

# INDECOMPOSABLE SUMMANDS OF FOULKES MODULES

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ABSTRACT. In this paper we study the modular structure of the permutation module  $H^{(2^n)}$  of the symmetric group  $S_{2n}$  acting on set partitions of a set of size  $2n$  into  $n$  sets each of size 2, defined over a field of odd characteristic  $p$ . In particular we characterize the vertices of the indecomposable summands of  $H^{(2^n)}$  and fully describe all of its indecomposable summands that lie in blocks of  $p$ -weight at most two. When  $2n < 3p$  we show that there is a unique summand of  $H^{(2^n)}$  in the principal block of  $S_{2n}$  and that this summand exhibits many of the extensions between simple modules in its block.

## 1. INTRODUCTION

The symmetric group  $S_{2n}$  acts on the collection of all set partitions of  $\{1, \dots, 2n\}$  into  $n$  sets each of size two. Let  $H^{(2^n)}$  denote the corresponding permutation representation of  $S_{2n}$ , defined over a field  $F$  of odd prime characteristic  $p$ . Equivalently,  $H^{(2^n)}$  is the  $FS_{2n}$ -module induced from the trivial representation of the imprimitive wreath product  $S_2 \wr S_n \leq S_{2n}$ . We call  $H^{(2^n)}$  a *Foulkes module*. In the main theorem of [8] the authors used results on the indecomposable summands of Foulkes modules to determine certain decomposition numbers of the symmetric group. In this paper we study the structure of Foulkes modules more closely. In particular, we characterize the vertices of the indecomposable summands of each  $H^{(2^n)}$  and give a precise description of all summands in blocks of  $p$ -weight at most two.

Let  $Q_t$  be a Sylow  $p$ -subgroup of  $S_2 \wr S_{tp}$ . By [8, Theorem 1.2], if  $U$  is an indecomposable summand of  $H^{(2^n)}$  then  $U$  has vertex  $Q_t$  for some  $t \in \mathbf{N}_0$ . Our first main theorem gives the converse.

**Theorem 1.1.** *Let  $n \in \mathbf{N}_0$ . For all  $t \in \mathbf{N}_0$  such that  $t \leq n/p$  there is an indecomposable summand of  $H^{(2^n)}$  with vertex  $Q_t$ .*

To state the second main theorem we need some more definitions from [8]. When defined over a field of characteristic zero,  $H^{(2^n)}$  has ordinary character  $\sum \chi^{2\lambda}$  where the sum is over all partitions  $\lambda$  of  $n$ , and  $2\lambda$  is the partition obtained from  $\lambda$  by doubling each part. (For an elegant proof of this fact

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with minimal prerequisites, see [12].) We say that such partitions are *even*. Given a  $p$ -core  $\gamma$ , let  $w(\gamma)$  be the minimum number of  $p$ -hooks that, when added to  $\gamma$ , give an even partition. Let  $\mathcal{E}(\gamma)$  be the set of even partitions that can be obtained by adding  $w(\gamma)$   $p$ -hooks to  $\gamma$ . For example, if  $p = 3$ , then  $w((3, 1)) = 2$  and  $\mathcal{E}((3, 1)) = \{(6, 4), (6, 2, 2), (4, 4, 2)\}$ . Let  $B(\gamma, w)$  denote the  $p$ -block of  $S_n$  with  $p$ -core  $\gamma$  and  $p$ -weight  $w$ , where  $n = |\gamma| + wp$ . As a convenient shorthand, we write  $\nu \in B(\gamma, w)$  to mean that the partition  $\nu$  has  $p$ -core  $\gamma$  and  $p$ -weight  $w$ . Let  $S^\mu$  denote the Specht module labelled by the partition  $\mu$ . For  $\mu$  a  $p$ -regular partition, let  $P^\mu$  denote the projective cover of the simple  $FS_n$ -module  $D^\mu$ , defined in [15, Corollary 12.2] as the unique top composition factor of  $S^\mu$ . Finally let  $d_{\mu\nu} = [S^\mu : D^\nu]$ .

**Theorem 1.2.** *Let  $n \in \mathbf{N}$ . Let  $\gamma$  be a  $p$ -core.*

- (i) *Suppose that  $|\gamma| = 2n$ . There is a summand of  $H^{(2^n)}$  in  $B(\gamma, 0)$  if and only if  $\gamma$  is even. In this case the unique summand is the simple projective Specht module  $S^\gamma$ .*
- (ii) *Suppose that  $|\gamma| = 2n - p$ . There is a summand of  $H^{(2^n)}$  in  $B(\gamma, 1)$  if and only if  $w(\gamma) = 1$ . In this case  $\mathcal{E}(\gamma) = \{2\lambda, 2\mu\}$  for partitions  $\lambda, \mu$  with  $\lambda \triangleleft \mu$ , and the unique summand is  $P^{2\mu}$ .*
- (iii) *Suppose that  $|\gamma| = 2(n - p)$ . There is a summand of  $H^{(2^n)}$  in  $B(\gamma, 2)$  if and only if  $w(\gamma) = 0$  or  $w(\gamma) = 2$ . If  $w(\gamma) = 2$  the unique summand is  $P^{2\mu}$ , where  $2\mu$  is the unique maximal element of  $\mathcal{E}(\gamma)$ . If  $w(\gamma) = 0$  the unique summand has vertex  $Q_1$  and its Green correspondent is  $P \otimes S^\gamma$  as a representation of  $N_{S_{2n}}(Q_1)/Q_1 \cong N_{S_{2p}}(Q_1)/Q_1 \times S_{n-2p}$ , where  $P$  is the projective cover of the trivial  $FN_{S_{2p}}(Q_1)/Q_1$ -module.*

If  $t = \lfloor n/p \rfloor$  then  $Q_t$  is a Sylow  $p$ -subgroup of  $S_{2t} \wr S_n$ . The projective cover of the trivial representation of  $N_{S_{2n}}(Q_t)/Q_t$  has vertex  $Q_t$  as a representation of  $N_{S_{2n}}(Q_t)$ ; its Green correspondent is a summand of  $H^{(2^n)}$  with vertex  $Q_t$  lying in the principal block of  $S_{2n}$ . This summand is an example of a Scott module: see [9] or [3] for their definition and basic properties. Our third main theorem describes these summands when  $2n < 3p$ .

**Theorem 1.3.** *Let  $2k < p$  and let  $n = p + k$ . There is a unique summand  $U$  of  $H^{(2^n)}$  in the principal block of  $S_{2n}$ . This summand is the Scott module with vertex  $Q_1$ . The module  $U$  has three Loewy layers and*

$$\text{soc } U \cong \text{top } U \cong \bigoplus_{\substack{2\nu \in B((2k), 2) \\ 2\nu \neq (2k, 2^p)}} D^{2\nu}.$$

*The Loewy layers of  $U$  are shown in Figure 1 overleaf, where each edge shows a non split extension exhibited by this module.*

Our final theorem counts the summands of  $H^{(2^n)}$  which, like the Scott module summand, have the largest possible vertex.

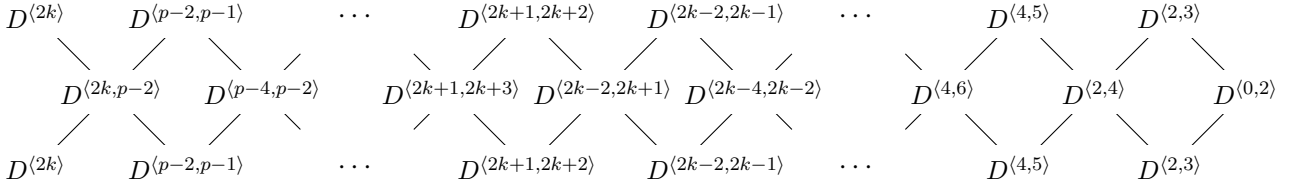


FIGURE 1. The three Loewy layers of the unique principal block summand of  $H^{(2^n)}$  when  $n = 2(p+k)$  and  $2 < 2k < p-1$ . The labels of simple modules are defined in Section 7. If  $2k \in \{0, 2, p-1\}$  then the structure is the same but minor changes must be made to the labels: see Figure 5 in Section 7.

**Theorem 1.4.** *Let  $t = \lfloor n/p \rfloor$  and let  $r = n - tp$ . The number of indecomposable summands of  $H^{(2^n)}$  with vertex  $Q_t$  is equal to the number of  $p$ -core partitions that can be obtained by removing  $p$ -hooks from even partitions of  $2r$ . Each such summand lies in a different block of  $S_{2n}$ .*

In particular we note that if  $0 \leq 2r < p$  then the number of indecomposable summands of  $H^{(2^n)}$  with vertex  $Q_t$  is simply the number of even partitions of  $2r$ . Another easy corollary of our theorems is the following.

