A COROLLARY OF STANLEY'S HOOK CONTENT FORMULA

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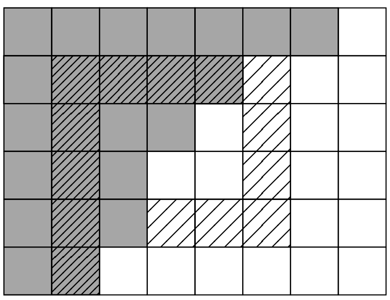
Abstract. We use the hook lengths of a partition to define two rectangular tableaux. We prove these tableaux have equal entries, first by elementary combinatorial arguments, and then using Stanley's Hook Content Formula and symmetric polynomials.

1. Introduction

This paper presents two proofs of an appealing corollary of Stanley's Hook Content Formula [3, Theorem 7.21.2] for the number of semistandard Young tableaux: the first is self-contained and entirely elementary, while the second uses Stanley's result and symmetric polynomials. The author hopes the paper will be useful as an introduction to this interesting circle of ideas.

The following definitions are standard. A *partition* of $n \in \mathbf{N}_0$ is a sequence $(\lambda_1, \ldots, \lambda_k)$ of natural numbers such that $\lambda_1 \geq \ldots \geq \lambda_k$ and $\lambda_1 + \cdots + \lambda_k = n$. The *size* of λ is n. We define $\ell(\lambda) = k$ and $a(\lambda) = \lambda_1$, setting $\ell(\varnothing) = a(\varnothing) = 0$. The *Young diagram* of λ, denoted $[\lambda]$, is the set of *boxes* $\{(i,j) : 1 \leq i \leq \ell(\lambda), 1 \leq j \leq \lambda_i\}$.

We fix throughout r, $c \in \mathbf{N}$. Let $D = \{(i,j) : 1 \leq i \leq r, 1 \leq j \leq c\}$. We orient D by compass directions, thus $(r,1)$ is the box in its south-west corner and $(1,c)$ is the box in its north-east corner. As a running example, the Young diagram of $(7,5,4,3,3,2)$, shown as a subset of D when $r = 6$ and $c = 8$, is below.



The hatched squares show the hooks on $(2,2)$ and $(5,6)$, as defined formally in the definition below.

Definition. Let λ be a partition with $\ell(\lambda) \leq r$ and $a(\lambda) \leq c$. Let $(i,j) \in D$.

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(i) The *hook* on (i,j), denoted $H_{(i,j)}(\lambda)$, is

$$\{(i,j)\} \cup \big\{(i',j) \in [\lambda] : i' > i\big\} \cup \big\{(i,j') \in [\lambda] : j' > j\big\}$$

if $(i,j) \in [\lambda]$ and

$$\{(i,j)\} \cup \big\{(i',j) \in D\backslash[\lambda] : i' < i\big\} \cup \big\{(i,j') \in D\backslash[\lambda] : j' < j\big\}$$

if $(i,j) \in D\backslash[\lambda]$. We define the *hook length* of (i,j), denoted $h_{(i,j)}(\lambda)$, to be $|H_{(i,j)}(\lambda)|$.

(ii) The *distance* of (i,j), denoted $d_{(i,j)}(\lambda)$, is the number of boxes in any walk by steps south and west to $(r,1)$ if $(i,j) \in [\lambda]$, or of any walk by steps north and east to $(1,c)$ if $(i,j) \in D\backslash[\lambda]$.

Our result concerns two ways to fill the boxes of D with natural numbers. Formally, these are specified by two functions from D to $\mathbf{N}$, assigning to each box of D a corresponding *entry* in $\mathbf{N}$.

