INVOLUTIVE RANDOM WALKS ON FINITE TOTAL ORDERS AND THE REAL INTERVAL

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Abstract. This paper studies a family of random walks defined on the finite total orders using their order reversing involutions: starting at $x \in \{0, 1, \ldots, n-1\}$, an element $y \leq x$ is chosen according to a prescribed probability distribution, and the walk then steps to $n-1-y$. We show that, provided every state is accessible, these walks are recurrent and ergodic. We then find the invariant distributions, eigenvalues and eigenvectors of a distinguished family of walks whose transition matrices have the global anti-diagonal eigenvalue property studied in earlier work by Ochiai, Sasada, Shirai and Tsuboi. One of our main theorems is that this family of walks is characterised by their reversibility. As a corollary we obtain the invariant distributions and rate of convergence of the random walk on the set of subsets of $\{1, \ldots, m\}$ in which steps are taken alternately to subsets and supersets, each chosen equiprobably. We then consider analogously defined random walks on the real interval $[0, 1]$ and use techniques from the theory of self-adjoint compact operators on Hilbert spaces to prove analogues of the main results in the discrete case.

1. Introduction

This paper studies a family of random walks defined on the finite total orders using their order reversing involutions: starting at $x \in \{0, 1, \ldots, n-1\}$, an element $y \leq x$ is chosen according to a prescribed probability distribution, and the walk then steps to $n-1-y$. Provided every state is accessible, we prove that these walks are recurrent and ergodic with a unique invariant distribution. We then consider a distinguished subfamily whose transition matrices have the global anti-diagonal eigenvalue property studied in earlier work by Ochiai, Sasada, Shirai and Tsuboi [6]. We find the invariant distributions, eigenvectors and eigenvalues and show that these walks are characterised by their reversibility. As a corollary we obtain the invariant distributions and rate of convergence of the random walk on the set of subsets of $\{1, \ldots, m\}$ in which steps are taken alternately to subsets and

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supersets, each chosen equiprobably. We then consider a related family of non-reversible walks, also with interesting spectral behaviour. In the second part we consider the analogous involutive walks on the real interval $[0, 1]$ and use techniques from the theory of self-adjoint compact operators on Hilbert spaces to prove analogues of all the results in the discrete case. We also prove further results on a trigonometrically-weighted random walk that do not appear to have discrete analogues.

Let $[y, x]$ denote the interval $\{y, y+1, \ldots, x\} \subseteq \mathbb{N}_0$ and let $[x]$ denote the *down-set* $\{0, 1, \ldots, x\}$. It is very convenient to specify the probabilities of steps using the following definition.

Definition 1.1. A *weight* with domain $\mathbb{N}_0$ is a function γ on the set of intervals of $\mathbb{N}_0$, taking values in the non-negative real numbers, such that $\gamma_\varnothing = 0$ and

(1) $0 < \sum_{y\in[x]} \gamma_{[y,x]} < \infty$ for all $x \in n$.
(2) If $y \leq x \leq x'$ and $\gamma_{[y,x]} \neq 0$ then $\gamma_{[y,x']} \neq 0$.

We say that a weight γ is *strictly positive* if $\gamma_{[y,x]} > 0$ for all $x, y \in n$, *atomic* if $\gamma_{[y,x]} = \gamma_{[y,y]}$ for all x, $y \in n$ with $y \leq x$, and $\star$*-symmetric* if $\gamma_{[y,x]} = \gamma_{[x^\star,y^\star]}$ for all x, $y \in n$. We write γ_y for $\gamma_{[y,y]}$ and $N(\gamma)$ for the function defined by $N(\gamma)_x = \sum_{y\in[x]} \gamma_{[y,x]}$.

For brevity, we shall use the ordinal notation in which n is equal to the subset $\{0, 1, \ldots, n-1\}$ of $\mathbb{N}_0$. We define a *weight with domain* n by replacing the ordinal $\mathbb{N}_0$ throughout with n in Definition 1.1. Let $\star : n \to n$ be the order anti-involution defined by $x^\star = n - 1 - x$; the n will always be clear from context.

Definition 1.2. Let γ be a weight defined on n. The γ*-weighted involutive walk* on n is the Markov chain with steps defined as follows: if the current state is $x \in n$ choose y from $[x]$ with probability proportional to $\gamma_{[y,x]}$, and then step to $y^\star$.

Writing $P(\gamma)$ for the transition matrix of the γ-weighted involutive walk on n, Definition 1.2 asserts that

$$P(\gamma)_{xz} = \frac{\gamma_{[z^\star,x]}}{N(\gamma)_x}. \tag{1.1}$$

for all x, $z \in n$. (Note that we number rows and columns of matrices from 0.)

The accessibility assumption in our first two main theorems is motivated by Lemma 2.2 below.

Theorem 1.3. *Let γ be a weight. A γ-weighted involutive walk on n in which every state is accessible is irreducible, recurrent and ergodic with a unique invariant distribution.*

We are particularly concerned with reversible involutive random walks.

Theorem 1.4. *Let γ be a weight. A γ-weighted involutive walk on n in which every state is accessible is reversible if and only if γ factorizes as a product $\alpha\beta$ where α is a strictly positive atomic weight and β is a $\star$-symmetric weight. Moreover, in this case the unique invariant distribution is proportional to $\alpha_{x^\star} N(\alpha\beta)_x$.*

An important class of strictly positive weights that factorize as in Theorem 1.4 is defined for a, $b \in \mathbb{R}^{>-1}$ by

$$\gamma^{(a,b)}_{[y,x]} = \binom{y+a}{y}\binom{b+x-y}{x-y}. \tag{1.2}$$

In particular, $\gamma^{(0,0)}$ is the constant weight, and $\gamma^{(1,0)}$ and $\gamma^{(0,1)}$ are the weights for which $y \in [x]$ is chosen with probability proportional to $y+1$ and $x-y+1$, respectively. Observe that $\gamma^{(a,b)}_{[y,x]}$ is asymptotic to $a^y b^{x-y}/y!(x-y)!$ as $a, b \to \infty$. Setting $b = ac$ and scaling by a^x we obtain $c^{x-y}/y!(x-y)!$. It is therefore natural to define $\gamma^{(c)}$ for $c \in \mathbb{R}^{>0}$ by

$$\gamma^{(c)}_{[y,x]} = \binom{x}{y} c^{x-y}. \tag{1.3}$$

As a further extension, we use that $\binom{y-a'}{y} = (-1)^y\binom{a'-1}{y}$ and $\binom{x-y-b'}{x-y} = (-1)^{x-y}\binom{b'-1}{x-y}$ to extend the weights $\gamma^{(a,b)}$ to a, $b \in \mathbb{R}^{<-1}$, by defining

$$\delta^{(a',b')}_{[y,x]} = \binom{a'-1}{y}\binom{b'-1}{x-y}. \tag{1.4}$$

Then $\delta^{(a',b')}$ is a strictly positive factorizable weight with domain n where $n = \min(\lceil a' \rceil, \lceil b' \rceil)$. Moreover $\delta^{(a',b')}$ is equal to $(-1)^x \gamma^{(-a,-b)}$, when $\gamma^{(-a',-b')}$ is defined as a function on intervals in the obvious way.

To orient the reader we show the transition matrices $P(\gamma^{(0,0)})$, $P(\gamma^{(1,0)})$, $P(\gamma^{(0,1)})$ and $P(\gamma^{(1)})$ below for $n = 4$. To emphasise their characteristic *anti-triangular* structure, we use dots to denote entries that are zero because the relevant interval is empty. This convention is in force throughout.

$$\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array}
\begin{pmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \frac{1}{2} & \frac{1}{2} \\ \cdot & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}
\begin{pmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \frac{2}{3} & \frac{1}{3} \\ \cdot & \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ \frac{2}{5} & \frac{3}{10} & \frac{1}{5} & \frac{1}{10} \end{pmatrix}
\begin{pmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \frac{1}{3} & \frac{2}{3} \\ \cdot & \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{10} & \frac{1}{5} & \frac{3}{10} & \frac{2}{5} \end{pmatrix}
\begin{pmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \frac{1}{2} & \frac{1}{2} \\ \cdot & \frac{1}{4} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \end{pmatrix}$$

In each case, the eigenvalues of P are $(-1)^d P_{dd^\star}$ for $d \in 4$. Up to signs, these are the entries on the anti-diagonal. This generalises as follows:

Theorem 1.5. *The $\gamma^{(a,b)}$-, $\delta^{(a',b')}$- and $\gamma^{(c)}$-weighted involutive walks on n are irreducible, recurrent, reversible and ergodic with unique invariant distribution π where π_x is proportional to $\binom{n-1-x+a}{n-1-x}\binom{x+a+b+1}{x}$ for $\gamma^{(a,b)}$, to*

$\binom{a'-1}{n-1-x}\binom{(a'-1)+(b'-1)}{x}$ for $\delta^{(a',b')}$ and to $\binom{n-1}{x}\left(\frac{c+1}{c}\right)^x$ for $\gamma^{(c)}$. The eigenvalues of $P(\gamma^{(a,b)})$ are

$$(-1)^d\binom{a+d}{d}\Big/\binom{a+b+d+1}{d}$$

for $d \in n$, the eigenvalues of $P(\delta^{(a',b')})$ are $(-1)^d\binom{a'-1}{d}/\binom{(a'-1)+(b'-1)}{d}$ for $d \in n$ and the eigenvalues of $P(\gamma^{(c)})$ are $(-1)^d/c^d$ for $d \in n$. In each case, up to the signs $(-1)^d$, these are the anti-diagonal entries of the transition matrix.

We remark that if a, $b \in \mathbb{N}_0$ then a more convenient form for the absolute values of the eigenvalues of $P(\gamma^{(a,b)})$ is $\binom{a+b+1}{a}/\binom{a+b+d+1}{b+1}$. We show after (5.6) that these eigenvalues are strictly decreasing in absolute value. Hence the second largest eigenvalue in absolute value is $-(a+1)/(a+b+2)$; this controls the rate of convergence of the $\gamma^{(a,b)}$-weighted involutive walk.