**Corollary 1.5.**  *$H^{(2^n)}$  is an indecomposable  $FS_{2n}$ -module if and only if  $n = 2$  and  $p = 3$ .*

*Proof.* By Theorem 1.1 there are summands of  $H^{(2^n)}$  with vertices  $Q_0$  and  $Q_1$  whenever  $n \geq p$ . When  $4 \leq n < p$  the partitions  $(2n)$ ,  $(2n-2, 2)$  and  $(2n-4, 4)$  are not all in the same block, so  $H^{(2^n)}$  has summands in two different blocks. By Theorem 1.2, if  $p = 3$  then  $H^{(2^2)}$  is indecomposable; if  $p > 3$  then  $H^{(2^2)} \cong S^{(4)} \oplus S^{(2,2)}$ . If  $p = 3$  then  $H^{(2^3)}$  has  $S^{(4,2)}$  as the unique summand in its block; if  $p = 5$  then  $H^{(2^3)}$  has  $S^{(2,2,2)}$  as the unique summand in its block; if  $p > 5$  then  $H^{(2^3)} \cong S^{(6)} \oplus S^{(4,2)} \oplus S^{(2,2,2)}$ .  $\square$

The results in this paper show that the behaviour of Foulkes modules is very different from the better studied Young permutation modules (see, for instance, [6], [5], [11] and [13]). In particular, by Theorem 1.2, each Foulkes module has at most one summand in each block of weight at most two. The Young permutation modules  $M^{(r-s,s)}$  also have multiplicity-free ordinary characters, but it follows easily from [11, Theorem 3.3] that if  $p$  is odd,  $0 \leq c < p$  and  $p \leq j \leq p + c/2$  then  $M^{(2p+c-j,j)}$  always has two summands, namely the Young modules  $Y^{(2p+c)}$  and  $Y^{(p+c,p)}$ , in the principal block of  $S_{2p+c}$ . By Corollary 1.5 there is only one non-trivial indecomposable Foulkes module, whereas  $M^{(p-1,1)}$  is always indecomposable. (The indecomposable Young permutation modules are classified in [5, Theorem 2].) It would be

interesting to know the decomposition of further permutation modules for symmetric groups in prime characteristic.

**Outline.** In §2 we give the prerequisite material from [8]. In §3 we find  $w(\gamma)$  for a special class of  $p$ -cores  $\gamma$ . This result is used in the proof of Theorem 1.1. In §4 we recall Richards' results [18] on decomposition numbers in blocks of  $p$ -weight 2, and show that even partitions of  $p$ -weight 2 have a surprisingly simple characterization using his definitions. These results are used in the proof of Theorem 1.2(iii) and Theorem 1.3. In §5, §6 and §7 we prove Theorem 1.1, Theorem 1.2 and Theorem 1.3 respectively. Theorem 1.4 is obtained as a corollary of Theorem 1.2 in §6. We end in §8 with an example to illustrate these theorems. The proofs in §3 and §4 are the most technical in this paper; we suggest the reader returns to the proofs after seeing the applications.

## 2. PREREQUISITES

The following results have short proofs using [8]. For background on local representation theory we refer the reader to [1].

**Proposition 2.1.** *Let  $\gamma$  be a  $p$ -core partition and let  $2n = |\gamma| + pw$ . The Green correspondence induces a bijection between the indecomposable summands of  $H^{(2^n)}$  in  $B(\gamma, w)$  with vertex  $Q_t$  and the indecomposable projective summands of  $H^{(2^{n-tp})}$  in  $B(\gamma, w - 2t)$ .*

*Proof.* Let  $U$  be an indecomposable non-projective summand of  $H^{(2^n)}$  in  $B(\gamma, w)$ . By [8, Theorem 1.2],  $U$  has  $Q_t$  as a vertex for some  $t \in \mathbf{N}$  and its Green correspondent has a tensor factorization  $V \boxtimes W$  as a representation of  $(N_{S_{2tp}}(Q_t)/Q_t) \times S_{2(n-tp)}$ . Here  $V$  and  $W$  are projective and  $W$  is an indecomposable summand of  $H^{(2^{n-tp})}$ . By [8, Theorem 2.7], which is proved using the results in [3],  $W$  lies in the block  $B(\gamma, w - 2t)$  of  $S_{2(n-tp)}$ . The map sending  $U$  to  $W$  therefore has the required properties.  $\square$

**Proposition 2.2.** *Let  $\gamma$  be a  $p$ -core partition and let  $2n = |\gamma| + pw$ .*

- (i) *All the summands of  $H^{(2^n)}$  in  $B(\gamma, w(\gamma))$  are projective.*
- (ii) *If  $2\mu$  is a maximal element of  $\mathcal{E}(\gamma)$  then  $\mu$  is  $p$ -regular and  $P^{2\mu}$  is a summand of  $H^{(2^n)}$  lying in  $B(\gamma, w(\gamma))$ .*
- (iii) *If  $P$  is an indecomposable projective summand of  $H^{(2^n)}$  in  $B(\gamma, w(\gamma))$  then  $P = P^{2\nu}$  for some  $2\nu \in \mathcal{E}(\gamma)$  and the ordinary character of  $P$  (defined using Brauer reciprocity) is  $\sum_{2\lambda \in E} \chi^{2\lambda}$  for some  $E \subseteq \mathcal{E}(\gamma)$ .*

*Proof.* Part (i) is Proposition 5.1 of [8]. Since this is a key result in this paper, we briefly sketch the proof: if  $U$  is a non-projective summand in  $B(\gamma, w(\gamma))$  then, by Proposition 2.1, there is a projective summand of  $H^{(2^{n-tp})}$

in  $B(\gamma, w(\gamma) - 2t)$  for some  $t \in \mathbf{N}$ . But this implies that there is an even partition with  $p$ -core  $\gamma$  and  $p$ -weight  $w(\gamma) - 2t$ , contradicting the definition of  $w(\gamma)$ . Parts (ii) and (iii) follow immediately from Proposition 1.3 of [8].  $\square$

We use the following proposition in the proof of Theorem 1.2 to show that, if  $w \leq 2$ , then the summand identified in Proposition 2.2(ii) is the unique summand of  $H^{(2^n)}$  lying in its block.

**Proposition 2.3.** *Let  $\gamma$  be a  $p$ -core partition of  $n$ . If  $|\mathcal{E}(\gamma)| < 2w(\gamma) + 1$  then there is a unique summand of  $H^{(2^n)}$  in  $B(\gamma, w(\gamma))$ .*

*Proof.* It was proved independently in [18, Theorem 2.8] and [2, Proposition 4.6(i)] that if  $\nu$  is a  $p$ -regular partition of  $n$  lying in a block of  $p$ -weight  $w$  then  $d_{\mu\nu} > 0$  for at least  $w + 1$  partitions  $\mu$ . By Proposition 2.2(i), all the summands of  $H^{(2^n)}$  in  $B(\gamma, w(\gamma))$  are projective. By Brauer reciprocity, the ordinary character of each such summand contains at least  $w + 1$  distinct irreducible characters. The result now follows from Proposition 2.2(iii).  $\square$

### 3. EVEN PARTITIONS AND THE ABACUS

We make extensive use of James' abacus notation, as defined in [14, page 78]. We number the abacus runners from 0 to  $p - 1$  and the abacus positions by integers. By convention, our abaci have infinitely many beads with only finitely many beads in positions after the first space. We say that beads before the first space are *initial*. For  $r \in \mathbf{Z}$ , we define *row*  $r$  to consist of positions  $pr, \dots, pr + p - 1$ . By a *single-step move* we mean a move of a bead into a space immediately below it.

**Remark 3.1.** Given a partition  $\lambda$  and an integer  $z \in \mathbf{Z}$ , there exists a unique abacus  $A_z(\lambda)$  representing  $\lambda$  and having first space in position  $z$ . Let  $x \in \mathbf{Z}$  and let  $\tilde{A}$  be the abacus obtained from  $A_z(\lambda)$  by moving the bead in position  $y$  of  $A_z(\lambda)$  to position  $y + x$ , for all  $y \in \mathbf{Z}$ . It is easy to see that  $\tilde{A} = A_{z+x}(\lambda)$ . We say that  $\tilde{A}$  is a *rotation* of  $A$ .

It is simple to recognise even partitions on the abacus.

**Definition 3.2.** Let  $A$  be an abacus. We say that beads in positions  $\beta$  and  $\beta'$  of  $A$  where  $\beta < \beta'$  are *consecutive* and form a *gap* if there are spaces in positions  $\beta + 1, \dots, \beta' - 1$ . This gap is *odd* if  $\beta' - \beta$  is even and *even* if  $\beta' - \beta$  is odd. If  $\beta' - \beta = 1$ , then we say that the beads are *adjacent*, or in *adjacent positions*.

**Lemma 3.3.** *Let  $A$  be an abacus representing a partition  $\nu$ . Then  $\nu$  is even if and only if  $A$  has no odd gaps.*  $\square$

The following proposition is helpful in determining  $w(\gamma)$ . Informally, it states that certain beads on an abacus representing  $\gamma$  need not be moved to find a partition in  $\mathcal{E}(\gamma)$ .

**Proposition 3.4.** *Let  $\gamma$  be a  $p$ -core partition and let  $A = A_0(\gamma)$  be the abacus representing  $\gamma$  with first space in position 0. Let  $B$  be an abacus representing a partition  $\lambda \in \mathcal{E}(\gamma)$  obtained by a sequence of single-step moves on  $A$ . If  $x < -1$  then  $B$  has a bead in position  $x$ .*

*Proof.* Suppose, for a contradiction, that there exists  $x < -1$  such that  $B$  has a space in position  $x$ . Choose  $x$  minimal with this property. Since the bead in position  $x - 1$  of  $A$  is not moved, to avoid an odd gap, the bead in position  $x + 1$  of  $A$  must be moved. By replacing the abacus  $A$  with  $A_{-x}(\gamma)$  we may assume that  $x = 0$ . Therefore positions 0 and 1 in  $B$  have spaces.