Definition. Let λ be a partition with $\ell(\lambda) \leq r$ and $a(\lambda) \leq c$. Let $(i,j) \in D$. The *hook/distance tableau for* λ has entry in box (i,j)

$$\begin{cases} h_{(i,j)}(\lambda) & \text{if } (i,j) \in [\lambda] \\ d_{(i,j)}(\lambda) & \text{if } (i,j) \in D\backslash[\lambda]. \end{cases}$$

The *distance/hook tableau for* λ has entry in box (i,j)

$$\begin{cases} d_{(i,j)}(\lambda) & \text{if } (i,j) \in [\lambda] \\ h_{(i,j)}(\lambda) & \text{if } (i,j) \in D\backslash[\lambda]. \end{cases}$$

Theorem 1. *For any partition λ with $\ell(\lambda) \leq r$ and $a(\lambda) \leq c$, the multisets of entries of the hook/distance tableau for λ and the distance/hook tableau for λ are equal.*

In our running example, the hook/distance tableau (below left) and distance/hook tableau (below right) both have, for instance, six entries of 1, three entries of 8, and 12 as their unique greatest entry.

12	11	9	6	4	2	1	1
9	8	6	3	1	4	3	2
7	6	4	1	6	5	4	3
5	4	2	8	7	6	5	4
4	3	1	9	8	7	6	5
2	1	11	10	9	8	7	6

6	7	8	9	10	11	12	1
5	6	7	8	9	1	2	4
4	5	6	7	1	3	4	6
3	4	5	1	3	5	6	8
2	3	4	2	4	6	7	9
1	2	1	4	6	8	9	11

In §2 we give an elementary combinatorial proof of Theorem 1, working by induction on the size of λ. Then in §3 we put the theorem in its proper context by giving a shorter algebraic proof using Stanley's Hook Content Formula and a bijection due to King [1, §4].

2. An elementary combinatorial proof of the main theorem

We work by induction on n, the size of λ. If $n = 0$, so $\lambda = \varnothing$, then the hook/distance tableau has the distances $i + c - 1, \ldots, i$ from west to east in row i, while the distance/hook tableau has the hook lengths $i, \ldots i + c - 1$ from west to east in row i. Therefore the multisets of entries agree.

Suppose the theorem holds for the partition λ of n where $n < rc$. Let $(a, b) \in D \backslash [\lambda]$ be a box such that $[\lambda] \cup \{(a, b)\}$ is the Young diagram of a partition, denoted λ'. As a visual aid, we define the *hook/hook tableau* to have entry $h_{(i,j)}(\lambda)$ in box (i, j) if $(i, j) \in [\lambda]$ and entry $h_{(i,j)}(\lambda')$ in box (i, j) if $(i, j) \in D \backslash [\lambda']$. No entry is assigned to the exceptional box (a, b). The hook/hook tableau in our running example with $(a, b) = (2, 6)$ is below.

12	11	9	6	4	**2**	1	1
9	**8**	**6**	**3**	**1**		**1**	**3**
7	6	4	1	1	**2**	4	6
5	4	2	1	3	**4**	6	8
4	3	1	2	4	**5**	7	9
2	1	1	4	6	**7**	9	11

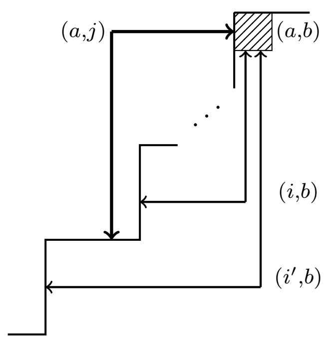
The following lemma is used below to express the hook lengths of λ' in terms of those of λ. The hook/hook tableau above shows that the sets $\mathcal{R}$, $\mathcal{R}'$, $\mathcal{C}$ and $\mathcal{C}'$ in this lemma are $\{1, 3, 6, 8, 9\}$, $\{2, 4, 5, 7\}$, $\{2\}$ and $\{1, 3\}$.

Lemma 2.