Given a weight γ, we define the *down-step* matrix $H(\gamma)$ by $H(\gamma)_{xy} = \gamma_{[y,x]}/N(\gamma)_x$. Note that $H(\gamma)$ is lower triangular and $P(\gamma)_{xz^\star} = H(\gamma)_{xz}$. Thus if $J(n)$ is the $n \times n$ matrix defined, using Iverson bracket notation, by $J(n)_{xy} = [x+y = n-1]$, having ones on its anti-diagonal and zeros elsewhere, then $P(\gamma) = H(\gamma)J(n)$ and Theorem 1.5 asserts that the eigenvalues of $P(\gamma)$ are, up to signs, equal to the eigenvalues of $H(\gamma)$, for each relevant weight γ.

More generally the authors of [6] say in their Definition 2.3 that a lower-triangular $n \times n$ matrix H has the *anti-diagonal eigenvalue property* if H is diagonalizable and the eigenvalues of $HJ(n)$ are precisely $(-1)^d H_{dd}$ for $d \in n$. By Theorem 1.5, the lower-triangular matrices $H(\gamma^{(a,b)})$, $H(\delta^{(a',b')})$ and $H(\gamma^{(c)})$ all have this property. The authors of [6] conclude after (2.8) that a complete classification of such matrices is infeasible. We agree: our Example 2.5 and the following remark give one indication of the difficulties. They therefore introduce the *global anti-diagonal eigenvalue property*, namely that for all $m \leq n$, the $m \times m$ top-left submatrix of H has the anti-diagonal property. Since the weights $\gamma^{(a,b)}$, $\gamma^{(c)}$ and $\delta^{(a',b')}$ are defined without reference to the size of the matrix, the corresponding downstep matrices all have this stronger property.

The main result of [6] is that a lower-triangular matrix H with distinct eigenvalues $\lambda_0, \ldots, \lambda_{d-1}$ has the global anti-diagonal eigenvalue property if and only if H is equal to the matrix H^λ defined by

$$H^\lambda = B(n)\mathrm{Diag}(\lambda_0, \ldots, \lambda_{n-1})B(n)^{-1} \tag{1.5}$$

where $B(n)$ is the $n \times n$ Pascal's Triangle matrix with entries $B_{xy} = \binom{x}{y}$. Our proof of Theorem 1.5 establishes the 'if' direction *en route* using Lemma 4.2. Example 2.4 is included to give a quick illustration of the main idea. Since [6] shows it is of independent interest, we state this result as a proposition below.

Proposition 1.6. *The down-step matrices $H(\gamma^{(a,b)})$, $H(\delta^{(a',b')})$, $H(\gamma^{(c)})$ are each equal to $B(n)DB(n)^{-1}$ where, in each case D is the diagonal matrix of the absolute values of the eigenvalues of the down-step matrix, arranged in decreasing order.*

Also motivated by [6], in §6, we give a necessary and sufficient condition for the transition matrix of a weighted involutive random walk to have the global anti-diagonal eigenvalue property. Given $\lambda_0, \ldots, \lambda_{n-1} \in \mathbb{R}$, let $P^\lambda = H^\lambda J(n)$.

Corollary 1.7. *Let $\lambda_0, \ldots, \lambda_{n-1} \in \mathbb{R}$. The matrix P^λ is the matrix of a weighted involutive random walk in which all states are accessible if and only if $\lambda_0 = 1$ and $\sum_{e=0}^{k}(-1)^e\binom{k}{e}\lambda_{k^\star+e} > 0$ for all $k \in n$. Moreover, in this case $P^\lambda_{xz} > 0$ if and only if $x + z \geq n - 1$.*

It is important to note that §2.2 of [6] considers the related problem of when a half-infinite lower-triangular matrix has the (infinite version) of the global anti-diagonal eigenvalue property. The main result is Theorem 2.34, which characterises such matrices as those whose diagonal entries $\lambda_0, \lambda_1, \ldots$ are *completely monotonic*: that is

$$\sum_{e=0}^{k}(-1)^e\binom{k}{e}\lambda_{m+e} > 0$$

for all m, $k \in \mathbb{N}_0$. The authors comment at the start of §2.2 that 'similar results hold in the finite case'. This is true, but as seen in the corollary above (and in Proposition 6.3), only a one parameter family of inequalities is required. Moreover, the inequalities in Corollary 1.7 are specific to each n: they may all hold for $\lambda_0, \ldots, \lambda_{n-1}$ and then fail for $\lambda_0, \ldots, \lambda_{n-1}, \lambda_n$. For instance, this is the case if λ_n is very large. Thus it is not possible to obtain our results directly from those stated in [6]; instead one must adapt the proofs, which require the use of the quite deep Hausdorf Moment Theorem. Another sign that the problems are somewhat different is that there is no 'limiting case' of the involutive random walk, because the ordinal $\mathbb{N}$ has no anti-involution.

The main result in §6 is not considered in [6]. To state it, we must define a *weak weight* by dropping property (2) in Definition 1.1. If $b' \in \mathbb{N}$ then $\delta^{(a',b')}$, defined exactly as in (1.4), is a weak weight with domain $\lceil a' \rceil$. Since property (1) still holds, the down-step and transition matrices for weak weights may be defined as before.

Theorem 1.8. *Let $n \geq 3$ and let P be the transition matrix of a reversible Markov chain on n in which all states are accessible. Then $PJ(n)$ has the global anti-diagonal property if and only if $P = P(\gamma^{(a,b)})$ for unique $a, b \in \mathbb{R}^{>-1}$, or $P = P(\delta^{(a',b')})$ for a unique $a' \in \mathbb{R}^{>n-1}$ and a unique b' with either $b' \in \mathbb{N}$ or $b' \in \mathbb{R}^{>n-1}$, or $P = P(\gamma^{(c)})$ for unique $c \in \mathbb{R}^{>0}$.*

Since the weights $\gamma^{(a,b)}$ were discovered by computer experiment, looking for involutive walks with rational eigenvalues, we find it somewhat remarkable that they, together with their natural generalizations, are characterised as in this theorem. In fact we prove a somewhat sharper result, stated in Proposition 6.6, which implies (see Example 6.7) that if $n = 10$ and the second largest eigenvalue (in absolute value) of P is $\frac{2}{3}$, then P is reversible if and only if the third largest eigenvalue of P is either in the interval $(\frac{1}{3}, \frac{10}{23})$, or is one of the four exceptional values $\frac{10}{23}$, $\frac{13}{30}$, $\frac{22}{51}$ and $\frac{3}{7}$. This gives one indication of the depth of this result.

Non-reversible involutive walks are also of considerable interest: in §7 we show that there is a family of involutive walks whose eigenvalues are 1 and $\pm\sqrt{\lambda_x \lambda_{x^\star}}$ for arbitrary $\lambda_1, \ldots, \lambda_{n-1}$ with $1 > \lambda_1 > \ldots > \lambda_{n-1} \geq 0$.

The continuous analogue of the weight-involutive walk is defined on the interval $[0, 1]$. Starting at a state $x \in [0, 1]$, an element $y \in [0, x]$ is chosen according to a probability distribution specified by a real weight (as defined in Definition 8.1); the walk then steps to $1 - y$. In §8 and §9 we prove analogues of all the results in §3 and §5 in the continuous setting. We mention here that, by Theorem 9.6, the analogue of the $\gamma^{(a,b)}$-weighted involutive walk has discrete spectrum $(-1)^d \binom{a+d}{d} / \binom{a+b+d+1}{d}$ for $d \in \mathbb{N}_0$; the Jacobi functions form the corresponding orthonormal basis of eigenfunctions. Thus the eigenvectors corresponding to the eigenvalues identified in Theorem 1.5 are the discrete analogue of these special functions. We believe these eigenvectors are worthy of further study and identify some of their properties in §5.4.

Taking two steps at a time in an involutive random walk gives the walk on n in which steps are alternately down and up. An immediate corollary of Theorem 1.3 is that, when weighted by $\gamma^{(a,b)}$, this down-up walk on n is also irreducible, reversible, ergodic and recurrent. Moreover its eigenvalues are the squares of the eigenvalues in Theorem 1.5. A similar result holds on $[0, 1]$ and for the up-down walk. Thus despite its more intricate definition, the involutive random walk is the more fundamental of the two random processes. This is seen in the corollary below.

Corollary 1.9. *Fix $m \in \mathbb{N}$ and let $0 < p < 1$. Let $\mathcal{P}$ be the set of subsets of $\{1, \ldots, m\}$.*

(i) *The random walk on $\mathcal{P}$ in which, starting from $X \in \mathcal{P}$, we choose $Y \subseteq X$ by putting each $x \in X$ in Y independently with probability p, and then step to $\{1, \ldots, m\} \backslash Y$, is irreducible, reversible, recurrent and ergodic with unique invariant distribution π where $\pi_X = p^{m-|X|}/(1+p)^m$ for each $X \in \mathcal{P}$. Its eigenvalues are $(-p)^e$ for $0 \leq e \leq m$, with multiplicities $\binom{m}{e}$.*

(ii) *Taking two steps at a time in the walk in (i) gives the walk on $\mathcal{P}$ in which steps are alternately down and up. This walk has the same invariant distribution, and its eigenvalues are p^{2e} for $0 \leq e \leq m$, again with multiplicities $\binom{m}{e}$.*

Finally we remark that property (2) in Definition 1.1 is particularly useful in the generalizations of Theorems 1.3 and 1.4 to random walks defined on an arbitrary poset $\mathcal{P}$ with anti-involution. The precise statement and proof of these results are postponed to a further paper with many examples of this type, including an analogue of Corollary 1.9 in which subsets are replaced with subspaces of a finite vector space.

Outline. We encourage all readers to begin in §2 where we give some examples intended to illustrate the main results and their proofs. In §3 we prove Theorems 1.3 and 1.4. In §4 we prove Theorem 1.5 for the weights $\gamma^{(c)}$. In §5 we extend the method of §4 to prove Theorem 1.5 for the weights $\gamma^{(a,b)}$ and $\delta^{(a',b')}$, and give many further results on the left- and right-eigenvectors of the transition matrix $P(\gamma^{(a,b)})$. We end with the proof of Corollary 1.9. In §6 we prove Corollary 1.7 and Theorem 1.8. In §7 we consider a different class of involutive walks with interesting spectral behaviour and characterise when they are reversible. In §8 we give a general setting using Hilbert spaces for weighted involutive walks on the interval $[0,1]$ and in §9 we prove analogues of the results for the $\gamma^{(a,b)}$-weighted discrete walks.