Let  $b$  be the highest bead on runner 0 of  $B$  below position 0, and let  $c$  be the highest bead on runner 1 of  $B$  below position 1. If  $b$  and  $c$  are on the same row then we may move  $b$  up to position 0 and  $c$  up to position 1 without changing the parity of any gap in  $B$ . (This holds because there is a bead in position  $-1$ , and the gap between this bead and the consecutive bead in  $B$  is even.) We therefore obtain an even partition with  $p$ -core  $\gamma$  and strictly smaller  $p$ -weight, a contradiction.

Assume that  $c$  is on a strictly lower row than  $b$ ; the other case is dealt with symmetrically. Let  $b'$  be the first bead in a greater numbered position than  $b$ , and let  $c'$  be the first bead in a greater numbered position than  $c$ . Suppose that  $b'$  is on runner  $u$  and  $c'$  is on runner  $v$ . There are at most  $2(p - 1)$  spaces between  $b$  and  $b'$  since otherwise  $b'$  can be moved two rows up without changing the parity of any gaps. Hence either  $b'$  is on the same row as  $b$ , in which case  $u$  is odd, or  $b'$  is on the row directly below, in which case  $u$  is even. A similar remark applies to  $c$  and  $c'$ , swapping odd and even, with one extra case when  $v = 0$ , in which case  $c'$  is two rows below  $c$ .

We claim that there is a sequence of bead moves using the beads  $b$  and  $b'$  and beads on runners  $u$  and  $v$  that gives a new abacus  $B^*$ , representing a partition of strictly smaller  $p$ -weight than  $\lambda$ , such that  $B^*$  has no odd gaps. This can be shown by considering the four cases for the parity of  $u$  and  $v$ . The different parity cases are split into subcases  $u < v$  and  $u > v$ , as shown in Figure 2, and the equal parity cases are split into subcases  $u = 0$  and  $u \neq 0$ , as shown in Figure 3. The indicated bead moves deal correctly with the exceptional cases when  $u = 0$ ,  $v = 0$  or  $v = 1$  and two runners coincide. (Note that  $u = 1$  is impossible because then  $b' = c$  and  $b$  and  $c$  are in the same row.) If  $v = 0$  and  $u$  is odd then  $c'$  is two rows below  $c$ , the bead moves in the top right diagram in Figure 2 apply and a bead is moved into the position vacated by  $b$ .

We give full details for the case where  $B$  is the top-left abacus in Figure 2. Thus  $c$  is on a strictly lower row than  $b$ , and  $b'$  and  $c'$  are in the same rows as  $b$  and  $c$ , respectively. Let  $s$  be the row of bead  $b$ ,  $t$  be the row of bead  $c$  and let  $w$  be the  $p$ -weight of the partition  $\lambda$ . Let  $B'$  be the abacus obtained from  $B$  by moving beads  $b$  and  $c$  up to positions 0 and 1 respectively by a sequence of single-step bead moves. Notice that  $B'$  has no odd gaps in rows  $1, \dots, s-1$ , and that  $B'$  represents a partition of  $p$ -weight  $w-s-t$ . Let  $B''$  be the abacus obtained from  $B'$  by making the unique series of single-step moves of beads on runner  $u$  that has the final effect of swapping bead  $b'$  with the space in position  $pt+u$ , in row  $t$  on runner  $u$ . (The bead that ends in position  $pt+u$  in  $B''$  is therefore the lowest bead on runner  $u$  of  $B'$  above this position.) It is easily seen that  $B''$  has no odd gaps and represents a partition of  $p$ -weight

$$w - s - t + (t - s) = w - 2s.$$

This is clearly strictly less than  $w$ , the  $p$ -weight of  $\lambda$ , as required. □

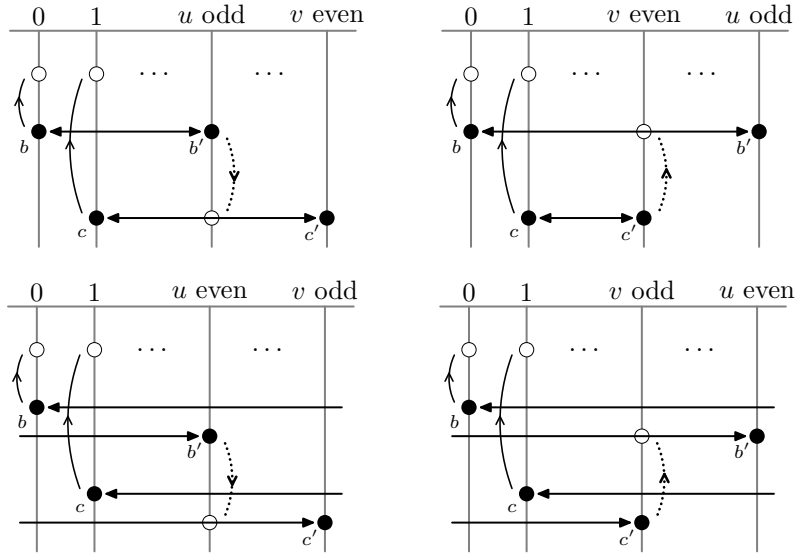


FIGURE 2. Bead moves on the abacus  $B$  used in the proof of Proposition 3.4 when  $u$  and  $v$  have different parity. White circles show spaces in  $B$  that are present by hypothesis and are the target of bead moves; the two adjacent spaces are positions 0 and 1. Thick arrows show even gaps in  $B$ . Solid curved arrows show moves of the beads  $b$  and  $c$  on runners 0 and 1, respectively. On runners  $u$  and  $v$  dotted curved arrows show the overall effect of a sequence of single-step upward or downward moves using the marked bead and the beads between the marked bead and the target space.

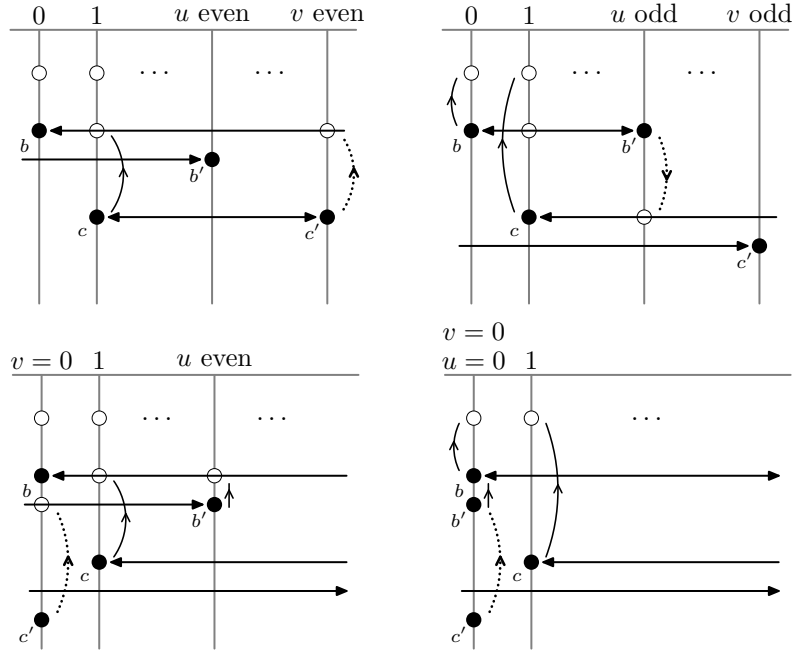


FIGURE 3. Bead moves on the abacus  $B$  used in the proof of Proposition 3.4 when  $u$  and  $v$  have equal parity. The conventions for arrows are as described in the caption of Figure 2. The cases  $u = 0, v \neq 0$  and  $u = v = 0$  are dealt with separately: in the latter case at least four beads move, bead  $b'$  is moved to the position vacated by bead  $b$  and a bead is moved into the position vacated by bead  $b'$ .

Despite its special nature, the following result appears to require the full power of Proposition 3.4.