(i) *Let* $\mathcal{R} = \{h_{(a,j)}(\lambda) : 1 \leq j < b\}$ *and let* $\mathcal{R}' = \{h_{(i,b)}(\lambda') : a < i \leq r\}$. *Then* $\mathcal{R} \cup \mathcal{R}' = \{1, \ldots r - a + b - 1\}$ *where the union is disjoint.*

(ii) *Let* $\mathcal{C} = \{h_{(i,b)}(\lambda) : 1 \leq i < a\}$ *and let* $\mathcal{C}' = \{h_{(a,j)}(\lambda') : b < j \leq c\}$. *Then* $\mathcal{C} \cup \mathcal{C}' = \{1, \ldots, c - b + a - 1\}$ *where the union is disjoint.*

Proof. It is clear from the diagram below and the hook/hook tableau that no hook length in $\mathcal{R}$ can equal a hook length in $\mathcal{R}'$.



Therefore $\mathcal{R}$ and $\mathcal{R}'$ are disjoint. If $\ell(\lambda) = r$ then the greatest hook length in $\mathcal{R} \cup \mathcal{R}'$ is $h_{(a,1)} = (r - a) + (b - 1) \in \mathcal{R}$, measured by walking north from $(r, 1)$ to $(a, 1)$ then east to $(a, b - 1)$. Otherwise it is $h_{(r,b)} = b + (r - a - 1) \in \mathcal{R}'$, measured by walking east from $(r,1)$ to (r,b) then north to $(a+1,b)$. Since $|\mathcal{R}|+|\mathcal{R}'| = (b-1)+(r-a)$, this proves (i); the proof of (ii) is entirely analogous. $\square$

Given a multiset $\mathcal{X}$ of natural numbers, let $\mathcal{X}^+ = \{x+1 : x \in \mathcal{X}\}$. Let $\cup$ denote the union of multisets. Thus $\{2,2,3\}^+ = \{3,3,4\}$ and $\{1,2\} \cup \{2,2,3\} = \{1,2,2,2,3\}$.

We are now ready for the inductive step. Define

$$\begin{aligned} \mathcal{H}_{\mathrm{NW}} &= \{h_{(i,j)}(\lambda) : (i,j) \in [\lambda]\} & \mathcal{H}'_{\mathrm{NW}} &= \{h_{(i,j)}(\lambda') : (i,j) \in [\lambda']\} \\ \mathcal{H}_{\mathrm{SE}} &= \{h_{(i,j)}(\lambda) : (i,j) \in D\backslash[\lambda]\} & \mathcal{H}'_{\mathrm{SE}} &= \{h_{(i,j)}(\lambda') : (i,j) \in D\backslash[\lambda']\} \end{aligned}$$

and let $\mathcal{D}_{\mathrm{NW}}, \mathcal{D}_{\mathrm{SE}}, \mathcal{D}'_{\mathrm{NW}}, \mathcal{D}'_{\mathrm{SE}}$ be defined analogously, replacing hook lengths with distances.

If $(i,j) \in [\lambda']$ then

$$h_{(i,j)}(\lambda') = \begin{cases} h_{(i,j)}(\lambda) & \text{if } i \neq a \text{ and } j \neq b \\ h_{(i,j)}(\lambda)+1 & \text{if } i = a \text{ or } j = b \text{ but not both} \\ 1 & \text{if } i = a \text{ and } j = b \end{cases}$$

and similarly if $(i,j) \in D\backslash[\lambda']$ then

$$h_{(i,j)}(\lambda') = \begin{cases} h_{(i,j)}(\lambda) & \text{if } i \neq a \text{ and } j \neq b \\ h_{(i,j)}(\lambda)-1 & \text{if } i = a \text{ or } j = b \text{ but not both.} \end{cases}.$$