2. Examples and motivation

This section is not logically essential; we hope it will be valuable to readers wanting a quick introduction to the main ideas of the proofs or a further taste of our results.

2.1. **The unweighted involutive walk.**

Example 2.1. When the weight γ is constant, each $y \in [x]$ is equally likely to be chosen in a down-step from x. For instance, if $n = 4$, from state 2, the walk steps to each of $0^\star = 3$, $1^\star = 2$ and $2^\star = 1$ with equal probability, and the transition matrix is as shown below.

$$\begin{pmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \frac{1}{2} & \frac{1}{2} \\ \cdot & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}.$$

For any n, Theorem 1.3 implies that the unweighted involutive walk is irreducible, ergodic and recurrent. Since the constant weight is both atomic and $\star$-symmetric, Theorem 1.4 implies that the unweighted involutive walk is reversible; since $N(\gamma)_x = x + 1$ for each $x \in n$, the unique invariant distribution π has π_x proportional to $x + 1$. (It is worth noting that this easy description is possible in part because our weights are not required to be normalized as probability distributions.) By the special case $a = b = 0$ of Theorem 1.5 the eigenvalues of the transition matrix are $1, -\frac{1}{2}, \frac{1}{3}, \ldots, (-1)^{n-1}\frac{1}{n-1}$, and so it has the anti-diagonal eigenvalue property considered

in [6]. Since the submatrices formed from the top-right e columns have the same special behaviour, the transition matrix has the (stronger) global anti-diagonal property. From the second largest eigenvalue (in absolute value), we see the absolute spectral gap is $\frac{1}{2}$. By Corollary 12.7 in [4] we have

$$\lim_{t\to\infty} \left|\left|\frac{P^t_{xy}}{\pi_y} - 1\right|\right|^{1/t} = \tfrac{1}{2}$$

for any $x \in n$.

We note that the transition matrix in Example 2.1 also appears in [5, Example 7.5] in the context of a random walk on the indecomposable non-projective representations of $\mathrm{SL}_2(\mathbb{F}_p)$ in characteristic p, as one of two blocks (after rearranging rows and columns) of the matrix shown in Figure 5, taking the parameter n in [5] to be $(p-1)/2$.

2.2. **Accessibility.** The following lemma and example show that it is reasonable to impose an accessibility assumption on the states in an involutive random walk.

Lemma 2.2. *Let P be the transition matrix of a Markov chain on n. There is a weak weight γ on $2n$ such that, restricted to its accessible states, the γ-weighted involutive random walk is isomorphic to the Markov chain specified by P.*

Proof. Writing $y^\star = n-1-y$ as usual, let γ be the function on intervals of $2n$ defined by

$$\gamma_{[y,x]} = \begin{cases} 1 & \text{if } y = 0 \text{ and } x < n \\ 0 & \text{if } 0 < y < n \text{ and } x < n \textit{ or } y \geq n \\ P_{xy^\star} & \text{if } 0 \leq y < n \text{ and } x \geq n. \end{cases}$$

By construction, γ is a weak weight on $2n$, the states $0, 1, \ldots, n-1$ are inaccessible, and the probability that the γ-weighted involutive walk steps from $n+x$ to $n+y$ is P_{xy}. □

We use this construction to show that the accessibility assumption in Theorem 1.4 cannot be dropped.

Example 2.3. Taking $n = 3$, the weak weight γ used to prove Lemma 2.2 is defined by the matrix below, in which $\gamma_{[y,x]}$ is in position (x, y); as usual dots indicate entries that are 0 because the interval is empty.

$$\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 0 & \cdot & \cdot & \cdot & \cdot \\ 1 & 0 & \mathbf{0} & \cdot & \cdot & \cdot \\ P_{02} & P_{01} & P_{00} & 0 & \cdot & \cdot \\ P_{12} & P_{11} & P_{10} & 0 & 0 & \cdot \\ P_{22} & P_{21} & P_{20} & 0 & 0 & 0 \end{pmatrix}$$

If $P_{xy} > 0$ for all x, $y \in 3$ then γ is a normalized weight and the matrix above is $H(\gamma)$. By Lemma 2.2, the γ-weighted involutive walk is reversible if and only if the Markov chain specified by P is reversible. Suppose this is the case and consider the modified weight γ' in which we change $\gamma_{[2,2]}$, shown in bold above, from 0 to 1. Since the accessible states are unaltered, the invariant distribution, π say, is supported on $\{3, 4, 5\}$. Suppose that γ factorizes as a product $\alpha\beta$ as in Theorem 1.4. By this theorem, $\pi_x = \alpha_{x^\star} N(\alpha\beta)_x$ for each x, and so α is supported on $\{3^\star, 4^\star, 5^\star\}$. In particular $\alpha_2 \neq 0$ and so $\beta_{[2,2]} = \gamma'_{[2,2]}/\alpha_2 \neq 0$. Similarly $\beta_{[1,2]} = \gamma'_{[1,2]}/\alpha_2 = 0$. If β is $\star$-symmetric, we must have $\beta_{[2^\star,2^\star]} = \beta_{[3,3]} \neq 0$ but $\beta_{[1^\star,2^\star]} = \beta_{[3,4]} = 0$. This contradicts property (2) in Definition 1.1.

2.3. **The anti-diagonal eigenvalue property.** We now illustrate the key Lemma 4.2 in the proof of Theorem 1.5.

Example 2.4. The calculation below verifies the identity $H(\gamma^{(0,1)})B(4) = B(4)\mathrm{Diag}(1, \frac{1}{3}, \frac{1}{6}, \frac{1}{10})$:

$$\begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \frac{2}{3} & \frac{1}{3} & \cdot & \cdot \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{6} & \cdot \\ \frac{2}{5} & \frac{3}{10} & \frac{1}{5} & \frac{1}{10} \end{pmatrix} \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot \\ 1 & 2 & 1 & \cdot \\ 1 & 3 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ 1 & \frac{1}{3} & \cdot & \cdot \\ 1 & \frac{2}{3} & \frac{1}{6} & \cdot \\ 1 & 1 & \frac{1}{2} & \frac{1}{10} \end{pmatrix} = \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot \\ 1 & 2 & 1 & \cdot \\ 1 & 3 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \frac{1}{3} & \cdot & \cdot \\ \cdot & \cdot & \frac{1}{6} & \cdot \\ \cdot & \cdot & \cdot & \frac{1}{10} \end{pmatrix}.$$

It follows that the columns of the 4×4 Pascal's Triangle are eigenvectors of $H(\gamma^{(0,1)})$. Observe that column d has the values of the polynomial $\binom{X}{d} = X(X-1)\dots(X-d+1)/d!$. Therefore, writing V_e for the subspace of $\mathbb{R}^4$ of all column vectors $\big(f(0), f(1), f(2), f(3)\big)^{\mathrm{t}}$ where f is a polynomial of degree $< e$, we see that $H(\gamma^{(0,1)})$ preserves the flag of subspaces $0 = V_0 \subset V_1 \subset V_2 \subset V_3 \subset V_4 = \mathbb{R}^4$. Now since $J(4)\big(f(0), f(1), f(2), f(3)\big)^{\mathrm{t}} = \big(f(3), f(2), f(1), f(0)\big)$ is the values of the polynomial $f(n-1-x)$, the matrix $J(4)$ preserves each subspace. Therefore so does $P(\gamma^{(0,1)}) = H(\gamma^{(0,1)})J(4)$. Moreover, since the leading coefficient in $(n-1-x)^d$ is $(-1)^d$, the scalar by which $P(\gamma^{(0,1)})$ acts on the quotient space V_e/V_{e-1} is $(-1)^e$ times the corresponding eigenvalue for $H(\gamma^{(0,1)})$. This shows that $B(4)^{-1}P(\gamma^{(0,1)})B(4)$ is upper-triangular with diagonal entries 1, $-\frac{1}{3}$, $\frac{1}{6}$, $-\frac{1}{10}$. In particular, $H(\gamma^{(0,1)})$ has the anti-diagonal eigenvalue property.

Using the idea in the previous example and Example 2.1, we now exhibit an infinite family of transition matrices for weighted involutive walks whose corresponding lower-triangular matrix has the anti-diagonal eigenvalue property with rational eigenvalues..

Example 2.5. For $\tau \in \mathbb{R}\backslash -1$, let L and U be the lower- and upper-triangular matrices shown below; their product is the third matrix.

$$\begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot \\ 1 & 2+\tau & 1 & \cdot \\ 1 & 3+\tau & \frac{3+\tau}{1+\tau} & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 3+\tau & \frac{3+\tau}{1+\tau} & 1 \\ \cdot & -1 & -\frac{2}{1+\tau} & -1 \\ \cdot & \cdot & 1 & 1+\tau \\ \cdot & \cdot & \cdot & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 3+\tau & \frac{3+\tau}{1-\tau} & 1 \\ 1 & 2+\tau & 1 & \cdot \\ 1 & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{pmatrix}.$$

The matrix $L\mathrm{Diag}(1, \lambda_1, \lambda_2, \lambda_3)L^{-1}$ has as its eigenvectors the columns of L. The matrix L was constructed so that $LU = J(4)L$. Therefore the subspaces of $\mathbb{R}^4$ spanned by the first k columns of L and $J(4)L$ are equal for all k. A generalization of the argument in Example 2.4 (or, equivalently, in Lemma 4.2) now shows that that $L\mathrm{Diag}(1, \lambda_1, \lambda_2, \lambda_3)L^{-1}$ has the anti-diagonal eigenvalue property for any λ_1, λ_2, $\lambda_3 \in \mathbb{R}$. Taking $\lambda_1 = \frac{1}{2}$, $\lambda_2 = \frac{1}{3}$ and $\lambda_3 = \frac{1}{4}$ and multiplying by $J(4)$, we obtain the family of anti-triangular matrices

$$\begin{pmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \frac{1}{2} & \frac{1}{2} \\ \cdot & \frac{1}{3} & \frac{2+\tau}{6} & \frac{2-\tau}{6} \\ \frac{1}{4} & \frac{3+\tau}{12(1+\tau)} & \frac{(3+\tau)(1+2\tau)}{12(1+\tau)} & \frac{3-2\tau}{12} \end{pmatrix}$$

all with eigenvalues 1, $-\frac{1}{2}$, $\frac{1}{3}$, $-\frac{1}{4}$. When $\tau = 0$ the matrix L is $B(4)$, and so we have a 1-parameter family of transition matrices of involutive random walks defined for $-\frac{1}{2} < \tau < \frac{3}{2}$, all with rational eigenvalues and having the (non-global) anti-diagonal eigenvalue property. These continuously deform the unweighted involutive walk seen in Example 2.1.