**Proposition 3.5.** *Let  $\gamma$  be a  $p$ -core partition. Suppose that the abacus  $A_0(\gamma)$  has  $t$  beads in positive positions on runner  $u$ ,  $t'$  beads in positive positions on runner  $u'$ , where  $u < u'$ , and no beads in any other positive position.*

- (i) *If  $u$  and  $u'$  are both even then  $w(\gamma) = tt'$  and a partition in  $\mathcal{E}(\gamma)$  can be obtained by moving each of the  $t'$  beads in positive positions on runner  $u'$  down  $t$  steps.*
- (ii) *If  $u$  is odd and  $u' = p - 1$  then  $w(\gamma) = t(t' + 1)$  and a partition in  $\mathcal{E}(\gamma)$  can be obtained by moving each of the  $t$  beads in positive positions on runner  $u$  down  $t' + 1$  steps.*

*Proof.* We prove case (i) and then indicate the changes needed for (ii). The  $p$ -core  $\gamma$  is represented by the abacus  $A = A_0(\gamma)$ , in which the first space is in position 0. The  $p$  initial beads occupying positions  $-p, \dots, -1$  are shown on the diagram of  $A_0(\gamma)$  below. (If  $t > t'$  then runners  $u$  and  $u'$  should be



swapped.) Let  $b$  and  $b'$  be the beads in positions  $u$  and  $u'$  respectively and let  $c$  and  $c'$  be the lowest beads on runners  $u$  and  $u'$  respectively.

$$\begin{array}{cccccccccccc}
 0 & \cdots & u & u+1 & \cdots & u'-1 & u' & \cdots & p-1 \\
 \bullet & \cdots & \bullet & \bullet & \cdots & \bullet & \bullet & \cdots & \bullet d \\
 \circ & \cdots & b\bullet & \circ & \cdots & \circ & \bullet b' & \cdots & \circ \\
 \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
 \circ & \cdots & c\bullet & \circ & \cdots & \circ & \bullet & \cdots & \circ \\
 \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
 \circ & \cdots & \circ & \circ & \cdots & \circ & \bullet c' & \cdots & \circ
 \end{array}$$

It is clear that  $t$  single-step moves of each of the  $t'$  beads between  $b'$  and  $c'$  (inclusive, starting by moving bead  $c'$ ) give an abacus representing an even partition. Hence  $w(\gamma) \leq tt'$ . It remains to show that  $tt'$  moves are necessary.

Let  $B$  be an abacus representing a partition in  $\mathcal{E}(\gamma)$  obtained from  $A$  by a sequence of single-step moves. No bead in a negative position other than  $-1$  is moved, by Proposition 3.4. If the bead labelled  $d$  in position  $-1$  is moved, then to avoid an odd gap, the beads  $b$  and  $b'$  must also be moved. Let  $r \in \mathbf{N}_0$  be the row containing bead  $d$  in  $B$ . Raising  $d$  back to position  $-1$  and raising all the remaining beads in rows  $0, 1, \dots, r$  in  $B$  up one row now gives an abacus representing an even partition of smaller  $p$ -weight. Hence the odd gaps between the beads on runners  $u$  and  $u'$  are removed by moving beads only on these runners. Therefore the abacus  $B$  has exactly  $t+t'$  beads in positive positions. Moreover, if two of these beads belong to the same row, then this row has the form

$$\begin{array}{cccccccccccc}
 0 & \cdots & u & u+1 & \cdots & u'-1 & u' & \cdots & p-1 \\
 \circ & \cdots & \bullet & \circ & \cdots & \circ & \bullet & \cdots & \circ
 \end{array}$$

in which the beads on runners  $u$  and  $u'$  form an odd gap, a contradiction. Hence no two beads in positive positions belong to the same row of  $B$ . Let  $r_1, \dots, r_{t+t'} \in \mathbf{N}_0$  be the  $t+t'$  distinct rows occupied by a bead in a positive position in  $B$ . We deduce that

$$\begin{aligned}
 w(\gamma) &= (r_1 + r_2 \cdots + r_{t+t'}) - (1 + \cdots + (t-1)) - (1 + \cdots + (t'-1)) \\
 &\geq (1 + \cdots + (t+t'-1)) - (1 + \cdots + (t-1)) - (1 + \cdots + (t'-1)) \\
 &= tt'
 \end{aligned}$$

hence we have  $w(\gamma) = tt'$ .

In case (ii) a similar argument applies. Since the bead in position  $-1$  is on runner  $p-1$ , it follows immediately from Proposition 3.4 that only beads on runner  $u$  and runner  $p-1$  are moved.  $\square$

4. EVEN PARTITIONS AND BLOCKS OF  $p$ -WEIGHT 2

Let  $\gamma$  be a  $p$ -core and let  $\nu \in B(\gamma, 2)$ . Thus  $\gamma$  is obtained from  $\nu$  by removing two rim  $p$ -hooks. In [18, page 397], Richards defines  $\delta(\nu)$  to be the absolute value of the difference of the leg-lengths of these rim  $p$ -hooks. (This difference is independent of the choice of rim  $p$ -hooks, when such a choice exists.) A partition  $\nu$  such that  $\delta(\nu) = 0$  either has two rim  $p$ -hooks, or a unique rim  $p$ -hook and a rim  $2p$ -hook. In the former case  $\nu$  is *black* if the larger leg-length is even, and *white* if it is odd. In the latter case  $\nu$  is *black* if the leg-length of the rim  $2p$ -hook is congruent to 0 or 3 modulo 4, and otherwise *white*. If  $\nu$  is  $p$ -regular, let  $\nu^\circ$  denote the largest partition such that  $\nu^\circ \trianglelefteq \nu$  and  $\delta(\nu^\circ) = \delta(\nu)$ , and if  $\delta(\nu) = 0$ , such that  $\nu$  and  $\nu^\circ$  have the same colour. (Thus  $\nu^\circ = \nu^{\circ'}$  in Richards' notation.)

The main result of [18] can be stated as follows.

**Theorem 4.1** (Theorem 4.4 in [18]). *Let  $\gamma$  be a  $p$ -core and let  $\nu \in B(\gamma, 2)$  be  $p$ -regular. Then  $d_{\nu\nu} = d_{\nu^\circ\nu} = 1$ . If  $\mu \in B(\gamma, 2)$  and  $\mu \notin \{\nu, \nu^\circ\}$  then  $d_{\mu\nu} = 1$  if  $\nu^\circ \triangleleft \mu \triangleleft \nu$  and  $\delta(\nu) - \delta(\mu) \in \{1, -1\}$ ; otherwise  $d_{\mu\nu} = 0$ .*

In the proof of Theorem 1.2(iii) we also need part (ii) of the following lemma. In the proof we use that the leg-length of a rim  $p$ -hook corresponding to a single-step upward move of a bead  $b$  in position  $\beta$  on an abacus is the number of beads in positions  $\beta - (p - 1), \dots, \beta - 1$ .

**Lemma 4.2.** *Let  $\gamma$  be an even  $p$ -core represented by an abacus  $A$  and let  $\nu \in B(\gamma, 2)$ .*

- (i) *If  $\nu$  is even then  $\nu$  is obtained either by single-step moves of two consecutive beads in  $A$ , or by two single-step moves of the final bead in  $A$ .*
- (ii) *If  $\nu$  is even then  $\delta(\nu) = 0$ .*

*Proof.* Suppose bead  $b$  in  $A$  is one of the beads moved to obtain an abacus representing  $\nu$ . If there is a bead, say  $c$ , after  $b$ , then a new odd gap is created between the bead before  $b$  in the abacus and bead  $c$ . Hence one of these beads must be moved. This proves (i). If the final bead is moved twice then the leg-length of the corresponding rim  $2p$ -hook is 0, so  $\delta(\nu) = 0$ . Otherwise consecutive beads  $b$  and  $b'$  are moved, where  $b$  and  $b'$  are in positions  $\beta$  and  $\beta'$  with  $\beta < \beta'$ . Let  $\mu$  be obtained from  $\nu$  by removing the rim  $p$ -hook corresponding to  $b'$ . By the observation before the lemma, the leg-lengths of the rim  $p$ -hooks in  $\nu$  and  $\mu$  corresponding to bead  $b'$  and  $b$ , respectively, are equal to the number of beads in  $A$  in positions  $\beta, \dots, \beta + (p - 1)$ . Hence  $\delta(\nu) = 0$ , as required for (ii).  $\square$

It is possible to sharpen Lemma 4.2(ii) so that the converse also holds. For this we need one more statistic. If  $\nu \in B(\gamma, 2)$  is obtained by single-step

moves of distinct beads  $b$  and  $b'$  on an abacus representing  $\gamma$ , let  $\Delta(\nu)$  be the number of beads strictly between  $b$  and  $b'$ . (This is clearly independent of the choice of abacus.) If  $\nu$  has a rim  $2p$ -hook then let  $\Delta(\nu) = \delta(\nu)$ . The proof of Lemma 4.2 shows that in all cases, if  $\Delta(\nu) = 0$  then  $\delta(\nu) = 0$ .

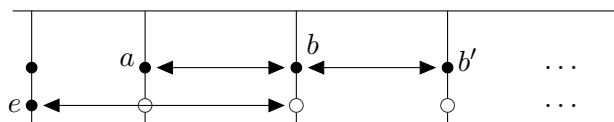
**Proposition 4.3.** *Let  $\gamma$  be an even  $p$ -core and let  $\nu \in B(\gamma, 2)$ . Then  $\nu$  is even if and only if  $\Delta(\nu) = 0$  and  $\nu$  is black.*

*Proof.* Let  $A$  be an abacus representing  $\gamma$ . Let  $\nu$  be a partition obtained from  $A$  by two single-step moves, represented by the abacus  $B$ . We consider two cases.

