acts only on B_i, leaving A_i unchanged. Indeed, B_i is an $\mathbb{F}$-vector subspace of $\operatorname{Sym}_{(n,m)} \operatorname{Sym}^d E$, the domain of $\widehat{\mathcal{M}}_j$ (which is best seen through the Schur basis in polynomial interpretation), so the composition $\widehat{\mathcal{M}}_j$ is well-defined on the elements of B_i. Expanding the second tensor factor we compute explicitly (again, we refer to diagram (1) to keep track of the tensor products):

$$\begin{aligned}
&(\diamondsuit_{(2^{k'})} \otimes \psi_{\tilde{n}'j})(\mathrm{wr}_{k'} \otimes 1)^{\otimes 2}(\spadesuit_{k'\tilde{n}'} \otimes \spadesuit_{k'j})B_i \\
&\qquad \leqslant \diamondsuit_{(2^{k'})} \mathrm{wr}_{k'}^{\otimes 2} \operatorname{Sym}_{(k',k')} \operatorname{Sym}^k E \otimes \psi_{\tilde{n}'j} \operatorname{Sym}_{(\tilde{n}',j)} \operatorname{Sym}^k E \\
&\qquad\qquad = \textstyle\bigwedge^{2k'} \operatorname{Sym}^{k+k'-1} E \otimes \Delta^{(\tilde{n}',j)} \operatorname{Sym}^k E = 0,
\end{aligned}$$

because the first tensor factor in the last line is 0, as $k + k' < 2k'$. We plug this into the previous computation to conclude

$$\widehat{\mathcal{M}}_j \pi(A_i \otimes B_i) = \pi(A_i \otimes 0) = 0,$$

and so S_i is a submodule of M_{j+1}. In particular, by induction, S_i is a submodule of M_i. $\square$

Lemma 4.2. *As a linear map, the restriction $\mathcal{M}_i : S_i \to \tilde{\pi}(L_i)$ is surjective.*

Proof. As in the proof above, we begin with observing that

$$\mathcal{M}_i(S_i) = \widehat{\mathcal{M}}_i \pi(A_i \otimes B_i) = \pi(A_i \otimes \widehat{\mathcal{M}}_i(B_i)). \tag{4.2}$$

We now compute explicitly the image under $\widehat{\mathcal{M}}_i$ of a basis element $s_\lambda \otimes s_\mu$ of B_i, where $\lambda \in L(n,k)$ and $\mu \in L(m,k)$, that is:

$$\widehat{\mathcal{M}}_i(s_\lambda \otimes s_\mu) = (\mathrm{wr}_{2k}^{-1} \otimes 1)(\diamondsuit_{(2^k)} \otimes \psi_{\tilde{n}i})(\mathrm{wr}_k \otimes 1)^{\otimes 2}(\spadesuit_{k\tilde{n}} \otimes \spadesuit_{ki}) s_\lambda \otimes s_\mu\,.$$

We can rewrite the right-hand side as

$$(\mathrm{wr}_{2k}^{-1} \otimes \psi_{\tilde{n}i})(\diamondsuit_{(2^k)} \otimes 1)(\mathrm{wr}_k \otimes 1)^{\otimes 2}(\spadesuit_{k\tilde{n}} \otimes \spadesuit_{ki}) s_\lambda \otimes s_\mu\,.$$

By Remark 2.6 and by (2.8) we compute

$$\begin{aligned}
&(\mathrm{wr}_k \otimes 1)^{\otimes 2}(\spadesuit_{k\tilde{n}} \otimes \spadesuit_{ki}) s_\lambda \otimes s_\mu \\
&\qquad = (\mathrm{wr}_k \otimes 1)^{\otimes 2} \sum_{\substack{\nu \in L(k,k) \\ \nu \subseteq \lambda \subseteq \nu \sqcup (k^{\tilde{n}})}} \sum_{\substack{\theta \in L(k,k) \\ \theta \subseteq \mu \subseteq \theta \sqcup (k^i)}} s_\nu \otimes s_{\lambda/\nu} \otimes s_\theta \otimes s_{\mu/\theta} \\
&\qquad = \sum_{\substack{\nu \in L(k,k) \\ \nu \subseteq \lambda \subseteq \nu \sqcup (k^{\tilde{n}})}} \sum_{\substack{\theta \in L(k,k) \\ \theta \subseteq \mu \subseteq \theta \sqcup (k^i)}} a_{\nu+\rho_k} \otimes s_{\lambda/\nu} \otimes a_{\theta+\rho_k} \otimes s_{\mu/\theta}\,.
\end{aligned}$$