We remark that $n \times n$ matrices with the anti-diagonal eigenvalue property may be constructed using any pair L, U satisfying the identity $LU = J(n)L$; matrices with the global anti-diagonal eigenvalue property are precisely those given by the pair $L = B(n)$ and $U = D^{\pm}J(n)B(n)J(n)$, where $D^{\pm}$ is the diagonal matrix with entries $D^{\pm}_{xx} = (-1)^x$. There are however many further examples of matrices with the anti-diagonal eigenvalue property that do not come from this construction, for example

$$\begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot \\ \frac{1}{2} & \frac{1}{2} & \cdot & \cdot & \cdot \\ \frac{2}{5} & \frac{1}{5} & \frac{2}{5} & \cdot & \cdot \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \cdot \\ \frac{5}{18} & \frac{1}{9} & \frac{2}{9} & \frac{1}{9} & \frac{5}{18} \end{pmatrix}$$

chosen at random from a computer search for such matrices with rational eigenvalues.

2.4. **The global anti-diagonal property and reversibility.** The proof of Theorem 1.8 uses the global anti-diagonal property and Kolomogorov's Criterion for reversibility (stated as Lemma 6.8(ii)). We illustrate the key idea in the inductive step.

Example 2.6. Suppose that P is the transition matrix of a reversible Markov chain on 5 and that P has the global anti-diagonal property. Thus, by the main result of [6], $P = B(5)\mathrm{Diag}(\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4)B(5)^{-1}J(5)$ for some $\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{R}$. In particular, the top-right 4×4-submatrix of P is equal to $B(4)\mathrm{Diag}(\lambda_0, \lambda_1, \lambda_2, \lambda_3)B(4)^{-1}J(4)$. Suppose we know inductively that this is the transition matrix of the unweighted involutive walk on 4. Since $P(\gamma^{(0,0)})$ has eigenvalues 1, $-\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{4}$ we deduce that $\lambda_0 = 1$, $\lambda_1 = \frac{1}{2}$, $\lambda_2 = \frac{1}{3}$ and $\lambda_3 = \frac{1}{4}$. By Lemma 6.1,

$$P = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \frac{1}{2} & \frac{1}{2} \\ \cdot & \cdot & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \cdot & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \lambda_5 & 1-4\lambda_4 & -1+6\lambda_4 & 1-4\lambda_4 & \lambda_4 \end{pmatrix}.$$

For instance, by this lemma, the entry in the bottom right is $\lambda_0 - 4\lambda_1 + 6\lambda_2 - 4\lambda_3 + \lambda_4$, which simplifies to λ_4. Consider the 4-cycle $4 \mapsto 1 \mapsto 3 \mapsto 2 \mapsto 4$ and its reverse $4 \mapsto 2 \mapsto 3 \mapsto 1 \mapsto 4$. By the reversibility hypothesis and Kolmogorov's Criterion, the probability of following the cycle is the same in each direction. In this example the non-zero probabilities are constant in each row, and so we may cancel $P_{13}P_{32}P_{24} = \frac{1}{2} \times \frac{1}{4} \times \frac{1}{3} = P_{23}P_{31}P_{14}$ to get

$$1 - 4\lambda_4 = P_{41} = P_{42} = -1 + 6\lambda_4.$$

Hence $\lambda_4 = \frac{1}{5}$. Therefore P is the transition matrix of the unweighted involutive walk on 5.

2.5. **The $\delta^{(a',b')}$-weighted involutive walk.** We give two examples of how the weights $\delta^{(a',b')}$ arise in the context of the base case of the inductive proof of Theorem 1.8.

Example 2.7. Let γ be a weak weight on 3 such that $H(\gamma)$ has the global anti-diagonal eigenvalue property and suppose that the eigenvalues of $H(\gamma)$ are 1, $\frac{2}{3}$ and ν. By the main result of [6], $H(\gamma) = B(3)\mathrm{Diag}(1, \frac{2}{3}, \nu)B(3)^{-1}$. Thus $H(\gamma)$ is the matrix

$$\begin{pmatrix} 1 & \cdot & \cdot \\ 1 & 1 & \cdot \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & \frac{2}{3} & \cdot \\ \cdot & \cdot & \nu \end{pmatrix} \begin{pmatrix} 1 & \cdot & \cdot \\ -1 & 1 & \cdot \\ 1 & -2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \cdot & \cdot \\ \frac{1}{3} & \frac{2}{3} & \cdot \\ -\frac{1}{3}+\nu & \frac{4}{3}-2\nu & \nu \end{pmatrix}.$$

Since $H(\gamma)$ is non-negative we have $\nu \geq \frac{1}{3}$. By Lemma 4.2, the right-hand side is the down-step matrix of the $\gamma^{(a,b)}$-weighted involutive walk if and only if the eigenvalues agree; since $H(\gamma^{(a,b)})$ has diagonal entries 1, $(a+1)/(a+b+2)$ and $(a+1)(a+2)/(a+b+2)(a+b+3)$, we must set $a = 2b+1$ and require $\frac{2}{3}(a+2)/(2a+3) = \nu$. If $\nu \neq \frac{4}{9}$ we have the unique solution $a = (15\nu - 8)/(9\nu - 4)$. If $\nu > \frac{4}{9}$ then a, $b > -1$ and we are in the usual case for the weights $\gamma^{(a,b)}$. In the exceptional case $\nu = \frac{4}{9}$ we have $H(\gamma) = H(\gamma^{(\frac{2}{3})})$. If $\nu < \frac{4}{9}$ then a, $b < -1$; by the remark

after (1.4), defining $\gamma^{(a,b)}$ as a function on intervals in the natural way, we have $H(\gamma) = H(\gamma^{(a,b)}) = H(\delta^{(-a,-b)})$, where $\delta^{(a',b')}_{[y,x]} = \binom{a'-1}{y}\binom{b'-1}{x-y}$. To make this more concrete, we consider three special cases.

(1) If $\nu = \frac{4}{9} + \frac{1}{18} = \frac{1}{2}$ then $a = 1$, $b = 0$ and γ is the weight $\gamma^{(1,0)}$; the matrix $P(\gamma^{(1,0)})$ appears in the introduction.
(2) If $\nu = \frac{4}{9} - \frac{1}{36} = \frac{5}{12}$ then $a = -7$ and $b = -4$, giving the $\delta^{(7,4)}$-weighted involutive walk. By (1.4), the domain of this weight is 4, and correspondingly, the unique ϑ such that $1, -\frac{2}{3}, \frac{5}{12}, -\vartheta$ are the eigenvalues of a reversible 4×4 transition matrix with the global anti-diagonal eigenvalue property is the absolute value of the fourth eigenvalue of $P(\delta^{(7,4)})$, namely $\frac{5}{42}$ by Theorem 1.5.
(3) Finally if $\nu = \frac{1}{3}$ then γ is the weak weight $\delta^{(3,2)}$; it was defined with domain 3 because $\delta^{(3,2)}_{[0,3]} = \binom{3}{4}\binom{2}{0} = 0$, and correspondingly the 0 state in the $\delta^{(3,2)}$-weighted involutive walk is inaccessible. Thus the walk is well-defined but degenerate, and so not one of those characterised in Theorem 1.8.

Example 2.8. A special case of the weak weight $\delta^{(a',2)}$ was seen in (3) above. In general this weak weight is defined with domain n where $n = \lceil a' \rceil$ and defines an involutive walk in which, starting at $x \in n$, we step to $x^\star$ with probability proportional to $\binom{a'-1}{x}$ and to $x^\star + 1$ with probability proportional to $\binom{a'-1}{x-1}$. Since $\binom{a'-1}{x-1} = \binom{a'-1}{x}\frac{x}{a'-x}$, the probabilities are x/a' and $1 - x/a'$, respectively. All states are accessible, and so by Theorem 1.3, all states are recurrent. By Theorem 1.4, the walk is reversible with invariant distribution proportional to $\binom{a'-1}{n-1-x}\binom{a'}{x}$. Moreover if the second largest eigenvalue (in absolute value) is μ then, from $\delta^{(a',2)}_{[0,1]} = 1 - 1/a' = \mu$, we see that $a' = 1/(1-\mu)$. The matrices for $\delta^{(1/(1-\mu),2)}$-weakly weighted involutive walk and the special cases when $a' = 5$ (and so $\mu = \frac{4}{5}$) are shown below.

$$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \mu & 1-\mu \\ \cdot & \cdot & 2(1-\mu) & 2\mu-1 & 0 \\ \cdot & 3(1-\mu) & 3\mu-2 & 0 & 0 \\ 4(1-\mu) & 4\mu-3 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \frac{4}{5} & \frac{1}{5} \\ \cdot & \cdot & \frac{3}{5} & \frac{2}{5} & 0 \\ \cdot & \frac{2}{5} & \frac{3}{5} & 0 & 0 \\ \frac{1}{5} & \frac{4}{5} & 0 & 0 & 0 \end{pmatrix}$$

(A convenient way to compute the first matrix by hand uses Lemma 6.1(ii).) The $\delta^{(5,2)}$-weakly weighted involutive walk cannot be extended to an involutive walk on 6 in which all states are accessible because $\binom{5-1}{5} = 0$; indeed by Lemma 6.5, the condition $n \leq \lceil a' \rceil$ is necessary and sufficient.