The equations for $h_{(i,j)}(\lambda')$ above show that $\mathcal{H}'_{\mathrm{NW}}$ is obtained from $\mathcal{H}_{\mathrm{NW}}$ by removing each hook length in $\mathcal{R} \cup \mathcal{C}$ and inserting each hook length in $\mathcal{R}^+ \cup \mathcal{C}^+ \cup \{1\}$. Similarly $\mathcal{H}'_{\mathrm{SE}}$ is obtained from $\mathcal{H}_{\mathrm{SE}}$ by removing each hook length in $R'^+ \cup C'^+ \cup \{1\}$ and inserting each hook length in $\mathcal{R}' \cup \mathcal{C}'$. Therefore, using the multiset union,

$$\begin{aligned} \mathcal{H}'_{\mathrm{NW}} \cup \mathcal{R} \cup \mathcal{C} &= \mathcal{H}_{\mathrm{NW}} \cup \mathcal{R}^+ \cup \mathcal{C}^+ \cup \{1\} \\ \mathcal{H}'_{\mathrm{SE}} \cup R'^+ \cup C'^+ \cup \{1\} &= \mathcal{H}_{\mathrm{SE}} \cup \mathcal{R}' \cup \mathcal{C}'. \end{aligned}$$

We now manipulate these equations so that the inductive hypothesis $\mathcal{H}_{\mathrm{NW}} \cup \mathcal{D}_{\mathrm{SE}} = \mathcal{D}_{\mathrm{NW}} \cup \mathcal{H}_{\mathrm{SE}}$ applies. Recall from Lemma 2 that $\mathcal{R} \cup \mathcal{R}' = \{1, \ldots, r-a+b-1\}$ and $\mathcal{C} \cup \mathcal{C}' = \{1, \ldots, c-b+a-1\}$. Hence, taking the multiset union of both sides of the two equations above with $\mathcal{R}' \cup \mathcal{C}'$ and $\mathcal{R}^+ \cup \mathcal{C}^+$, respectively, we get

$$\begin{aligned} \mathcal{H}'_{\mathrm{NW}} \cup \{1, \ldots, r-a+b-1\} \cup \{1, \ldots, c-b+a-1\} &= \mathcal{H}_{\mathrm{NW}} \cup \mathcal{Y} \cup \{1\} \\ \mathcal{H}'_{\mathrm{SE}} \cup \{2, \ldots, r-a+b\} \cup \{2, \ldots, c-b+a\} \cup \{1\} &= \mathcal{H}_{\mathrm{SE}} \cup \mathcal{Y} \end{aligned}$$

where $Y = \mathcal{R}^+ \cup \mathcal{C}^+ \cup \mathcal{R}' \cup \mathcal{C}'$. Setting $\mathcal{Z} = \{1, \ldots, r-a+b-1\} \cup \{2, \ldots, c-b+a\}$, it follows that

$$\mathcal{H}'_{\mathrm{NW}} \cup \mathcal{Z} \cup \{1\} = \mathcal{H}_{\mathrm{NW}} \cup \mathcal{Y} \cup \{1, c-b+a\} \tag{1}$$

$$\mathcal{H}'_{\mathrm{SE}} \cup \mathcal{Z} \cup \{r-a+b\} = \mathcal{H}_{\mathrm{SE}} \cup \mathcal{Y}. \tag{2}$$

Since $d_{(a,b)}(\lambda') = r-a+b$ and $d_{(a,b)}(\lambda) = c-b+a$ we have $\mathcal{D}'_{\mathrm{SE}} \cup \{c-b+a\} = \mathcal{D}_{\mathrm{SE}}$ and $\mathcal{D}'_{\mathrm{NW}} = \mathcal{D}_{\mathrm{NW}} \cup \{r-a+b\}$. Taking the multiset union of both sides of (1) with $\mathcal{D}'_{\mathrm{SE}} \cup \{c-b+a\}$ and (2) with $\mathcal{D}'_{\mathrm{NW}}$, we get

$$\begin{aligned}\mathcal{H}'_{\mathrm{NW}} \cup \mathcal{D}'_{\mathrm{SE}} \cup \mathcal{Z} \cup \{1, c-b+a\} &= \mathcal{H}_{\mathrm{NW}} \cup \mathcal{D}_{\mathrm{SE}} \cup \mathcal{Y} \cup \{1, c-b+a\} \\ \mathcal{H}'_{\mathrm{SE}} \cup \mathcal{D}'_{\mathrm{NW}} \cup \mathcal{Z} \cup \{r-a+b\} &= \mathcal{H}_{\mathrm{SE}} \cup \mathcal{D}_{\mathrm{NW}} \cup \mathcal{Y} \cup \{r-a+b\}.\end{aligned}$$