Apply $\diamondsuit_{(2^k)} \otimes 1$ to the resulting expression using Remark 2.7 to obtain

$$\sum_{\substack{\nu,\theta \in L(k,k) \\ \nu \subseteq \lambda \subseteq \nu \sqcup (k^{\tilde{n}}) \\ \theta \subseteq \mu \subseteq \theta \sqcup (k^i)}} \mathrm{sgn}_{kk}(\nu,\theta) a_{(\nu+\rho_k)\sqcup(\theta+\rho_k)} \otimes s_{\lambda/\nu} \otimes s_{\mu/\theta} = a_{\rho_{2k}} \otimes \varphi(s_\lambda \otimes s_\mu),$$

where we have used Corollary 3.5 with φ as defined in (3.2). Finish by applying $\mathrm{wr}_{2k}^{-1} \otimes \psi_{\tilde{n}i}$ to get

$$\widehat{\mathcal{M}}_i(s_\lambda \otimes s_\mu) = 1 \otimes \psi_{\tilde{n}i}\, \varphi(s_\lambda \otimes s_\mu)\,. \tag{4.3}$$

Now, combining (4.2), (4.3), and the surjectivity of φ from Corollary 3.8, we conclude:

$$\begin{aligned}
\mathcal{M}_i(S_i) &= \pi(A_i \otimes \widehat{\mathcal{M}}_i(B_i)) = \pi(A_i \otimes \psi_{\tilde{n}i}\varphi(B_i)) \\
&= \pi(A_i \otimes \psi_{\tilde{n}i}(\mathrm{Sym}_{(\tilde{n},i)} \mathrm{Sym}^k E)) = \tilde{\pi}(A_i \otimes \Delta^{(\tilde{n},i)} \mathrm{Sym}^k E) = \tilde{\pi}(L_i). \quad \square
\end{aligned}$$

4.4. **Dimension counting and exactness.**

Proposition 4.3. *The* $\mathrm{SL}_2(\mathbb{F})$*-homomorphisms* $\tilde{\pi}^{-1}_{(\tilde{n},i),2k,k,d-k} \circ \mathcal{M}_i$ *realise the surjections* $M_i \twoheadrightarrow L_i$.

Proof. In the previous section we showed that

$$L_i \cong \tilde{\pi}(L_i) = \mathcal{M}_i(S_i) \leqslant \mathrm{im}\, \mathcal{M}_i \tag{4.4}$$

as $\mathbb{F}$-vector spaces. Hence, by definition of M_{i+1} as the kernel of $\mathcal{M}_i$ and by rank-nullity,

$$\dim L_i \leqslant \dim(\mathrm{im}\, \mathcal{M}_i) = \dim M_i - \dim M_{i+1}$$

for $i = 0, 1, \ldots, m-1$. Summing these inequalities, we get

$$\sum_{i=0}^{m-1} \dim L_i \leqslant \dim M_0 - \dim M_m \leqslant \dim M_0\,.$$

Recall that $M_0 = \Delta^{(n,m)} \mathrm{Sym}^d E$ by definition, and by equation (1.2) with $q = 1$, $\sum \dim L_i = \dim \Delta^{(n,m)} \mathrm{Sym}^d E$, so every inequality above is in fact an equality, in particular $\dim L_i = \dim(\mathrm{im}\, \mathcal{M}_i)$ and also $\dim M_m = 0$. Together with the relation (4.4) this gives $\operatorname{im} \mathcal{M}_i \cong L_i$, now as $\mathrm{SL}_2(\mathbb{F})$-modules because $\tilde{\pi}$ and $\mathcal{M}_i$ are $\mathrm{SL}_2(\mathbb{F})$-homomorphisms. □

Proof of Theorem 1.1. By Proposition 4.3, for $i = 0, 1, \ldots, m-1$ there is a short exact sequence

$$0 \to M_{i+1} \to M_i \xrightarrow{\tilde{\pi}^{-1} \circ \mathcal{M}_i} L_i \to 0,$$

which proves the claimed filtration of $\Delta^{(n,m)} \operatorname{Sym}^d E$. □

As a corollary of our main result, we get the following structure theorem.

Theorem 4.4. *There is a long exact sequence of* $\mathrm{SL}_2(\mathbb{F})$*-modules*

$$\cdots \to \Delta^{(2,2)} \operatorname{Sym}^d E \to \Delta^{(2,2)} \operatorname{Sym}^{d-1} E \to \cdots \to \Delta^{(2,2)} \operatorname{Sym}^2 E \to \mathbb{F}.$$

Proof. Theorem 1.1 with $n = m = 2$ gives a short exact sequence

$$0 \to \operatorname{Sym}_4 \operatorname{Sym}^{d-1} E \to \Delta^{(2,2)} \operatorname{Sym}^d E \xrightarrow{\overline{\mathcal{M}}_1^{(d)}} \operatorname{Sym}_4 \operatorname{Sym}^{d-2} E \to 0$$

for every $d \geqslant 2$. In particular

$$\ker \overline{\mathcal{M}}_1^{(d)} = \operatorname{Sym}_4 \operatorname{Sym}^{d-1} E = \operatorname{im} \overline{\mathcal{M}}_1^{(d+1)},$$

so the homomorphisms $\overline{\mathcal{M}}_1^{(d)}$ for $d \geqslant 2$ form the long exact sequence above. □

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References

[AFP+19] M. Aprodu, G. Farkas, Ş. Papadima, C. Raicu, and J. Weyman. Koszul modules and Green's conjecture. *Inventiones mathematicae*, 218(3):657–720, 2019.