*Case 1: Distinct beads.* Suppose that distinct beads  $b$  and  $b'$  are moved to obtain  $B$ . Let  $b$  and  $b'$  be in positions  $\beta$  and  $\beta'$  of  $A$ , respectively, where  $\beta < \beta'$ . If  $\nu$  is even then, by Lemma 4.2(i),  $b$  and  $b'$  are consecutive and so  $\Delta(\nu) = 0$ . Conversely, if  $b$  and  $b'$  are consecutive then  $\Delta(\nu) = 0$ . It remains to show that, if  $b$  and  $b'$  are consecutive, then  $\nu$  is even if and only if  $\nu$  is black.

The rim  $p$ -hook in  $\nu$  with the longer leg-length corresponds to bead  $b'$  in  $B$ . Its leg-length,  $\ell$  say, is the number of beads in positions  $\beta' + 1, \dots, \beta + p$  of  $B$ . Let  $a$  be the bead in the greatest position before position  $\beta$  in  $A$  and let  $e$  be the bead in the greatest position before position  $\beta + p$  in  $B$ . Thus  $a$  and  $b$  are consecutive beads in  $A$ , as are  $b$  and  $e$ .

If bead  $e$  is bead  $a$  then there are no beads except for  $b$  and  $b'$  between bead  $a$  and position  $\beta + p$  in  $B$ . In this case  $\ell = 1$  and so  $\nu$  is white, and, because of the gap in  $B$  between beads  $a$  and  $b$ ,  $\nu$  is not even. In the remaining case bead  $e$  is in a position  $\varepsilon$  strictly between  $\beta'$  and  $\beta + p$ . By replacing the abaci  $A$  and  $B$  with appropriate rotations, we may assume that bead  $e$  is on runner 0, and so the relevant rows of  $A$  are as shown in the abacus below, where arrows indicate gaps affected by the moves of  $b$  and  $b'$ .



Observe that between positions  $\beta$  and  $\varepsilon$  (inclusive) in  $A$  there are  $\ell + 1$  beads. Since  $A$  represents an even partition, all gaps between these beads are even. Hence  $\ell$  and  $\varepsilon - \beta$  have the same parity and  $\ell$  and  $\varepsilon - \beta + p$  have opposite parity. Therefore the gap between beads  $e$  and  $b$  in  $B$  of length  $\beta + p - \varepsilon - 1$  is even if and only if  $\ell$  is even, so if and only if  $\nu$  is black. If there is no bead after position  $\beta + p$  in  $B$  then it is now clear that  $\nu$  is even if and only if  $\nu$  is black. Suppose that bead  $a'$  is the first bead after position  $\beta + p$  in  $B$ . Let  $a'$  be in position  $\alpha'$ . Since  $A$  has a gap between positions  $\varepsilon$

and  $\alpha'$ , we see that

$$\alpha' - \varepsilon = (\alpha' - (\beta' + p)) + (\beta' - \beta) + ((\beta + p) - \varepsilon)$$

is odd. Hence  $\alpha' - (\beta' + p)$  and  $(\beta + p) - \varepsilon$  have the same parity. Thus the gaps between beads  $e$  and  $b$  and beads  $b'$  and  $a'$  in  $B$  have the same parity. The former gap has length  $\beta + p - \varepsilon - 1$ ; since this has the same parity as  $\ell$ , it again follows that  $\nu$  is even if and only if  $\nu$  is black.

*Case 2: One bead.* Suppose that bead  $b$  is moved twice. If  $\nu$  is even then, by Lemma 4.2(i), bead  $b$  is the final bead in  $A$ , the leg-length of the rim  $2p$ -hook is 0 and  $\delta(\nu) = 0$ . Hence  $\Delta(\nu) = 0$  and  $\nu$  is black.

Conversely, suppose that  $\Delta(\nu) = 0$  and  $\nu$  is black. Let  $b$  be in position  $\beta$  of  $A$ . Since  $\delta(\nu) = 0$ , for each  $j$  such that  $1 \leq j \leq p - 1$ , either there are beads in both positions  $\beta + j$  and  $\beta + j + p$ , or spaces in both these positions. Hence the leg-length of the rim  $2p$ -hook corresponding to bead  $b$  is even, and so is congruent to 0 modulo 4. The number of beads in positions  $\beta + 1, \dots, \beta + (p - 1)$  of  $A$  is therefore even.

Suppose, for a contradiction, that there is a bead after position  $\beta$ . Let  $e$  be the first bead after position  $\beta$ , and suppose that  $e$  is in position  $\varepsilon$ . The positions  $\beta, \dots, \varepsilon + p$  in  $A$  begin and end with beads. So we have evenly many beads in  $A$  in

$$((\varepsilon + p) - \varepsilon) + (\varepsilon - \beta) + 1$$

positions. Since  $\varepsilon - \beta$  is odd, this number is odd. Therefore two of the beads in these positions form an odd gap, a contradiction. Hence  $b$  is the final bead in  $A$  and  $\nu$  is even.  $\square$

## 5. PROOF OF THEOREM 1.1

In this section we completely characterize the vertices of all the indecomposable summands of  $H^{(2^n)}$ . The following lemma is required.

**Lemma 5.1.** *Let  $n \in \mathbf{N}$ . There exist  $k, \ell \in \mathbf{N}_0$  such that  $\ell \leq (k+1)(p-1)/2$  and*

$$(\star) \quad 2n = (k+1)\left(2 + \frac{p-1}{2}k\right) + 2\ell.$$

*Proof.* For  $k \in \mathbf{N}_0$  define  $\theta_k = (k+1)\left(2 + \frac{p-1}{2}k\right)$ . Note that  $\theta_k$  is even. Choose  $k$  so that  $\theta_k \leq 2n < \theta_{k+1}$  and then define  $\ell$  so that  $(\star)$  holds. Since  $\theta_{k+1} - \theta_k = 2 + (k+1)(p-1)$ , we have  $2\ell \leq (k+1)(p-1)$ , as required.  $\square$

*Proof of Theorem 1.1.* By Proposition 2.1 it is sufficient to prove that every Foulkes module has a projective summand. In turn, by Proposition 2.2(i), it is sufficient to prove that for all  $n$  there is an even partition  $2\lambda$  of  $2n$  with  $p$ -core  $\gamma$  such that  $2n = |\gamma| + w(\gamma)p$ , or, equivalently, such that  $2\lambda \in \mathcal{E}(\gamma)$ .

We now construct such partitions. Let  $k$  and  $\ell$  be as in Lemma 5.1. Define

$$\lambda_{k,0} = (2 + k(p-1), 2 + (k-1)(p-1), \dots, 2 + (p-1), 2).$$

Note that  $\lambda_{k,0}$  is an even  $p$ -core partition of  $(k+1)(2 + (p-1)k/2)$ . The abacus  $A_0(\lambda_{k,0})$  has  $k+1$  beads in positive positions on runner 2 and no beads in any other positive position. Define

$$\lambda_{k,\ell} = \lambda_{k,0} + ((2(s+1))^r, (2s)^{k+1-r})$$

where  $s$  and  $r$  are the unique natural numbers such that  $0 \leq r < k+1$ ,  $0 \leq s \leq \frac{p-1}{2}$  and  $\ell = (k+1)s + r$ . Note that  $\lambda_{k,\ell}$  is an even partition of  $2n$ . Let  $\gamma$  be the  $p$ -core of  $\lambda_{k,\ell}$ . We consider three cases.

- (1) If  $r = 0$  and  $0 \leq s < \frac{p-1}{2}$  then  $\lambda_{k,\ell}$  is a  $p$ -core partition. The abacus  $A_0(\lambda_{k,\ell})$  has  $k+1$  beads in positive positions on runner  $2s+2$ . Therefore  $\lambda_{k,\ell} = \gamma \in \mathcal{E}(\gamma)$ .
- (2) If  $r = 0$  and  $s = \frac{p-1}{2}$  then  $A_0(\lambda_{k,\ell})$  has  $k+1$  beads in positive positions on runner 1, in rows  $1, \dots, k+1$ , and no beads in any other positive position. Now  $\lambda_{k,\ell}$  is obtained from its  $p$ -core  $\gamma$  by the sequence of bead moves specified in Proposition 3.5(ii), taking  $t = k+1$ ,  $t' = 0$  and  $u = 1$ . Therefore  $\lambda_{k,\ell} \in \mathcal{E}(\gamma)$ .
- (3) If  $r > 0$  then  $(k+1)s + r = \ell \leq (k+1)(p-1)/2$  implies that  $s < \frac{p-1}{2}$ .
  - (a) If  $2s+2 < p-1$  then  $A_0(\lambda_{k,\ell})$  has  $k+1-r$  beads in positive positions on runner  $2s+2$  in rows  $0, \dots, k-r$  and  $r$  beads in positive positions on runner  $2s+4$  in rows  $k+1-r, \dots, k$ . Now  $\lambda_{k,\ell}$  is obtained from its  $p$ -core  $\gamma$  by the sequence of bead moves specified in Proposition 3.5(i), taking  $t = k+1-r$ ,  $t' = r$ ,  $u = 2s+2$  and  $u' = 2s+4$ . Therefore  $\lambda_{k,\ell} \in \mathcal{E}(\gamma)$ .
  - (b) If  $2s+2 = p-1$  then  $A_0(\lambda_{k,\ell})$  has  $k+1-r$  beads on runner  $p-1$  in rows  $0, \dots, k-r$  and  $r$  beads on runner 1 in rows  $k+2-r, \dots, k+1$ . Now  $\lambda_{k,\ell}$  is obtained from its  $p$ -core  $\gamma$  by the sequence of bead moves specified in Proposition 3.5(ii), taking  $t = r$ ,  $t' = k+1-r$ ,  $u = p-1$  and  $u' = 1$ . Therefore  $\lambda_{k,\ell} \in \mathcal{E}(\gamma)$ .

