EXERCISES FOR 'THE COUNTER-INTUITIVE BEHAVIOUR OF HIGH DIMENSIONAL SPACES'

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These exercises fill in some details in my blog post .

1. Euclidean spaces

Recall that S^n is the n-dimensional sphere defined by

$$S^n = \{(x_1, \ldots, x_n, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + \cdots + x_n + x_{n+1}^2 = 1\}$$

and $B^n \subseteq \mathbb{R}^n$ is the n-dimensional unit ball. Let A denote volume, so $A(S^n)$ is the surface area of S^n, and $A(B^n)$ is the volume of B^n.

1. Let $T \subseteq \mathbb{R}^n$ and let $f : T \to \mathbb{R}$ be a smooth function. Suppose that a patch U of an n-manifold $M \subseteq \mathbb{R}^{n+1}$ has a chart $\phi : T \to U$ of the form

$$\phi(x_1, \ldots, x_n) = \big(x_1, \ldots, x_n, f(x_1, \ldots, x_n)\big).$$

For $\mathbf{x} \in T$, let $u(\mathbf{x}) \in \mathbb{R}^{n+1}$ be a unit normal to U at the point $\phi(\mathbf{x})$.

(a) Show that the volume of $(n+1)$-dimensional parallelepiped with sides $\frac{\partial \phi}{\partial x_1}(\mathbf{x}), \ldots, \frac{\partial \phi}{\partial x_n}(\mathbf{x}), u(\mathbf{x})$ is $\sqrt{1 + ||\nabla f||^2}$.

(b) Deduce that the surface area of U is

$$A(U) = \int_T \sqrt{1 + ||\nabla f(\mathbf{x})||^2}\, \mathrm{d}\mathbf{x}.$$

[*Hint:* use that u is a unit normal vector to interpret the volume of the parallelepiped in (a) as an area.]

(c) By taking $T = B^n$ and $f(\mathbf{x}) = \sqrt{1 - x_1^2 - \cdots - x_n^2}$ deduce that the surface area of S^n is

$$\int_{B^n} \frac{1}{\sqrt{1 - x_1^2 - \cdots - x_n^2}}\, \mathrm{d}x_1 \ldots \mathrm{d}x_n.$$

(d) Hence show that

$$A(S^n) = \int_{-1}^{1} \frac{A\big(S^n(z)\big)}{\sqrt{1 - z^2}} \mathrm{d}z$$

where $S^n(z)$ is the subset of S^n of all points at height (i.e. the final coordinate) z. [*Hint:* evaluate the integral in (c) taking $x_n = z$; I'm sorry that the notation makes this an unobvious choice.]

(e) Deduce that, as claimed in the blog post, the density under the uniform measure on S^n as z varies is proportional to $(\sqrt{1 - z^2})^{n-2}$.

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Remark. The formula in (b) is sometimes taken as the *definition* of the surface area of a patch of a manifold, but since it only applies when the chart ϕ has a rather special form, I prefer to start one step before. Readers preferring to go all the way back to the general theory of integration of differential forms on manifolds clearly do not need my assistance.

2. Discrete spaces

2. Suppose that $0 < p < \frac{1}{2}$ and that $pn \in \mathbb{N}$. Show that

$$1 \geq \sum_{s=0}^{pn} \binom{n}{s} p^s (1-p)^{n-s} \geq p^{pn}(1-p)^{(1-p)n} \sum_{s=0}^{pn} \binom{n}{s}$$

and deduce that

$$\sum_{s=0}^{pn} \binom{n}{s} \leq 2^{hn}$$

where $h = -p \log_2 p - (1-p)\log_2(1-p)$ is the entropy of a flip of a single coin, biased to land heads with probability p.

3. Solutions

1. (a) For $\mathbf{x} \in B^n$, define an $(n+1) \times (n+1)$-matrix $U(\mathbf{x})$ by

$$U(\mathbf{x}) = \begin{pmatrix} 1 & 0 & \dots & 0 & \frac{\partial f}{\partial x_1}(\mathbf{x}) \\ 0 & 1 & \dots & 0 & \frac{\partial f}{\partial x_2}(\mathbf{x}) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & \frac{\partial f}{\partial x_n}(\mathbf{x}) \\ -\frac{\partial f}{\partial x_1}(\mathbf{x}) & -\frac{\partial f}{\partial x_2}(\mathbf{x}) & \dots & -\frac{\partial f}{\partial x_n}(\mathbf{x}) & 1 \end{pmatrix}.$$

Note that the final row is perpendicular to the first n rows and so is a normal to U at $\mathbf{x}$. Let $\ell(\mathbf{x})$ be the length of this normal vector. The volume of the required parallelepiped is then $|\det U(\mathbf{x})|/\ell(\mathbf{x})$.