Cancelling the elements of $\{1, c-b+a\}$ and $\{r-a+b\}$, we get $\mathcal{H}'_{\mathrm{NW}} \cup \mathcal{D}'_{\mathrm{SE}} \cup \mathcal{Z} = \mathcal{H}_{\mathrm{NW}} \cup \mathcal{D}_{\mathrm{SE}} \cup \mathcal{Y}$ and $\mathcal{H}'_{\mathrm{SE}} \cup \mathcal{D}'_{\mathrm{NW}} \cup \mathcal{Z} = \mathcal{H}_{\mathrm{SE}} \cup \mathcal{D}_{\mathrm{NW}} \cup \mathcal{Y}$. By our inductive hypothesis, the right-hand sides are equal. Therefore $\mathcal{H}'_{\mathrm{NW}} \cup \mathcal{D}'_{\mathrm{SE}} = \mathcal{D}'_{\mathrm{NW}} \cup \mathcal{H}'_{\mathrm{SE}}$, as required.

3. A symmetric polynomials proof of the main theorem

3.1. **Background.** Fix a partition λ. A λ-*tableau* is a function $[\lambda] \to \mathbf{N}$ assigning to each box of $[\lambda]$ an entry in $\mathbf{N}$. A λ-tableau t is *semistandard* if its rows are weakly increasing, when read from west to east, and its columns are strictly increasing, when read from north to south. Let $\mathrm{SSYT}_r(\lambda)$ denote the set of semistandard λ-tableaux with maximum entry at most our fixed number r. For $t \in \mathrm{SSYT}_r(\lambda)$, let x^t denote the monomial $x_1^{e_1} \ldots x_r^{e_r}$ where e_k is the number of entries of t equal to k. By definition, the *Schur polynomial* s_λ in r variables is

$$s_\lambda(x_1, \ldots, x_r) = \sum_{t \in \mathrm{SSYT}_r(\lambda)} x^t.$$

A fundamental result states that $s_\lambda(x_1, \ldots, x_r)$ is symmetric under permutation of $x_1, \ldots, x_r$. This has an elegant proof by the Bender–Knuth involution: see for instance Theorem 7.10.2 in [3].

Let $|t|$ denote the sum of the entries of $t \in \mathrm{SSYT}_r(\lambda)$. Specializing s_λ by $x_k \mapsto q^k$ we obtain

$$s_\lambda(q, \ldots, q^r) = \sum_{t \in \mathrm{SSYT}_r(\lambda)} q^{|t|}. \tag{3}$$

The minimum possible value of $|t|$ for $t \in \mathrm{SSYT}_r(\lambda)$ is $B(\lambda) = \sum_{i=1}^{\ell(\lambda)} i\lambda_i$. Stanley's Hook Content Formula may be stated as

$$s_\lambda(q, \ldots, q^r) = q^{B(\lambda)} \prod_{(i,j) \in [\lambda]} \frac{q^{r+j-i} - 1}{q^{h_{(i,j)}(\lambda)} - 1} \tag{4}$$

where the hook lengths $h_{(i,j)}(\lambda)$ for $(i,j) \in [\lambda]$ are as we have defined. (The term 'content' refers to the powers of q in the numerators.) Stanley's formula was first proved in [2, Theorem 15.3]; for the statement above see [3, Theorem 7.21.2] and the following discussion. For example,