This completes the proof of Theorem 1.1.  $\square$

**Remark 5.2.** Let  $a, n \in \mathbf{N}$ . Generalizing the Foulkes modules  $H^{(2^n)}$  already defined, let  $H^{(a^n)}$  denote the  $FS_{an}$ -module induced from the trivial representation of  $S_a \wr S_n$ . For  $t \in \mathbf{N}_0$  let  $P_t$  be a Sylow  $p$ -subgroup of  $S_a \wr S_{tp}$ . In [7] it is shown that if  $a < p$  and  $U$  is an indecomposable summand of  $H^{(a^n)}$ , then there exists  $t \in \mathbf{N}_0$  such that  $t \leq n/p$  and  $P_t$  is a vertex of  $U$ . It would be interesting to know if an analogue of Theorem 1.1 holds in this general setting. More precisely, is it true that for every  $t \in \mathbf{N}_0$  such that

$t \leq n/p$ , there is an indecomposable summand of  $H^{(a^n)}$  with vertex  $P_t$ ? The main obstacle in proving this for arbitrary  $a > 2$  is the lack of knowledge of the ordinary character of the Foulkes module  $H^{(a^n)}$ .

## 6. PROOF OF THEOREMS 1.2 AND 1.4

Throughout this section let  $\gamma$  be a  $p$ -core. By reduction modulo  $p$ , the composition factors of  $H^{(2^n)}$  are precisely the composition factors of the Specht modules  $S^{2\lambda}$  for  $\lambda$  a partition of  $n$ . A composition factor of  $S^{2\lambda}$  lies in the block  $B(\gamma, w)$  if and only if  $2\lambda \in B(\gamma, w)$ .

**Blocks of  $p$ -weight 0.** The unique module in the block  $B(\gamma, 0)$  is the simple projective Specht module  $S^\gamma$ . It is a composition factor of  $H^{(2^n)}$  if and only if  $\gamma$  is an even  $p$ -core. In this case, since  $S^\gamma$  is projective, it splits off as a direct summand. This proves Theorem 1.2(i).

**Blocks of  $p$ -weight 1.** Suppose that there is a summand in the block  $B(\gamma, 1)$ . Then an even partition can be obtained from  $\gamma$  by adding a single  $p$ -hook. Fix an abacus representing  $\gamma$ . Suppose the highest odd gap is between beads  $b$  and  $b'$  in positions  $\beta$  and  $\beta'$  where  $\beta < \beta'$ . The only single-step moves that can lead to an even partition are moves of  $b$  and  $b'$ . If  $b'$  has a space below it then moving  $b'$  gives an even partition only if  $b$  also has a space below it. A similar argument applies if  $b$  has a space below it. Therefore both  $b$  and  $b'$  have spaces below them and a single-step move of either bead gives an even partition. The two possible configurations are as shown in Figure 4 below.

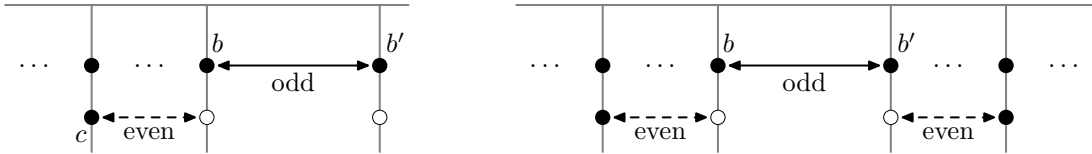


FIGURE 4. A  $p$ -core  $\gamma$  such that  $w(\gamma) = 1$  has an abacus of one of the forms shown above. In the left diagram, there is a single odd gap, bead  $b'$  is on runner  $p - 1$ , and the bead marked  $c$  is the final bead on the abacus. (It is possible that  $c = b'$ .) In the right diagram there are two odd gaps. In each diagram a single-step move of either  $b$  or  $b'$  gives an even partition.

Let  $2\lambda$  and  $2\mu$  be the even partitions obtained by moving  $b$  and  $b'$ , respectively. Then  $\mathcal{E}(\gamma) = \{2\lambda, 2\mu\}$ . It now follows from Propositions 2.2(ii) and 2.3 that  $P^{2\mu}$  is the unique summand of  $H^{(2^n)}$  in  $B(\gamma, 1)$ . This completes the proof of Theorem 1.2(iii).

**Remark 6.1.** By [19, §6, Example 1],  $B(\gamma, 1)$  is Morita equivalent to the principal block of  $FS_p$  by a Scopes functor. Since there are no beads between  $b$  and  $b'$ , the partitions  $2\lambda$  and  $2\mu$  are neighbours in the dominance order on partitions in  $B(\gamma, 1)$ . Since Scopes functors preserve the dominance order on partitions, it follows that  $P^{2\mu}$  has a Specht filtration with factors  $S^{2\mu}$  (at the top) and  $S^{2\lambda}$  (at the bottom). Thus  $P^{2\mu}$  has the Loewy structure

$$\begin{array}{c} D^{2\mu} \\ D^\nu \oplus D^{2\lambda} \\ D^{2\mu} \end{array}$$

where  $\nu$  is the smallest partition greater than  $2\mu$  in the dominance order on partitions in  $B(\gamma, 1)$ . (If  $2\mu$  is greatest then omit  $D^\nu$ .)

**Blocks of  $p$ -weight 2.** Suppose that there is a summand in the block  $B(\gamma, 2)$ . Then either  $w(\gamma) = 2$ , or  $w(\gamma) = 0$  and  $\gamma$  is an even  $p$ -core.

*Case  $w(\gamma) = 2$ .* Let  $A$  be an abacus for  $\gamma$ . Suppose that the consecutive beads  $b$  and  $b'$  form the highest odd gap in  $A$ . Since  $\gamma$  is a  $p$ -core, we can assume (by replacing  $A$  with an appropriate rotation) that  $b$  and  $b'$  lie in the same row  $r$ . Suppose, for a contradiction, that neither bead  $b$  nor  $b'$  is moved in a sequence of single-step bead moves leading to an abacus  $B$  representing a partition in  $\mathcal{E}(\gamma)$ . Then there exists a bead  $c$ , in an earlier position than bead  $b$ , that is moved into the gap between  $b$  and  $b'$ . Let  $c'$  be the greatest numbered bead before  $c$  and let  $c''$  be the least numbered bead after  $c$ . Thus  $c'$  and  $c$  are consecutive, as are  $c$  and  $c''$ . If  $c$  is not in row  $r - 1$ , then it is in row  $r - 2$  and is moved twice. This leaves an odd gap between beads  $c'$  and  $c''$ . Hence  $c$  lies in row  $r - 1$ . Let  $A'$  be the abacus obtained from  $A$  by a single-step move of bead  $c$ . The odd gap between  $c'$  and  $c''$  in  $A'$  cannot be removed by moving a bead  $d$  from row  $r - 2$ , since this creates a new odd gap between the two beads consecutive to  $d$  in  $A'$ . Hence either  $c'$  or  $c''$  is moved. Therefore  $B$  has two beads in the positions between  $b$  and  $b'$ , and so there is an odd gap in  $B$  involving  $b$  or  $b'$ .

We have shown that either bead  $b$  or bead  $b'$  is moved. Suppose that there is a bead  $d$  immediately below bead  $b'$ . Then there must be spaces in the positions immediately below bead  $b$  and bead  $d$ . The partitions in  $\mathcal{E}(\gamma)$  are obtained as follows:

- (i) two single-step moves of bead  $b$ ,
- (ii) single-step moves of beads  $b$  and  $d$ ,
- (iii) a single-step move of bead  $d$  followed by a single-step move of bead  $b'$ .

Hence  $|\mathcal{E}(\gamma)| = 3$  and the unique maximal partition in  $\mathcal{E}(\gamma)$  is the partition  $2\mu$  given by the moves in (iii). By Propositions 2.2(ii) and 2.3,  $P^{2\mu}$  is the unique summand of  $H^{(2^n)}$  in  $B(\gamma, 2)$ . A similar result holds if there is a

bead immediately below bead  $b$ ; in this case the unique maximal partition is given by two single-step moves of bead  $b'$ . (This case occurs in the block  $B((3, 1), 2)$  when  $p = 3$  in the example in §8 below.)