2.6. **The power-weighted involutive walk.** We consider an easily analysed special case of the weights in §7.

Example 2.9. Fix $r \in \mathbb{R}^{>0}$ with $e \neq 1$ and let $\rho^{(r)}$ be the weight defined by

$$\rho^{(r)}_{[y,x]} = \begin{cases} 1 & \text{if } x = y \\ r^{x-y} - c^{x-y-1} & \text{if } x > y. \end{cases}.$$

Since $N(\varepsilon)_x = r^x$, the corresponding probabilities are obtained by dividing by r^x. In particular, $P(\rho^{(r)})$ has anti-diagonal entries 1, $1/r$, ..., $1/r^{n-1}$. The matrices $P(\rho^{(2)})$ and $P(\rho^{(3)})$ are shown below when $n = 4$.

$$\begin{pmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \frac{1}{2} & \frac{1}{2} \\ \cdot & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{4} & \frac{1}{2} \end{pmatrix}, \quad \begin{pmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \frac{1}{3} & \frac{2}{3} \\ \cdot & \frac{1}{9} & \frac{2}{4} & \frac{2}{3} \\ \frac{1}{27} & \frac{2}{27} & \frac{2}{9} & \frac{2}{3} \end{pmatrix}.$$

Since the weight $\rho^{(r)}$ depends only on x and y through $x - y$, it is $\star$-symmetric. By Theorem 1.4, the unique invariant distribution for the $\rho^{(r)}$-weighted involutive walk is proportional to $N(\rho)_x = r^x$. It is therefore $\pi_x = r^x(r-1)/(r^n - 1)$. Since $\rho^{(r)}$ is the weight ε^λ defined in §7 taking $\lambda_x = r^x$, a special case of Theorem 7.4 implies that $P(\rho^{(r)})$ is diagonalizable and its eigenvalues, other than 1, are $1/r^{\frac{n}{2}}$ and $-1/r^{\frac{n}{2}}$, with multiplicities $\lfloor \frac{n}{2} \rfloor - 1$ and $\lfloor \frac{n}{2} \rfloor$, respectively. In particular, the matrices above have eigenvalues 1, $\frac{1}{4}$, $-\frac{1}{4}$, $-\frac{1}{4}$ and 1, $\frac{1}{9}$, $-\frac{1}{9}$, $-\frac{1}{9}$.

2.7. **Involutive random walks on the interval** $[0, 1]$**.**

Example 2.10. The continuous analogue of the weight $\gamma^{(1,1)}$ defined by $\gamma^{(1,1)}_{[y,x]} = (y+1)(x-y+1)$ is $\kappa^{(1,1)}$ where $\kappa^{(1,1)}_{[y,x]} = y(x-y)$ for all non-empty intervals $[y, x] \subseteq [0, 1]$. Again this weight factorizes as a product of an atomic and $\star$-symmetric weight. By Definition 8.1, $N(\kappa^{(1,1)})_x = \int_0^x y(x-y)\mathrm{d}y = \frac{x^3}{3}$. By the analogue of Theorem 1.4 stated in Proposition 8.2, the unique invariant distribution for the $\kappa^{(1,1)}$-weighted involutive walk is proportional to $(1-x)N(\kappa^{(1,1)})_x$, or equivalently, to $x(1-x)^3$. This may be compared with the discrete case, where, by Theorem 1.5 the unique invariant distribution for the $\gamma^{(1,1)}$-weighted involutive walk on n is proportional $(n-x)\binom{x+3}{x}$. Replacing x here with nX and scaling by n, one sees that as $n \to \infty$, the invariant distribution converges to $(1-X)X^3$, in agreement with the continuous case. Moreover, by Theorem 9.6, the spectrum of the transition map for the continuous random walk (as defined as a self-adjoint compact operator on Hilbert space in §8.2) is $\lambda(\gamma^{(1,1)})_d$ for $d \in \mathbb{N}$.

3. Factorizable weights and proofs of Theorems 1.3 and 1.4

Definition 1.1 is chosen in part so that the following lemma holds.

Lemma 3.1. *In the γ-weighted involutive walk on n any state steps to $n-1$ with non-zero probability. A state is recurrent if and only if it accessible.*

Proof. By property (1) in Definition 1.1, $\gamma_{[0,0]} = N(\gamma)_0 > 0$. By property (2) and (1.1), for *any* state x, the step $x \mapsto n-1$ has non-zero probability $\gamma_{[0,x]}/N(\gamma)_x$. Therefore $n-1$ is a recurrent state. Suppose that $z \in n$ is an accessible state, and so the step $x \mapsto z$ has non-zero probability for some state x. Hence $z^\star \leq x$ and $\gamma_{[z^\star,x]}/N(\gamma)_x > 0$. By property (2), $\gamma_{[z^\star,n-1]} > 0$, and so the step $n-1 \mapsto z$ also has non-zero probability. Since $n-1$ is recurrent, the lemma follows. □

We are now ready to prove Theorem 1.3.

Proof of Theorem 1.3. By assumption every state is accessible, and so by Lemma 3.1 every state is recurrent. Also by this lemma, the step $n-1 \mapsto n-1$ has non-zero probability, so the walk is aperiodic. Therefore there is a unique invariant distribution to which the walk converges by the ergodic theorem. □

We now turn to reversibility. The detailed-balance equations for an invariant probability distribution π on n are

$$\pi_x \frac{\gamma_{[z^\star,x]}}{N(\gamma)_x} = \pi_z \frac{\gamma_{[x^\star,z]}}{N(\gamma)_z}. \tag{3.1}$$

Since $\gamma_\varnothing = 0$ and $[z^\star, x]$ is non-empty if and only if $[x^\star, z]$ is non-empty, we may assume that $z^\star \leq x$ in this equation.

Lemma 3.2.

(i) *If α is atomic then the α-weighted involutive walk is reversible with unique invariant distribution $\pi(\alpha)$ such that $\pi(\alpha)_x \propto \alpha_{x^\star} N(\alpha)_x$. A state x is recurrent if and only if $\alpha_{x^\star} \neq 0$.*

(ii) *If β is $\star$-symmetric then the β-weighted involutive walk is reversible with unique invariant distribution $\pi(\beta)$ such that $\pi(\beta) \propto N(\beta)_x$ and every state is recurrent.*

Proof. When α is atomic the equations (3.1) simplify to $\pi_x \alpha_{z^\star}/N(\alpha)_x = \pi_z \alpha_{x^\star}/N(\alpha)_z$ for x, $z \in \mathcal{P}$ such that $x^\star \leq z$. Clearly one solution has each π_x proportional to $\alpha_{x^\star} N(\alpha)_x$. If β is $\star$-symmetric then $\beta_{[z^\star,x]} = \beta_{[x^\star,z]}$, and so the step $x \mapsto z$ has non-zero probability if and only if the step $z \mapsto x$ has non-zero probability. By Lemma 3.1, $n-1$ is a recurrent state and $x \mapsto n-1$ has-zero probability for any x. Hence all states are recurrent. Again using $\beta_{[z^\star,x]} = \beta_{[x^\star,z]}$, we see that one solution to (3.1) has each π_x proportional to $N(\alpha)_x$. In either case our solution assigns a non-zero probability to a state if and only if it is accessible. Therefore the invariant distributions just found are unique. □

Reversibility is preserved by products of weights.

Lemma 3.3. *Let α and β be weights. If the α-weighted and β-weighted involutive walks on $\mathcal{P}$ are reversible with respect to invariant distributions*

proportional to ϑ and ϕ, respectively, then the $\alpha\beta$-weighted involutive walk on $\mathcal{P}$ is reversible with respect to an invariant distribution proportional to $\vartheta_x\phi_x N(\alpha\beta)_x/N(\alpha)_x N(\beta)_x$.

Proof. Multiplying the two cases of (3.1) for α and β we obtain

$$\vartheta_x\phi_x\frac{\alpha_{[z^\star,x]}\beta_{[z^\star,x]}}{N(\alpha)_x N(\beta)_x} = \vartheta_z\phi_z\frac{\alpha_{[x^\star,z]}\beta_{[x^\star,z]}}{N(\alpha)_z N(\beta)_z}$$

for all x, $z \in \mathcal{P}$. Therefore $\pi_x = \vartheta_x\phi_x N(\alpha\beta)_x/N(\alpha)_x N(\beta)_x$ solves the case of (3.1) for $\alpha\beta$. □

We are now ready to prove Theorem 1.4

Proof of Theorem 1.4. Suppose that $\gamma = \alpha\beta$ factorizes as in the theorem. By Lemma 3.2 the α- and β-weighted involutive walks are reversible with respect to invariant distributions proportional to $\alpha_{x^\star}N(\alpha)_x$ and $N(\beta)_x$, respectively. Therefore, by Lemma 3.3, the γ-weighted involutive walk is reversible with respect to an invariant distribution proportional to

$$\alpha_{x^\star}N(\alpha)_x N(\beta)_x \times \frac{N(\alpha\beta)_x}{N(\alpha)_x N(\beta)_x} = \alpha_{x^\star}N(\alpha\beta)_x$$

as required.