$$s_{(3,2,1)}(q, q^2, q^3) = q^{10} + 2q^{11} + 2q^{12} + 2q^{13} + q^{14}$$

enumerates the semistandard tableaux

$$\begin{array}{|c|c|c|}\hline 1&1&1\\\hline 2&2\\\cline{1-2} 3\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|c|}\hline 1&1&1\\\hline 2&3\\\cline{1-2} 3\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|c|}\hline 1&1&2\\\hline 2&2\\\cline{1-2} 3\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|c|}\hline 1&1&2\\\hline 2&3\\\cline{1-2} 3\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|c|}\hline 1&1&3\\\hline 2&2\\\cline{1-2} 3\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|c|}\hline 1&2&2\\\hline 2&3\\\cline{1-2} 3\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|c|}\hline 1&1&3\\\hline 2&3\\\cline{1-2} 3\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 2&3\\\cline{1-2} 3\\\cline{1-1}\end{array}\,.$$

The central symmetry about x^{12} in the coefficients in this example is a special case of the following basic and well-known lemma, left to the reader in Exercise 7.75 in [3].

Lemma 3. *Let λ be a partition of n. Then*

$$s_\lambda(q, q^2, \ldots, q^r) = q^{(r+1)n} s_\lambda(q^{-1}, q^{-2}, \ldots, q^{-r}).$$

Proof. Let $f(x_1, x_2, \ldots, x_r)$ be a symmetric polynomial. If $x_1^{e_1} \ldots x_r^{e_r}$ is a monomial in f then so is $x_1^{e_r} \ldots x_r^{e_1}$, and the coefficients agree. Under the specialization $x_k \mapsto q^k$ the first becomes $q^{e_1+\cdots+re_r}$ and the second $q^{e_r+\cdots+re_1}$. Observe that the sum of exponents is $(r+1)(e_1 + \cdots + e_r) = (r+1)n$. Therefore the coefficients of q^d and $q^{(r+1)n-d}$ in $f(q, q^2, \ldots, q^r)$ agree for each d. Taking $f = s_\lambda$ this easily implies the lemma. $\square$

The end of our proof requires the following unique factorization theorem, implicitly used in (4.8) in [1].

Lemma 4. *Let $\mathcal{E}$ and $\mathcal{E}'$ be finite multisubsets of $\mathbf{N}$. In the ring $\mathbf{C}[q]$, $\prod_{e\in\mathcal{E}}(q^e - 1) = \prod_{e'\in\mathcal{E}'}(q^{e'} - 1)$ if and only if $\mathcal{E} = \mathcal{E}'$.*

Proof. If either $\mathcal{E}$ or $\mathcal{E}'$ is empty the result is obvious. In the remaining cases, let u be maximal such that $\prod_{e\in\mathcal{E}}(q^e - 1)$ has $\mathrm{e}^{2\pi \mathrm{i}/u}$ as a root. By maximality, $q^u - 1$ is a factor in the left-hand side. Since $\mathrm{e}^{2\pi\mathrm{i}/u}$ is then also a root of $\prod_{e'\in\mathcal{E}'}(q^{e'} - 1)$ the same argument shows that $q^u - 1$ is a factor in the right-hand side. Therefore $u = \max \mathcal{E} = \max \mathcal{E}'$. It follows inductively by cancelling $q^u - 1$ from both sides that $\mathcal{E} = \mathcal{E}'$. $\square$

Let λ° denote the partition defined by deleting any final zeroes from $(c - \lambda_r, \ldots, c - \lambda_1)$; here if $i > \ell(\lambda)$ we take $\lambda_i = 0$.

We require the following bijection which is indicated in [1, §4]; we give a complete proof.

Proposition 5. *There is a bijection*

$$\mathrm{SSYT}_r(\lambda) \to \mathrm{SSYT}_r(\lambda^\circ)$$

defined by sending $t \in \mathrm{SSYT}_r(\lambda)$ to the unique λ°-tableau t° having as its entries in column j the complement in $\{1, \ldots, r\}$ of the entries of t in column $c + 1 - j$, arranged in increasing order from north to south.