Now suppose there are spaces below beads  $b$  and  $b'$ . Let  $B$  be the abacus obtained from  $A$  by a single-step move of bead  $b'$ . Let  $C$  be the abacus obtained from  $B$  by swapping the bead  $b$  in  $B$  with the space in  $B$  in the position occupied by bead  $b'$  in  $A$ . Let  $\gamma^*$  be the  $p$ -core represented by  $C$ . The abaci  $B$  and  $C$  have the same pattern of odd and even gaps. Moreover, a single-step move of bead  $b$  in  $B$  does not give an even partition, since it restores the odd gap between beads  $b$  and  $b'$  present in  $A$ . Therefore  $w(\gamma^*) = 1$  and there is a bijection between the sequences of single-step moves on  $B$  and on  $C$  that give even partitions. From the  $p$ -weight one case we have  $|\mathcal{E}(\gamma^*)| = 2$ . Hence exactly two even partitions can be obtained by starting with a single-step move of bead  $b'$ , leaving bead  $b$  fixed. A similar argument deals with the case where bead  $b$  is moved. Therefore  $|\mathcal{E}(\gamma)| = 4$ , and by Propositions 2.2(ii) and 2.3, it follows that if  $2\mu$  is a maximal element of  $\mathcal{E}(\gamma)$  then  $P^{2\mu}$  is the unique summand of  $H^{(2^n)}$  in  $B(\gamma, 2)$ .

**Example 6.2.** We pause to give an example of the case where  $w(\gamma) = 2$  and there are spaces below beads  $b$  and  $b'$ . Let  $p = 5$  and let  $\gamma = (5, 1, 1, 1)$ . An abacus  $A$  representing  $\gamma$ , and the abacus  $C$  defined in the proof, are shown below left and below right.

$$\begin{array}{c}
 \begin{array}{ccccc}
 & 0 & 1 & 2 & 3 & 4 \\
 A & \begin{array}{c} b \bullet \\ \circ \\ \circ \end{array} & \begin{array}{c} \circ \\ \circ \\ \circ \end{array} & \begin{array}{c} \bullet b' \\ \circ \\ \circ \end{array} & \begin{array}{c} \bullet \\ \circ \\ \circ \end{array} & \begin{array}{c} \bullet \\ \bullet \\ \circ \end{array}
 \end{array}
 \qquad
 \begin{array}{ccccc}
 & 0 & 1 & 2 & 3 & 4 \\
 C & \begin{array}{c} \circ \\ \circ \\ \circ \end{array} & \begin{array}{c} \circ \\ \circ \\ \circ \end{array} & \begin{array}{c} \bullet b \\ \bullet b' \\ \circ \end{array} & \begin{array}{c} \bullet \\ \circ \\ \circ \end{array} & \begin{array}{c} \bullet \\ \bullet d \\ \circ \end{array}
 \end{array}
 \end{array}$$

Moving bead  $b'$  or bead  $d$  in  $C$  gives an even partition. The corresponding elements of  $\mathcal{E}(\gamma)$ , obtained by moving bead  $b$  in  $C$  back to position 0 are  $(8, 6, 2, 2)$  and  $(10, 4, 2, 2)$ , respectively. The other partitions in  $\mathcal{E}(\gamma)$  are found by moving  $b$  first; they are  $(6, 6, 2, 2, 2)$  and  $(10, 2, 2, 2, 2)$ . Thus the unique summand of  $H^{(2^9)}$  in  $B(\gamma, 2)$  is  $P^{(10, 4, 2, 2)}$ .

*Case  $w(\gamma) = 0$ .* Since  $H^{(2^{n-p})}$  has  $S^\gamma$  as its unique summand in the block  $B(\gamma, 0)$ , it follows from Proposition 2.1 that there is a unique summand of  $H^{(2^n)}$  in  $B(\gamma, 2)$  with vertex  $Q_1$ . Since any summand in  $B(\gamma, 2)$  has a vertex contained in the  $p$ -weight 2 defect group  $\langle (1, 2, \dots, p), (p+1, \dots, 2p) \rangle$  (see [14, Theorem 6.2.45]), any other summand in this block must be projective.

Suppose that  $P^\nu$  is such a projective summand. By Proposition 2.2(iii),  $\nu$  is an even partition. By Lemma 4.2(ii),  $\delta(\nu) = 0$ . By Theorem 4.1 and Proposition 2.3, the column of the decomposition matrix labelled by  $\nu$  has a non-zero entry in a row labelled by partition  $\mu$  with  $\delta(\mu) = 1$ . By another application of Lemma 4.2(ii), this partition  $\mu$  is not even, contradicting Proposition 2.2(iii). This completes the proof.  $\square$





If  $2 < 2k < p - 1$  then

$$\begin{aligned} \langle \mathbf{2k} \rangle \triangleright \langle p - 1 \rangle \triangleright \langle 2k, p - 2 \rangle \triangleright \langle \mathbf{p - 2}, \mathbf{p - 1} \rangle \triangleright \cdots \\ \cdots \triangleright \langle \mathbf{2t + 1}, \mathbf{2t + 2} \rangle \triangleright \langle 2t, 2t + 2 \rangle \triangleright \langle 2t - 1, 2t + 1 \rangle \triangleright \langle \mathbf{2t - 1}, \mathbf{2t} \rangle \triangleright \cdots \\ \cdots \triangleright \langle \mathbf{2k + 1}, \mathbf{2k + 2} \rangle \triangleright \langle 2k - 1, 2k + 2 \rangle \triangleright \langle 2k - 2, 2k + 1 \rangle \triangleright \langle \mathbf{2k - 2}, \mathbf{2k - 1} \rangle \triangleright \cdots \\ \cdots \triangleright \langle \mathbf{2s}, \mathbf{2s + 1} \rangle \triangleright \langle 2s - 1, 2s + 1 \rangle \triangleright \langle 2s - 2, 2s \rangle \triangleright \langle \mathbf{2s - 2}, \mathbf{2s - 1} \rangle \triangleright \cdots \\ \cdots \triangleright \langle \mathbf{2}, \mathbf{3} \rangle \triangleright \langle 1, 3 \rangle \triangleright \langle 0, 2 \rangle \triangleright \langle \mathbf{0}, \mathbf{1} \rangle \triangleright \langle 1, 1 \rangle; \end{aligned}$$

if  $2k = 0$  then

$$\begin{aligned} \langle \mathbf{p - 1} \rangle \triangleright \langle p - 2 \rangle \triangleright \langle p - 3, p - 1 \rangle \triangleright \langle \mathbf{p - 3}, \mathbf{p - 2} \rangle \triangleright \langle p - 4, p - 2 \rangle \triangleright \cdots \\ \cdots \triangleright \langle 2, 4 \rangle \triangleright \langle \mathbf{2}, \mathbf{3} \rangle \triangleright \langle 1, 3 \rangle \triangleright \langle 0, 2 \rangle \triangleright \langle \mathbf{0}, \mathbf{1} \rangle \triangleright \langle 1, 1 \rangle; \end{aligned}$$

if  $2k = 2$  then

$$\begin{aligned} \langle \mathbf{2} \rangle \triangleright \langle p - 1 \rangle \triangleright \langle 2, p - 2 \rangle \triangleright \langle \mathbf{p - 2}, \mathbf{p - 1} \rangle \triangleright \langle p - 3, p - 1 \rangle \triangleright \cdots \\ \cdots \triangleright \langle \mathbf{3}, \mathbf{5} \rangle \triangleright \langle \mathbf{3}, \mathbf{4} \rangle \triangleright \langle 1, 4 \rangle \triangleright \langle 0, 3 \rangle \triangleright \langle \mathbf{0}, \mathbf{1} \rangle \triangleright \langle 1, 1 \rangle; \end{aligned}$$

and if  $2k = p - 1$  then

$$\begin{aligned} \langle \mathbf{p - 1} \rangle \triangleright \langle p - 2, p - 1 \rangle \triangleright \langle p - 3 \rangle \triangleright \langle \mathbf{p - 3}, \mathbf{p - 2} \rangle \triangleright \langle p - 4, p - 2 \rangle \triangleright \cdots \\ \cdots \triangleright \langle 2, 4 \rangle \triangleright \langle \mathbf{2}, \mathbf{3} \rangle \triangleright \langle 1, 3 \rangle \triangleright \langle 0, 2 \rangle \triangleright \langle \mathbf{0}, \mathbf{1} \rangle \triangleright \langle 1, 1 \rangle. \end{aligned}$$

The even partitions are shown in bold type.

*Proof.* It is routine to check that the partitions  $\nu$  such that  $\delta(\nu) = 1$  are as claimed. Lemma 4.4 in [18], which states that  $\langle u, v \rangle \trianglelefteq \langle u', v' \rangle$  if and only if  $u \leq u'$  and  $v \leq v'$ , then gives the total order of the chains, except for the cases involving partitions of the form  $\langle u \rangle$ , which have to be checked separately.  $\square$

**Proposition 7.2.** *Let  $\mu \in B((2k), 2)$  be even. Suppose that, in the relevant chain of partitions in Lemma 7.1, we have adjacent partitions  $\zeta \triangleright \rho \triangleright \nu \triangleright \mu \triangleright \eta$  where  $\zeta$  is even. Then*

- (i)  $\mu$  is even and  $\zeta^\circ = \mu$  and  $\nu^\circ = \eta$ ;
- (ii) if  $\mu \neq \langle 0, 1 \rangle$  then the composition factors of  $S^\mu$  are  $D^\zeta$ ,  $D^\nu$  and  $D^\mu$ ;
- (iii) the composition factors of  $S^{\langle 0, 1 \rangle}$  are  $D^{\langle 2, 3 \rangle}$  and  $D^{\langle 0, 2 \rangle}$ ;
- (iv)  $\text{Ext}^1(D^\zeta, D^\nu) = \text{Ext}^1(D^\nu, D^\mu) = F$ ;
- (v) if  $\lambda \in B((2k), 2)$  and  $\text{Ext}^1(D^\mu, D^\lambda) \neq 0$  then  $\delta(\lambda) = 1$ .