Suppose conversely that the γ-weighted involutive walk is reversible with respect to an invariant distribution π. We must define an atomic weight α so that γ/α is $\star$-symmetric. By (3.1) we have

$$\pi_x\frac{\gamma_{[y,x]}}{N(\gamma)_x} = \pi_{y^\star}\frac{\gamma_{[x^\star,y^\star]}}{N(\gamma)_{y^\star}}$$

for each $x, y \in n$. By the recurrence assumption, π is strictly positive. We therefore require an atomic weight α such that $\gamma_{[y,x]}/\alpha_y = \gamma_{[x^\star,y^\star]}/\alpha_{x^\star}$ for all states x and y. Comparing with the displayed equation, we see that

$$\frac{\alpha_y}{\alpha_{x^\star}} = \frac{N(\gamma)_x}{\pi_x}\frac{\pi_{y^\star}}{N(\gamma)_{y^\star}}$$

and so, up to a multiplicative constant, the unique choice is $\alpha_y = \pi_{y^\star}/N(\gamma)_{y^\star}$. We therefore define

$$\alpha_{[y,x]} = \frac{\pi_{y^\star}}{N(\gamma)_{y^\star}}, \quad \beta_{[y,x]} = \frac{N(\gamma)_{y^\star}}{\pi_{y^\star}}\gamma_{[y,x]}.$$

By construction, α is atomic, β is $\star$-symmetric and $\gamma = \alpha\beta$. It only remains to check property (2) in Definition 1.1. This property always holds for an atomic weight. Suppose that $y \le x \le x'$. By property (2) for γ, we have $\beta_{[y,x]} = \frac{N(\gamma)_{y^\star}}{\pi_{y^\star}}\gamma_{[y,x]} \le \frac{N(\gamma)_{y^\star}}{\pi_{y^\star}}\gamma_{[y,x']} = \beta_{[y,x']}$, as required. □

4. The $\gamma^{(c)}$-weighted involutive walk

Let $c \in \mathbb{R}^{>0}$. Recall that $\gamma^{(c)}$ is the weight defined by $\gamma^{(c)}_{[y,x]} = \binom{x}{y}c^{x-y}$. Since $N(\gamma^{(c)})_x = (c+1)^x$ we have

$$H(\gamma^{(c)})_{xy} = \binom{x}{y}\frac{c^{x-y}}{(c+1)^x}. \tag{4.1}$$

In particular, $H(\gamma^{(1)})$ is the stochastic version of the Pascal's Triangle matrix. The matrices $P(\gamma^{(1)})$, $P(\gamma^{(2)})$ and $P(\gamma^{(\frac{1}{2})})$ are shown below when $n = 4$.

$$\begin{pmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \frac{1}{2} & \frac{1}{2} \\ \cdot & \frac{1}{4} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \end{pmatrix}, \quad \begin{pmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \frac{1}{3} & \frac{2}{3} \\ \cdot & \frac{1}{9} & \frac{4}{9} & \frac{4}{9} \\ \frac{1}{27} & \frac{2}{9} & \frac{4}{9} & \frac{8}{27} \end{pmatrix}, \quad \begin{pmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \frac{2}{3} & \frac{1}{3} \\ \cdot & \frac{4}{9} & \frac{4}{9} & \frac{1}{9} \\ \frac{8}{27} & \frac{2}{9} & \frac{4}{9} & \frac{1}{27} \end{pmatrix},$$

The symmetry in the matrices $P(\gamma^{(a,b)})$ under swapping a and b seen after (1.2) becomes a symmetry in the matrices $P(\gamma^{(c)})$ under swapping c and $1/c$. Observe that the eigenvalues of $H(\gamma^{(c)})$, and so the anti-diagonal entries of $P(\gamma^{(c)})$, are $1/(c+1)^d$ for $d \in n$.

4.1. **Reversibility and invariant distribution.** The interpretation of $P(\gamma^{(c)})$ as the limiting case of $P(\gamma^{(a,ac)})$ perhaps makes it less surprising that $\gamma^{(c)}$ is factorizable. From the required form $\binom{x}{y}c^{x-y} = \alpha_y\beta_{[y,x]}$, where β is $\star$-symmetric, we require α_y such that $x!c^{x-y}/y!(x-y)!\alpha_y$ is invariant under $(x,y) \mapsto (y^\star, x^\star)$. This already holds for c^{x-y} and $(x-y)!$, and we may cancel $y!$ and introduce the necessary factor of $y^\star!$ by taking α_y proportional to $1/y!(n-1-y)!$. We therefore define $\alpha_y = \binom{n-1}{y}$ as a weight with domain n, and find that

$$\frac{\binom{x}{y}c^{x-y}}{\binom{n-1}{y}} = \frac{x!(n-1-y)!}{(x-y)!(n-1)!}c^{x-y} \tag{4.2}$$

is indeed $\star$-symmetric. By Theorem 1.4, the $\gamma^{(c)}$-weighted involutive walk on n is reversible with unique invariant distribution proportional to $\alpha_{x^\star}N(\gamma^{(c)})_x$, that is

$$\pi_x \propto \binom{n-1}{x}\Big(\frac{c+1}{c}\Big)^x.$$

4.2. **The polynomial basis of $\mathbb{R}^n$.** We now introduce our approach to the anti-diagonal eigenvalue property. Example 2.4 gives a quick overview of the method. Fix $n \in \mathbb{N}$. For $e \in \{1, \dots, n\}$, let V_e denote the subspace of $\mathbb{R}^n$ of all column vectors of the form $\big(f(0), \dots, f(n-1)\big)^t$ where f is a polynomial of degree strictly less than e. Thus V_e is e-dimensional. Let $V_0 = \{0\}$. Define

$$v(d) = \Big(\binom{0}{d}, \binom{1}{d}, \dots, \binom{n-1}{d}\Big)^t \tag{4.3}$$

for $0 \leq d \leq n-1$. Hence, numbering positions in vectors from 0, we have $v(d)_x = \binom{x}{d}$ for $1 \leq x < n$. Let

$$w(d) = \left(\binom{n-1}{d}, \ldots, \binom{1}{d}, \binom{0}{d}\right)^t$$

denote the reversal of $v(d)$. Thus $w(d)_x = \binom{n-1-x}{d}$ for $0 \leq x < n$. Recall that $B(n)$ denotes the $n \times n$ Pascal's Triangle matrix. It has columns $v(0), \ldots, v(n-1)$. In particular $B(n)$ is unitriangular, so $v(0), \ldots, v(n-1)$ is a basis of $\mathbb{R}^n$. Hence so is $w(0), \ldots, w(d-1)$. Moreover, since $\binom{X}{d}$ and $\binom{n-1-X}{d}$ are each polynomials of degree d in the indeterminate X, we have

$$V_e = \langle v(0), \ldots, v(e-1) \rangle = \langle w(0), \ldots, w(e-1) \rangle \tag{4.4}$$

for each $e \in \{1, \ldots, n\}$.

Lemma 4.1. *For each $d \in n$ we have $w(d) \in (-1)^d v(d) + V_d$.*

Proof. Observe that the coefficient of X^d in $\binom{n-1-X}{d}$ is $(-1)^d/d!$, whereas the coefficient of X^d in $\binom{X}{d}$ is $1/d!$. Therefore $\binom{n-X}{d} = (-1)^d\binom{X}{d} + f$ where f is a polynomial of degree strictly less than d. The lemma now follows from (4.4). □

The following lemma is proved, in a different way, as Lemma 2.9 in [6]; we include a proof to make the article self-contained.

Lemma 4.2. *Suppose that H is a $n \times n$ matrix such that $Hv(d) = \mu_d v(d)$ for each d. Then*

$$H = B(n)\mathrm{Diag}(\mu_0, \ldots, \mu_{n-1})B(n)^{-1}$$

and $HJ(n)$ has eigenvalues $(-1)^d\mu_d$ for $d \in n$. Moreover H is lower-triangular, $HJ(n)$ is anti-triangular and H has the global anti-diagonal eigenvalue property.

Proof. Since $J(n)v(d) = w(d)$, Lemma 4.1 implies that $HJ(n)w(d) = Hv(d) = \lambda_d v(d) \in (-1)^d w(d) + \langle w(0), \ldots, w(d-1) \rangle$ for each $d \in n$. Hence $HJ(n)$ has eigenvalues $(-1)^d\mu_d$, as claimed. In terms of the matrix $B(n)$ we have $HB(n) = D(\mu_0, \ldots, \mu_{n-1})B(n)$, so $H = B(n)^{-1}D(\mu_0, \ldots, \mu_{n-1})B(n)$ is lower-triangular, and therefore $HJ(n)$ is anti-triangular. Hence H has the anti-diagonal eigenvalue property, and since the analogous result holds for all $m < n$, it has the (stronger) global anti-diagonal eigenvalue property. □

4.3. **Proofs of Theorem 1.5 and Proposition 1.6 for $\gamma^{(c)}$.** We now apply the general result in Lemma 4.2 to show that $H(\gamma^{(c)})$ has the anti-diagonal eigenvalue property, and so its eigenvalues are $(-1)^d/c^d$ for $d \in n$.

Lemma 4.3. *Let $c \in \mathbb{R}^{>-1}$. Then*

$$H(\gamma^{(c)})v(d) = v(d)/(c+1)^d.$$

Proof. Since $H(\gamma^{(c)})_{xy} = \binom{x}{y}c^{x-y}$ we have

$$\big(H(\gamma^{(c)})v(d)\big)_x = \sum_{y=0}^{n-1}\binom{x}{y}\frac{c^{x-y}}{(c+1)^x}\binom{y}{d} = \binom{x}{d}\frac{c^{x-d}}{(c+1)^x}\sum_{y=0}^{n-1}\binom{x-d}{y-d}\frac{1}{c^{y-d}}$$
$$= \binom{x}{d}\frac{c^{x-d}}{(c+1)^x}\Big(1+\frac{1}{c}\Big)^{x-d} = \binom{x}{d}\frac{1}{(c+1)^d} = \frac{v(d)_x}{(c+1)^d}$$

as required. □

This proves Proposition 1.6 for $H(\gamma^{(c)})$. By Lemma 4.2, $H(\gamma^{(c)})$ has the global anti-diagonal eigenvalue property and so the eigenvalues of $P(\gamma^{(c)})$ are $(-1)^d/(c+1)^d$, as claimed in Theorem 1.5. Since $\gamma^{(c)}$ is a strictly positive reversible weight, the remaining claims in Theorem 1.5 for the $\gamma^{(c)}$-weighted involutive walk follow from Theorems 1.3 and Theorem 1.4.

5. The $\gamma^{(a,b)}$-weighted and $\delta^{(a',b')}$-weighted involutive walks

Recall that for a, $b \in \mathbb{R}^{>-1}$ we defined the strictly positive weight $\gamma^{(a,b)}$ with domain $\mathbb{N}_0$ by $\gamma^{(a,b)}_{[y,x]} = \binom{y+a}{y}\binom{x-y+b}{x-y}$. It is convenient in this section to extend this definition to define $\gamma^{(a,b)}$, as above, for all a, $b \in \mathbb{R}$.