Proof. It suffices to prove that t° is semistandard. Suppose, for a contradiction, that columns $c - j - 1$ and $c - j$ of t° have entries $\ell_1^\circ \le k_1^\circ$, …, $\ell_{i-1}^\circ \le k_{i-1}^\circ$ and $\ell_i^\circ > k_i^\circ$ read from north to south. Let columns j and $j + 1$ of t read from north to south have entries $k_1 \le \ell_1$, …, $k_h \le \ell_h$ where h is maximal such that $\ell_h < \ell_i^\circ$. Then $\{\ell_1^\circ, \ldots, \ell_{i-1}^\circ, \ell_1, \ldots, \ell_h\}$ are all the numbers strictly less than ℓ_i° in $\{1, \ldots, r\}$, since, by choice of h, if ℓ_{h+1} is defined then $\ell_{h+1} > \ell_i^\circ$. But from the chain $\ell_i^\circ > k_i^\circ > \ldots > k_1^\circ$ and the inequalities $\ell_i^\circ > \ell_h \geq \ell_j \geq k_j$ for $j \in \{1, \ldots, h\}$, we see that ℓ_i is strictly greater than $i + h$ distinct numbers, a contradiction. □

3.2. **Proof of Theorem 1.** Observe that if $t \in \mathrm{SSYT}_r(\lambda)$ then $|t| + |t^\circ| = r(1+\cdots+c) = rc(c+1)/2$. Therefore by (3), the bijection in Proposition 5, and then Lemma 3, we have

$$\begin{aligned} s_{\lambda^\circ}(q, \ldots, q^r) &= \sum_{u \in \mathrm{SSYT}_r(\lambda^\circ)} q^{|u|} \\ &= \sum_{t \in \mathrm{SSYT}_r(\lambda)} q^{(r+1)rc/2 - |t|} \\ &= q^{r(r+1)c/2} s_\lambda(q^{-1}, \ldots, q^{-r}) \\ &= q^{(r+1)(rc/2-n)} s_\lambda(q, \ldots, q^r). \end{aligned}$$

Applying Stanley's Hook Content Formula (4) to each side we obtain

$$q^{B(\lambda^\circ)} \prod_{(i,j)\in[\lambda^\circ]} \frac{q^{r+j-i}-1}{q^{h_{(i,j)}(\lambda^\circ)}-1} = q^{(r+1)(rc/2-n)+B(\lambda)} \prod_{(i,j)\in[\lambda]} \frac{q^{r+j-i}-1}{q^{h_{(i,j)}(\lambda)}-1}.$$

We now relate the numerators to the distances in Theorem 1, using the bijection from $[\lambda^\circ]$ to $D \backslash [\lambda]$ defined by $(i, j) \mapsto (i', j')$ where $i' = r + 1 - i$ and $j' = c+1-j$. We have $h_{(i',j')}(\lambda) = h_{(i,j)}(\lambda^\circ)$. Moreover, $d_{(i',j')}(\lambda)$ is the number of boxes in any walk by steps north and east from $(r+1-i, c+1-j)$ to $(1, c)$, namely $r - i + j$. Therefore the left-hand side is

$$q^{B(\lambda^\circ)} \prod_{(i',j')\in D\backslash[\lambda]} \frac{q^{d_{(i',j')}(\lambda)}-1}{q^{h_{(i',j')}(\lambda)}-1}.$$

If $(i, j) \in [\lambda]$ then $d_{(i,j)}(\lambda)$ is the number of boxes in any walk by steps south and west to $(r, 1)$, again $r - i + j$. Therefore, cancelling the powers of q we obtain

$$\prod_{(i',j')\in D\backslash[\lambda]} \frac{q^{d_{(i',j')}(\lambda)}-1}{q^{h_{(i',j')}(\lambda)}-1} = \prod_{(i,j)\in[\lambda]} \frac{q^{d_{(i,j)}(\lambda)}-1}{q^{h_{(i,j)}(\lambda)}-1}.$$

Theorem 1 now follows by multiplying through by the denominators and applying Lemma 4.

References

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