Suppose we expand the determinant by picking the entry in row $n+1$ and column i. If $i = n+1$ we get a contribution of 1; otherwise we must also choose the entry in row i and column $n+1$, getting a contribution of $(\frac{\partial f}{\partial x_i})^2$. (Note the sign from the matrix cancels with the sign from the transposition.) Hence

$$\det U(\mathbf{x}) = 1 + ||\nabla f||^2.$$

Since $\ell(\mathbf{x}) = \sqrt{1 + ||\nabla f||^2}$ we deduce that the volume of the parallelepiped is $\sqrt{1 + ||\nabla f||^2}$, as required.

(b) Since $u(\mathbf{x})$ is a unit vector, the surface area element of U at $\mathbf{x}$ is $|\det U(\mathbf{x})| \mathrm{d}x_1 \dots \mathrm{d}x_n$. The result now follows from (a).

(c) The half of the sphere S^n with $x_{n+1} > 0$ is parametrised by the chart with domain B^n defined by

$$\phi(x_1, \dots, x_n) = (x_1, \dots, x_n, \sqrt{1 - x_1^2 - \dots - x_n^2}).$$

This chart has the form in (a) with $f(\mathbf{x}) = \sqrt{1 - ||\mathbf{x}||^2}$. We have

$$\nabla f(\mathbf{x}) = -\frac{\mathbf{x}}{\sqrt{1 - ||\mathbf{x}||^2}}$$

and so

$$1 + ||\nabla f||^2 = 1 + \frac{||\mathbf{x}||^2}{1 - ||\mathbf{x}||} = \frac{1}{1 - ||\mathbf{x}||^2}.$$

Applying (b), we get that the surface area of the half sphere is

$$\int_{B^n} \frac{1}{\sqrt{1 - ||\mathbf{x}||^2}} \mathrm{d}\mathbf{x}.$$

(d) By (c), the surface area $A(S^n)$ of the unit n-sphere is

$$\begin{aligned}
A(S^n) &= \int_{-1}^{1} \Big(\int_{(1-z^2)B_{n-1}} \frac{1}{\sqrt{1 - x_1^2 - \dots - x_{n-1}^2 - z^2}} \mathrm{d}x_1 \dots \mathrm{d}x_{n-1} \Big) \mathrm{d}z \\
&= \int_{-1}^{1} \Big(\int_{B_{n-1}} \frac{\big(\sqrt{1-z^2}\big)^{n-1}}{\sqrt{1 - z^2 - (1-z^2)y_1^2 - \dots - (1-z^2)y_{n-1}^2}} \mathrm{d}y_1 \dots \mathrm{d}y_{n-1} \Big) dz \\
&= \int_{-1}^{1} \frac{1}{\sqrt{1-z^2}} \int_{B_{n-1}} \frac{\big(\sqrt{1-z^2}\big)^{n-1}}{1 - y_1^2 - \dots - y_{n-1}^2} \mathrm{d}y_1 \dots \mathrm{d}y_{n-1} \\
&= \int_{-1}^{1} \frac{1}{\sqrt{1-z^2}} \big(\sqrt{1-z^2}\big)^{n-1} A(S^{n-1}) \\
&= \int_{-1}^{1} \frac{1}{\sqrt{1-z^2}} A\Big(\big(\sqrt{1-z^2}\big)^{n-1} S^{n-1} \Big)
\end{aligned}$$

where to go from line 1 to line 2 we make the change of variables $x_i = \sqrt{1-z^2}\, y_i$, so $\mathrm{d}x_1 \dots \mathrm{d}x_n$ becomes $(\sqrt{1-z^2})^{n-1} \mathrm{d}y_1 \dots \mathrm{d}y_n$, and the penultimate line follows by induction. Since the part of the n-sphere with final coordinate z is $(1-z^2)S^{n-1}$, the result follows.

(e) This follows immediately from the penultimate line in the displayed equation in the answer to (d).

2. Let $X \sim \mathrm{Bin}(n, p)$ be the distribution of n flips of a coin biased to land heads with probability p. Then

$$1 \geq \mathbf{P}[X \leq pn] = \sum_{s=0}^{pn} \binom{n}{s} p^s (1-p)^{n-s}$$

giving the first inequality. Since $p < \frac{1}{2}$,

$$p^s (1-p)^{n-s} = \big(\frac{p}{1-p}\big)^s (1-p)^n$$

decreases with s, and so the right-hand side in the first displayed equation is at least $\binom{n}{pn}p^{pn}(1-p)^{(1-p)n}$. Taking logs we get

$$0 \geq \log_2 \binom{n}{pn} + pn \log_2 p + (1-p)n \log_2(1-p)$$

which rearranges to $\log_2 npn \leq 2^{hn}$ where $h = -p\log_2 p - (1-p)\log_2(1-p)$, as required.