5.1. **Preliminaries on multisubsets.** A combinatorial interpretation of $\gamma^{(a,b)}$ is very helpful. Recall that $\binom{m+c-1}{c}$, or equally $(-1)^c\binom{-m}{c}$, is the number of c-multisubsets of a set of size m, or equivalently, the number of *chains* $0 \le y_1 \le \ldots \le y_c \le m-1$ in the ordinal m. Thus when $a, b \in \mathbb{N}_0$,

$$\gamma^{(a,b)}_{[y,x]} = \big|\{(y_1,\ldots,y_a,w_1,\ldots,w_b) : 0 \le y_1 \le \ldots \le y_a \le y \le w_1 \le \ldots \le w_b \le x\}\big|.$$

It is convenient to write $\big(\!\binom{m}{c}\!\big)$ for $\binom{m+c-1}{c}$. Note that $\big(\!\binom{m}{c}\!\big) = \binom{m+c-1}{c} = \binom{m+c-1}{m-1} = \big(\!\binom{c+1}{m-1}\!\big)$, and so, unlike the normal binomial coefficient, $\big(\!\binom{m}{c}\!\big)$ is, separately, a polynomial function of both m and c. We use this fact repeatedly below to deduce results on $\gamma^{(a,b)}$ for general a, $b \in \mathbb{R}$ from the special cases when a, $b \in \mathbb{N}_0$. The following lemma collects some results relevant to integral a and b.

Lemma 5.1. *Let m, $n \in \mathbb{N}$.*

(i) *If $c \in \mathbb{N}_0$ then* $\sum_{y=0}^{x}\big(\!\binom{y+1}{c}\!\big) = \big(\!\binom{x+1}{c+1}\!\big)$.
(ii) *If c, $d \in \mathbb{N}_0$ then* $\sum_{x=0}^{n-1}\big(\!\binom{n-x}{c}\!\big)\big(\!\binom{x+1}{d}\!\big) = \big(\!\binom{n}{c+d+1}\!\big)$.
(iii) *If c, $d \in \mathbb{N}_0$ then* $\sum_{y=0}^{x}\big(\!\binom{x-y+1}{c}\!\big)\binom{y}{d} = \binom{x+c+1}{c+d+1}$.
(iv) *If $c \in \mathbb{N}_0$ then* $\sum_{y=0}^{x}\big(\!\binom{x-y+1}{c}\!\big)(-1)^y\binom{m}{y} = (-1)^x\binom{m-c-1}{x}$.

Proof. For (i) we count $(c+1)$-multisubsets of $\{0,\ldots,x\}$: the $(c+1)$-multisubsets having y as their greatest element are in bijection with the c-multisubsets of $\{0,\ldots,y\}$; these are counted by $\big(\!\binom{y+1}{c}\!\big)$. For (ii), use $\big(\!\binom{z}{d}\!\big) = \big(\!\binom{d+1}{z-1}\!\big)$ for $x \in \mathbb{N}$ to rewrite the left-hand side as $\sum_{x=0}^{n-1}\big(\!\binom{c+1}{n-x-1}\!\big)\big(\!\binom{d+1}{x}\!\big)$. This counts the $(n-1)$-multisubsets of the disjoint union of sets of size $c+1$ and $d+1$, and so is $\big(\!\binom{c+d+2}{n-1}\!\big) = \big(\!\binom{n}{c+d+1}\!\big)$. For (iii), the non-zero summands

occur for $y \geq d$. Rewrite $\binom{y}{d}$ as $\left(\!\binom{y-d+1}{d}\!\right)$ and apply (ii) to the right-hand side of

$$\sum_{y=d}^{x} \left(\!\!\binom{x-y+1}{c}\!\!\right) \left(\!\!\binom{y-d+1}{d}\!\!\right) = \sum_{z=0}^{x-d} \left(\!\!\binom{x-d+1-z}{c}\!\!\right) \left(\!\!\binom{z+1}{d}\!\!\right)$$

to get $\left(\!\binom{x-d+1}{c+d+1}\!\right)$, which is $\binom{x+c+1}{c+d+1}$, as required. Finally for (iv), use $\left(\!\binom{x-y+1}{c}\!\right) = \binom{x-y+c}{c} = \binom{-c-1}{x-y}$ to rewrite the right-hand side as $(-1)^x \sum_{y=0}^{x} \binom{-c-1}{x-y}\binom{m}{y}$; by Vandermonde's convolution this is $(-1)^x \binom{m-c-1}{x}$. □

5.2. **Invariant distribution and eigenvalues of $P(\gamma^{(a,b)})$.** The function $\gamma^{(a,b)}$ factorizes as the product $\alpha^{(a)}\beta^{(b)}$ where $\alpha^{(a)}$ is defined by $\alpha^{(a)}_y = \binom{y+a}{y}$ and $\beta^{(b)}$ is defined by $\beta^{(b)}_{[y,x]} = \binom{x-y+b}{x-y}$. When a, $b \in \mathbb{R}^{>-1}$, $\alpha^{(a)}$ is a strictly positive atomic weight and $\beta^{(b)}$ is a strictly positive $\star$-symmetric weight. Moreover, since $\binom{c'-1}{y} = (c'-1)\dots(c'-y)/y!$ and $c'-(n-1) > 0$ if and only if $\lceil c' \rceil \geq n$, the weight $\delta^{(a',b')}$ with domain $\min(\lceil a' \rceil, \lceil b' \rceil)$ defined earlier by $\delta^{(a',b')}_{[y,x]} = \binom{a'-1}{y}\binom{b'-1}{x-y}$ has a similar factorization.

Lemma 5.2. *Let x, $y \in n$ with $y \leq x$. For a, $b \in \mathbb{R}$ we have*

(i) $\alpha^{(a)}_{[y,x]} = \left(\!\binom{y+1}{a}\!\right)$ *and* $N(\alpha^{(a)})_x = \left(\!\binom{x+1}{a+1}\!\right)$*;*

(ii) $\beta^{(b)}_{[y,x]} = \left(\!\binom{x-y+1}{b}\!\right)$ *and* $N(\beta^{(b)})_x = \left(\!\binom{x+1}{b+1}\!\right)$*;*

(iii) $\gamma^{(a,b)}_{[y,x]} = \left(\!\binom{y+1}{a}\!\right)\left(\!\binom{x-y+1}{b}\!\right)$ *and* $N(\gamma^{(a,b)})_x = \left(\!\binom{x+1}{a+b+1}\!\right)$.

Proof. The first parts are immediate from the (extended) definition. Since $N(\alpha^{(a)})_x$, $N(\beta^{(b)})_x$ and $N(\gamma^{(a,b)}_x$ are polynomials in a and b, it suffices to prove the remaining results when a and b are integral. In this case the normalization factors $N(\alpha^{(a)})_x$ and $N(\beta^{(b)})_x$ follow immediately from Lemma 5.1(i). For $\gamma^{(a,b)}$ we have

$$N(\gamma^{(a,b)})_x = \sum_{y\in[x]} \gamma^{(a,b)}_{[y,x]} = \sum_{y=0}^{x} \left(\!\!\binom{y+1}{a}\!\!\right) \left(\!\!\binom{x-y+1}{b}\!\!\right) = \left(\!\!\binom{x+1}{a+b+1}\!\!\right)$$

by Lemma 5.1(ii). □

Restated in more usual notation, (iii) says that $N(\gamma^{(a,b)})_x = \binom{a+b+1+x}{x}$. Thus if a, $b \in \mathbb{R}^{>-1}$ then

$$H(\gamma^{(a,b)})_{xy} = [x \geq y] \left(\!\!\binom{y+1}{a}\!\!\right) \left(\!\!\binom{x-y+1}{b}\!\!\right) \Big/ \left(\!\!\binom{x+1}{a+b+1}\!\!\right), \tag{5.1}$$

or equivalently, $H(\gamma^{(a,b)})_{xy} = [x \geq y]\binom{a+y}{y}\binom{b+x-y}{x-y}\Big/\binom{a+b+1+x}{x}$. This shows that the $n \times n$ matrix $H(\gamma^{(a,b)})$ is well defined whenever $a+b+2$, $\dots$, $a+b+n \neq 0$, or equivalently,

$$a + b \notin \{-2, \dots, -n\}. \tag{5.2}$$

For $\delta^{(a',b')}$ we simply compute

$$N(\delta^{(a',b')})_x = \sum_y \binom{a'-1}{y}\binom{b'-1}{x-y} = \binom{(a'-1)+(b'-1)}{x}$$

using Vandermonde's convolution. Therefore

$$H(\delta^{(a,b)})_{xy} = [x \geq y]\binom{a'-1}{y}\binom{b'-1}{x-y}\Big/\binom{(a'-1)+(b'-1)}{x}. \tag{5.3}$$

Since $N(\delta^{(a',b')})_x = (-1)^x N(\gamma^{(-a,-b)})$, we may also obtain $N(\delta^{(a,b)})_x$ from Lemma 5.1(iii). Hence the definition of $H(\delta^{(a,b)})$ in (5.3) agrees with that obtained from (5.1) by replacing a and b with $-a'$ and $-b'$.

Invariant distributions. By Theorem 1.4, if a, $b \in \mathbb{R}^{>-1}$, the unique invariant distribution π for the $\gamma^{(a,b)}$-weighted involutive walk on n is proportional to $\alpha^{(c)}_{n-1-x}N(\gamma^{(a,b)})$. Therefore, by Lemma 5.2,

$$\pi_x \propto \left(\!\!\binom{n-x}{a}\!\!\right)\left(\!\!\binom{x+1}{a+b+1}\!\!\right). \tag{5.4}$$

The normalization factor for π is $\left(\!\!\binom{n}{2a+b+2}\!\!\right)$ by Lemma 5.1(ii). Similarly for $\delta^{(a',b')}$ the unique invariant distribution on n, for $n \leq \min(\lceil a'\rceil, \lceil b'\rceil)$, is ϕ_x where

$$\phi_x \propto \binom{a'-1}{x}\binom{(a'-1)+(b'-1)}{x}. \tag{5.5}$$

Rewriting the first binomial as $\binom{a'-1}{a'-1-x}$ and applying Vandermonde's convolution, we see that the normalization factor is $\binom{2a'+b'-3}{n}$.