*Proof.* Part (i) follows from the definition of the map  $\circ$  and inspection of the chains in Lemma 7.1. Then (ii) and (iii) are easy deductions from Lemma 7.1 and Theorem 4.1. The Ext quivers of  $p$ -weight two principal blocks of symmetric groups were found by Martin in [16]: parts (iv) and (v)

can be read off from Figures 9 and 10 in the appendix. Alternatively these parts follow from [4, Theorem 6.1].  $\square$

We are now ready to prove Theorem 1.3.

*Proof of Theorem 1.3.* By Theorem 1.2(iii), there is a unique indecomposable summand of  $H^{(2^{p+k})}$  in the block  $B((2k), 2)$  of  $S_{2(p+k)}$ . This summand,  $U$  say, is the Scott module of vertex  $Q_1$ . By [3, (2.1)],  $U$  is self-dual and has the trivial module in its socle. By Proposition 7.2(ii) and (iii), each  $D^\nu$  labelled by a partition  $\nu$  such that  $\delta(\nu) = 1$  appears exactly once in  $U$ . Hence if  $2 < 2k < p - 1$  then the heart of  $U$  is

$$D^{(2k, p-2)} \oplus D^{(p-4, p-2)} \oplus \dots \\ \dots \oplus D^{(2k+1, 2k+3)} \oplus D^{(2k-2, 2k+1)} \oplus D^{(2k-4, 2k-2)} \oplus \dots \oplus D^{(0, 2)}.$$

If  $2k \in \{0, 2, p - 1\}$  then an analogous result holds with minor changes to the labels, as indicated in Figure 5 below. Similarly, each simple module  $D^{2\lambda}$  labelled by a  $p$ -regular even partition  $2\lambda$  appears exactly twice in  $U$ . By Proposition 7.2(iv), the only extensions in  $U$  are between modules in the heart of  $U$  and these  $D^{2\lambda}$ . It is easily seen, either from the chains in Lemma 7.1, or from Lemma 4.3 in [18] and our Lemma 4.2, that the unique even partition in  $B((2k), 2)$  that is not  $p$ -regular is  $(2k, 2^p)$  (interpreted as  $(2^p)$  if  $k = 0$ ). Hence  $U$  has three Loewy layers and

$$\text{soc } U \cong \text{top } U \cong \bigoplus_{\substack{2\lambda \in B((2k), 2) \\ 2\lambda \neq (2k, 2^p)}} D^{2\lambda}.$$

If  $2 < 2k < p - 1$  then the structure of  $U$  is as shown in Figure 1. The exceptional cases are shown in Figure 5 below.  $\square$

**Remark 7.3.** In [17], Paget proved that  $H^{(2^n)}$  has a filtration  $0 = V_0 \subset V_1 \subset \dots \subset V_d = H^{(2^n)}$  such that  $V_i/V_{i-1} \cong S^{2\lambda^{(i)}}$ , where  $\lambda(1) < \dots < \lambda(d)$  are the partitions of  $n$ . By Proposition 13 in [21], which was proved using the Hemmer–Nakano homological characterization of modules with a Specht filtration (see [10]), it follows that the summand  $U$  has a Specht filtration, provided  $p \geq 5$ . This filtration can be seen in Figures 1 and 5: for  $2k > 2$ , the bottom Specht factor is  $S^{(0,1)}$  (with composition factors  $D^{(0,2)}$  and  $D^{(2,3)}$ ) the next is  $S^{(2,3)}$  (with composition factors  $D^{(2,3)}$ ,  $D^{(2,4)}$ ,  $D^{(4,5)}$ ), and on on, ending with the trivial module  $D^{(2k)}$  at the top.

It would be interesting to know other cases where a single indecomposable module exhibits a significant proportion of the extensions between two classes of simple modules for the symmetric group.

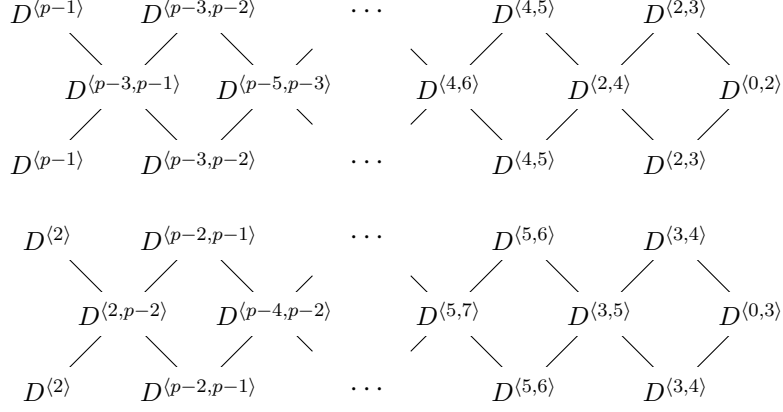


FIGURE 5. Loewy layers of the Scott module summand  $U$  of  $H^{(2n)}$  in the block  $B((2k), 2)$  in the cases  $2k = 0$  (top) and  $2k = 2$  (bottom). If  $2k = p - 1$  then the diagram for  $2k = 0$  applies, replacing  $\langle p - 3, p - 1 \rangle$  with  $\langle p - 3 \rangle$ .

## 8. EXAMPLE: $H^{(2^7)}$ IN CHARACTERISTIC 5

Let  $p = 5$ . The even partitions of 14 lie in the 5-blocks  $B((4), 2)$ ,  $B((2, 2), 2)$ ,  $B((3, 1), 2)$ ,  $B((1^4), 2)$ ,  $B((5, 2, 2), 1)$  and  $B((4, 4, 2, 2, 2), 0)$  of  $S_{14}$ . The principal block summand,  $U$  say, is dealt with by Theorem 1.3. Since  $w((3, 1)) = w((1^4)) = 2$ ,  $w((5, 2, 2)) = 1$  and  $w((4, 4, 2, 2, 2)) = 0$ , it follows from Theorem 1.2 that the summands in these blocks are the projective modules  $P^{(10,4)}$ ,  $P^{(8,2,2,2)}$ ,  $P^{(10,2,2)}$  and  $S^{(4,4,2,2,2)}$ , respectively. By Theorem 1.2(iii), there is a unique summand,  $V$  say, in  $B((2, 2), 2)$ . This summand has vertex  $Q_1$ . By Remark 7.3,  $V$  has a Specht filtration by the Specht factors in this block, namely  $S^{(12,2)}$ ,  $S^{(6,4,4)}$  and  $S^{(2^7)}$ . Using Theorem 4.1 to get the required decomposition numbers, one finds that

$$\begin{aligned}
& U \oplus V \oplus P^{(10,4)} \oplus P^{(8,2,2,2)} \\
&= \begin{pmatrix} S^{(14)} \\ S^{(4,4,4,2)} \\ S^{(4,2^5)} \end{pmatrix} \oplus \begin{pmatrix} S^{(12,2)} \\ S^{(6,4,4)} \\ S^{(2^7)} \end{pmatrix} \oplus \begin{pmatrix} S^{(10,4)} \\ S^{(8,6)} \\ S^{(8,4,2)} \end{pmatrix} \oplus \begin{pmatrix} S^{(8,2,2,2)} \\ S^{(6,4,2,2)} \\ S^{(6,2^4)} \end{pmatrix} \\
&= \begin{pmatrix} D^{(14)} \oplus D^{(4,4,4,2)} \\ D^{(7,5,1,1)} \oplus D^{(4,3,2,2,2,1)} \\ D^{(14)} \oplus D^{(4,4,4,2)} \end{pmatrix} \oplus \begin{pmatrix} D^{(12,2)} \oplus D^{(6,4,4)} \\ D^{(7,4,3)} \oplus D^{(3,3,3,2,1,1,1)} \\ D^{(12,2)} \oplus D^{(6,4,4)} \end{pmatrix} \\
&\quad \oplus \begin{pmatrix} D^{(10,4)} \\ D^{(13,1)} \oplus D^{(8,6)} \\ D^{(10,4)} \oplus D^{(8,4,2)} \\ D^{(13,1)} \oplus D^{(8,6)} \\ D^{(10,4)} \end{pmatrix} \oplus \begin{pmatrix} D^{(8,2,2,2)} \\ D^{(9,2,2,1)} \oplus D^{(6,4,2,2)} \\ D^{(6,5,2,1)} \oplus D^{(8,2,2,2)} \oplus D^{(6,2,2,2,2)} \\ D^{(9,2,2,1)} \oplus D^{(6,4,2,2)} \\ D^{(8,2,2,2)} \end{pmatrix}.
\end{aligned}$$

The Loewy layers of  $P^{(10,2,2)}$  are given by Remark 6.1, while the other summand,  $S^{(4,4,2,2,2)}$  is simple. By Proposition 6.5 in [8], all these summands have abelian endomorphism rings. We note that  $P^{(10,4)}$  is the projective summand used in the proof of Theorem 1.1.

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