How to Read Proofs: The 'Self-Explanation' Strategy

The "self-explanation" strategy has been found to enhance problem solving and comprehension in learners across a wide variety of academic subjects. It can help you to better understand mathematical proofs: in one recent research study students who had worked through these materials before reading a proof scored 30% higher than a control group on a subsequent proof comprehension test.

How to Self-Explain

To improve your understanding of a proof, there is a series of techniques you should apply.

After reading each line:

- Try to identify and elaborate the main ideas in the proof.
- Attempt to explain each line in terms of previous ideas. These may be ideas from the information in the proof, ideas from previous theorems/proofs, or ideas from your own prior knowledge of the topic area.
- Consider any questions that arise if new information contradicts your current understanding.

Before proceeding to the next line of the proof you should ask yourself the following:

- Do I understand the ideas used in that line?
- Do I understand why those ideas have been used?
- How do those ideas link to other ideas in the proof, other theorems, or prior knowledge that I may have?
- Does the self-explanation I have generated help to answer the questions that I am asking?

On the next page you will find an example showing possible self-explanations generated by students when trying to understand a proof (the labels "(L1)" etc. in the proof indicate line numbers). Please read the example carefully in order to understand how to use this strategy in your own learning.

Example Self-Explanations

Theorem

No odd integer can be expressed as the sum of three even integers.

Proof

(L1) Assume, to the contrary, that there is an odd integer x, such that $x = a+b+c$, where a, b, and c are even integers.

(L2) Then $a = 2k, b = 2l$, and $c = 2p$, for some integers k, l, and p.

(L3) Thus $x = a + b + c = 2k + 2l + 2p = 2(k + l + p)$.

(L4) It follows that x is even; a contradiction.

(L5) Thus no odd integer can be expressed as the sum of three integers. □

After reading this proof, one reader made the following self-explanations:

- "This proof uses the technique of proof by contradiction."
- "Since a, b and c are even integers, we have to use the definition of an even integer, which is used in L2."
- "The proof then replaces a, b and c with their respective definitions in the formula for x."
- "The formula for x is then simplified and is shown to satisfy the definition of an even integer also; a contradiction."
- "Therefore, no odd integer can be expressed as the sum of three even integers."

Self-Explanation Compared with Other Comments

You must also be aware that the self-explanation strategy is not the same as *monitoring* or *paraphrasing*. These two methods will not help your learning to the same extent as self-explanation.

Paraphrasing

"a, b and c have to be positive or negative, even whole numbers."

There is no self-explanation in this statement. No additional information is added or linked. The reader merely uses different words to describe what is already represented in the text by the words "even integers". You should avoid using such paraphrasing during your own proof comprehension. Paraphrasing will not improve your understanding of the text as much as self-explanation will.

Monitoring

"OK, I understand that $2(k + l + p)$ is an even integer."

This statement simply shows the reader's thought process. It is not the same as self-explanation, because the student does note relate the sentence to additional information in the text or to prior knowledge. Please concentrate on self-explanation rather than monitoring.

A possible self-explanation of the same sentence would be:

"OK, $2(k + l + p)$ is an even integer because the sum of 3 integers is an integer and 2 times an integer is an even integer."

In this example the reader identifies and elaborates the main ideas in the text. They use information that has already been presented to understand the logic of the proof

This is the approach you should take after reading every line of a proof in order to improve your understanding of the material.

Practice Proof 1

Now read this short theorem and proof and self-explain each line, either in your head or by making notes on a piece of paper, using the advice from the preceding pages.

Theorem

There is no smallest positive real number.

Proof

Assume, to the contrary, that there exists a smallest positive real number.

Therefore, by assumption, there exists a real number r such that for every positive number s, $0 < r < s$.

Consider $m = \frac{r}{2}$.

Clearly, $0 < m < r$.

This is a contradiction because m is a positive real number that is smaller than r.

Thus there is no smallest positive real number. □

Practice Proof 2

Here's another more complicated proof for practice. This time, a definition is provided too. Remember: use the self-explanation training after *every* line you read, either in your head or by writing on paper.

Definition

An *abundant* number is a positive integer n whose divisors add up to more than $2n$.

For example, 12 is abundant because $1+2+3+4+6+12 > 24$.

Theorem

The product of two distinct primes is not abundant.

Proof

Let $n = p_1p_2$, where p_1 and p_2 are distinct primes.

Assume that $2 \leq p_1$ and $3 \leq p_2$.

The divisors of n are $1, p_1, p_2$ and p_1p_2.

Note that $\frac{p_1+1}{p_1-1}$ is a decreasing function of p_1.

So $\max\left\{\frac{p_1+1}{p_1-1}\right\} = \frac{2+1}{2-1} = 3$.

Hence $\frac{p_1+1}{p_1-1} \leq p_2$.

So $p_1 + 1 \leq p_1p_2 - p_2$.

So $p_1 + 1 + p_2 \leq p_1p_2$.

So $1 + p_1 + p_2 + p_1p_2 \leq 2p_1p_2$. $\square$

Remember. . .

Using the self-explanation strategy had been shown to substantially improve students' comprehension of mathematical proofs. Try to use it every time you read a proof in lectures, in your notes or in a book.