[ALRS13] F. Ardila, E. León, M. Rosas, and M. Skandera. Tres lecciones en combinatoria algebraica. II. Las funciones simétricas y la teoría de representaciones. 2013. arXiv:1301.3988.

[BB05] A. Bjorner and F. Brenti. *Combinatorics of Coxeter groups*. Springer, 2005.

[BFK99] J. Bernstein, I. Frenkel, and M. Khovanov. A categorification of the Temperley–Lieb algebra and Schur quotients of $U(\mathfrak{sl}_2)$ via projective and Zuckerman functors. *Selecta Mathematica*, 5:199–241, 1999.

[CF94] L. Crane and I. B. Frenkel. Four-dimensional topological quantum field theory, Hopf categories, and the canonical bases. *Journal of Mathematical Physics*, 35(10):5136–5154, 1994.

[FGGK26] I. Fischer, M. Gangl, Á. Gutiérrez, and N. Kumari. Affine Jacobi–Trudi formulas. In preparation, 2026+.

[FKS07] I. Frenkel, M. Khovanov, and C. Stroppel. A categorification of finite-dimensional irreducible representations of quantum sl_2 and their tensor products. *Selecta Mathematica*, 12(3):379, 2007.

[Ful97] William Fulton. *Young tableaux*, volume 35 of *London Mathematical Society student texts*. CUP, 1997.

[GK25] Á. Gutiérrez and C. Krattenthaler. Schur log-concavity and the quantum Pascal triangle. arXiv:2509.22648, 2025.

[GMSW25] Á. Gutiérrez, Á. Martínez, M. Szwej, and M. Wildon. Modular isomorphisms of $\mathrm{SL}_2(\mathbb{F})$-plethysms for Weyl modules labelled by hook partitions. arXiv:2509.01490, 2025.

[GMSW26a] Á. Gutiérrez, Á. Martínez, M. Szwej, and M. Wildon. A new bijective proof of the q-Pfaff–Saalschütz identity with applications to quantum groups. *European Journal of Combinatorics*, 133:104321, 2026.

[GMSW26b] Á. Gutiérrez, Á. Martínez, M. Szwej, and M. Wildon. Plethystic lifts of q-binomial identities. *To appear in the Proceedings of FPSAC, Séminaire Lotharingien Combinatoire*, 2026.

[Gri23] D. Grinberg. Comments on arxiv:2105.00538v3. Available at `https://www.cip.ifi.lmu.de/~grinberg/algebra/sln-equiv.pdf`, 2023.

[IOT25] C. Ikenmeyer, H. Omar, and D. Tsintsilidas. Field-independent Kronecker-plethysm isomorphisms, 2025. arXiv:2509.10069.

[KL09] M. Khovanov and A. D. Lauda. A diagrammatic approach to categorification of quantum groups I *Representation Theory of the American Mathematical Society*, 13(14), 2009.

[Lau10] A. D. Lauda. A categorification of quantum sl(2). *Advances in Mathematics*, 225(6), 2010.

[LR11] Nicholas A. Loehr and Jeffrey B. Remmel. A computational and combinatorial exposé of plethystic calculus. *J. Algebraic Combin.*, 33(2):163–198, 2011.

[Mac95] I. G. Macdonald. *Symmetric functions and Hall polynomials*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, second edition, 1995. With contributions by A. Zelevinsky, Oxford Science Publications.

[McD25] E. McDowell. An explicit construction of the Weyl module as a quotient of symmetric tensors by dual Garnir relations. arXiv:2508.14788, 2025.

[McDW22] E. McDowell and M. Wildon. Modular plethystic isomorphisms for two-dimensional linear groups. *Journal of Algebra*, 602, 2022.

[MW25] Á. Martínez and M. Wildon. A new modular plethystic $\mathrm{SL}_2(\mathbb{F})$-isomorphism $\mathrm{Sym}^{N-1}E \otimes \wedge^{N+1}\mathrm{Sym}^{d+1}E \cong \Delta^{(2,1^{N-1})}\mathrm{Sym}^d E$. *Journal of Algebra*, 682, 2025.

[Rou08] R. Rouquier. 2-Kac-Moody algebras. arxiv:0812.5023, 2008.

[Sta99] R. P. Stanley. *Enumerative combinatorics Vol. 2*. Cambridge Studies in Adv. Math., 1999.

[Str05] C. Stroppel. Categorification of the Temperley–Lieb algebra, tangles and cobordisms via projective functors. *Duke Mathematical Journal*, 126(3):547–596, 2005.

[Wey03] J. Weyman. *Cohomology of Vector Bundles and Syzygies*. Cambridge Tracts in Mathematics. Cambridge University Press, 2003.

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