Spectrum of $P(\gamma^{(a,b)})$ and $P(\delta^{(a',b')})$. By the remark after (5.3), we may deal with the two cases in a uniform way. Whenever (5.2) holds, the eigenvalues of $H(\gamma^{(a,b)})$ are its diagonal entries, namely $\lambda_0^{(a,b)}, \ldots, \lambda_{n-1}^{(a,b)}$ where

$$\lambda(\gamma^{(a,b)})_d = \left(\!\!\binom{d+1}{a}\!\!\right)\Big/\left(\!\!\binom{d+1}{a+b+1}\!\!\right). \tag{5.6}$$

The following lemma, analogous to Lemma 4.3, reveals the eigenvectors. Recall from (4.3) that $v(d) \in \mathbb{R}^n$ is the column vector with entries $v(d)_x = \binom{x}{d}$.

Lemma 5.3. *Let a, $b \in \mathbb{R}$ and let $n \in \mathbb{N}$ be such that $a+b \notin \{-2, \ldots, -n\}$. Then*

$$\big(H(\gamma^{(a,b)})v(d)\big)_x = \frac{\left(\!\!\binom{d+1}{a}\!\!\right)}{\left(\!\!\binom{d+1}{a+b+1}\!\!\right)}v(d)_x.$$

Proof. We have

$$\big(H(\gamma^{(a,b)})v(d)\big)_x = \sum_{y=0}^{x}\left(\!\!\binom{y+1}{a}\!\!\right)\left(\!\!\binom{x-y+1}{b}\!\!\right)\binom{y}{d}\Big/\left(\!\!\binom{x+1}{a+b+1}\!\!\right). \tag{5.7}$$

When a, $b \in \mathbb{N}_0$, the numerator on the right-hand side is

$$\begin{aligned}\sum_{y=0}^{x}\binom{y+a}{a}\binom{y}{d}\left(\!\!\binom{x-y+1}{b}\!\!\right) &= \binom{a+d}{a}\sum_{y=0}^{x}\binom{y+a}{a+d}\left(\!\!\binom{x-y+1}{b}\!\!\right)\\ &= \binom{a+d}{a}\binom{x+a+b+1}{a+b+d+1}\end{aligned}$$

by $\binom{y+a}{a}\binom{y}{d} = \binom{a+d}{a}\binom{y+a}{a+d}$ and an instance of Lemma 5.1(iii). We now use $\binom{x+a+b+1}{a+b+d+1}/\left(\!\binom{x+1}{a+b+1}\!\right) = \binom{x+a+b+1}{a+b+d+1}/\binom{x+a+b+1}{a+b+1} = \binom{x}{d}/\binom{a+b+d+1}{a+b+1}$ to get that the right-hand side of (5.7) is

$$\binom{a+d}{a}\binom{x}{d}/\binom{a+b+d+1}{a+b+1} = \binom{x}{d}\left(\!\!\binom{d+1}{a}\!\!\right)/\left(\!\!\binom{d+1}{a+b+1}\!\!\right)$$
$$= v(d)_x\left(\!\!\binom{d+1}{a}\!\!\right)/\left(\!\!\binom{d+1}{a+b+1}\!\!\right).$$

This proves the equality claimed in the lemma for each position of the vector when a, $b \in \mathbb{N}_0$, and since, by multiplying through by the non-zero quantities $N(\gamma^{(a,b)})_x$ and $\left(\!\binom{d+1}{a+b+1}\!\right)$ each side becomes polynomial in a and b, it therefore holds for all claimed a and b. □

This proves Proposition 1.6 for the down-step matrices $H(\gamma^{(a,b)})$ and $H(\delta^{(a',b')})$; note that for the latter we have $a' = -a$, $b' = -b$ and $n \geq \min(\lceil a' \rceil, \lceil b' \rceil)$ and so the hypothesis for Lemma 4.2 holds.

5.3. **Proof of Theorems 1.5 for $\gamma^{(a,b)}$ and $\delta^{(a,b)}$.** Combining Lemmas 5.3 and 4.2 we see that whenever (5.2) holds, $H(\gamma^{(a,b)})$ has the global anti-diagonal property. Hence the eigenvalues of $P(\gamma^{(a,b)}) = H(\gamma^{(a,b)})J(n)$ are $(-1)^d\left(\!\!\binom{d+1}{a}\!\!\right)/\left(\!\!\binom{d+1}{a+b+1}\!\!\right)$ for $d \in n$. Rewriting this as $(-1)^d\binom{a+d}{d}/\binom{a+b+d+1}{d}$ we obtain the eigenvalues $\lambda(\gamma^{(a,b)})_d$ claimed in Theorem 1.5 for $P(\gamma^{(a,b)})$. For $P(\delta^{(a',b')})$ we simply substitute $a' = -a$ and $b' = -b$ and use $\binom{z}{d} = \binom{d-z-1}{z}$ to get $(-1)^d\binom{a'-1}{d}\binom{(a'-1)+(b'-1)}{d}$, again as claimed. The rest of the theorem follows from Theorem 1.3 and, using the factorizations at the start of this section, Theorem 1.4.

Further remarks. Since

$$\frac{\lambda(\gamma^{(a,b)})_d}{\lambda(\gamma^{(a,b)})_{d+1}} = \frac{\binom{d+a-1}{d-1}/\binom{d+a+b}{d-1}}{\binom{d+a}{d}/\binom{d+a+b+1}{d}} = \frac{d}{d+a}\,\frac{d+a+b+1}{d} = \frac{a+b+1+d}{a+d}$$

the sequence of eigenvalues of $P(\gamma^{(a,b)})$ is decreasing in absolute value. The same holds for $P(\delta^{(a',b')})$ rewriting the right-hand side as $(a'+b'-d-1)/(a'-d)$. The ratio above also implies that $\lambda(\gamma^{(a,b)})_d \to 0$ as $d \to \infty$. The rate of convergence of the involutive walk is controlled by the second largest eigenvalue (in absolute value) of $P(\gamma^{(a,b)})$, namely $\lambda(\gamma^{(a,b)})_1 = (a+1)/(a+b+2)$; again for $P(\delta^{(a,b)})$, the rewriting $(a'-1)/\big((a'-1)+(b'-1)\big)$ is most convenient.

5.4. **Eigenvectors of $P(\gamma^{(a,b)})$.** Fix a, $b \in \mathbb{R}^{>-1}$. By a basic result, the right eigenvectors of $P(\gamma^{(a,b)})$ are orthogonal with respect to the inner product $\langle v, w\rangle = \sum_{x=0}^{n-1}\pi_x v_x w_x$ where $\pi_x = \left(\!\!\binom{n-x}{a}\!\!\right)\left(\!\!\binom{x+1}{a+b+1}\!\!\right)/\left(\!\!\binom{n}{2a+b+2}\!\!\right)$ is the invariant distribution in (5.4). For $0 \leq d < n$, let $g(d) \in \mathbb{R}^n$ be the right-eigenvector of $P(\gamma^{(a,b)})$ with eigenvalue $(-1)^d\lambda(\gamma^{(a,b)})_d$, normalized with respect to $\langle\ ,\ \rangle$. We have seen that $g(d) \in \langle v(0), \ldots, v(d)\rangle$ for $0 \leq d < n$.

Therefore $g(0), \dots, g(n-1)$ are the Gram–Schmidt orthonormalizations of $v(0), \dots, v(n-1)$, with respect to $\langle\ ,\ \rangle$. In particular $g(0)$ is constant. A routine calculation shows that, up to a constant factor, $g(1)_x = (a+b+2)(n-1)-(2a+b+3)x$ for $0 \le x < n$; extending the $\propto$ notation, we write this as $g(1)_x \propto (a+b+2)(n-1)-(2a+b+3)x$. Since the eigenvalues $(-1)^d\lambda(\gamma^{(a,b)})_d$ of $P(\gamma^{(a,b)})$ are distinct, we also have

$$g(e) \in \Big\langle \prod_{d=0}^{e-1} \big(P(\gamma^{(a,b)}) - (-1)^d\lambda(\gamma^{(a,b)})_d\big) v(e) \Big\rangle$$

for $0 \le e < n$. Using either of these observations it is easy to compute the right-eigenvectors for any particular n. By Remark 9.8 below, the eigenvectors $g(d)$ are the discrete analogue of the Jacobi polynomials arising in the theory of special functions and eigenvalue problems. We believe they deserve further combinatorial study.

The left-eigenvectors can be obtained using the following lemma; we include a proof for completeness.

Lemma 5.4. *Let P be the transition matrix of a finite reversible Markov chain with strictly positive invariant distribution proportional to π. Let $v \in \mathbb{R}^n$ and let $w \in \mathbb{R}^n$ be defined by $w_x = \pi_x v_x$. Then v is a right-eigenvector for P with eigenvalue λ if and only if w is a left-eigenvector for P with eigenvalue λ.*

Proof. We have

$$(wP)_y = \sum_x w_x P_{xy} = \sum_x v_x \pi_x P_{xy} = \sum_x v_x \pi_y P_{yx}$$

where the final equality uses the detailed balance equation $\pi_x P_{xy} = \pi_y P_{yx}$. Now v is a right-eigenvector for P with eigenvalue λ if and only if the right-hand side is $\lambda \pi_y v_y$ for each y. Since $w_y = \pi_y v_y$ and $\pi_y > 0$, this holds if and only if v is a left-eigenvector for P with eigenvalue λ. □

Remarkably enough, the left-eigenspace for the final eigenvalue $\lambda(\gamma^{(a,b)})_{n-1} = (-1)^{n-1}\big(\!\binom{n}{a}\!\big)/\big(\!\binom{n}{a+b+1}\!\big)$ does not depend on either a or b; up to signs, it is a row of Pascal's Triangle.

Proposition 5.5. *Let $w_x = (-1)^x\binom{n-1}{x}$. Then for all a, $b \in \mathbb{R}^{>-1}$ we have $wP = (-1)^{n-1}\big(\!\binom{n}{a}\!\big)/\big(\!\binom{n}{a+b+1}\!\big)w$.*

Proof. Using $P(\gamma^{(a,b)})J(n) = H(\gamma^{(a,b)})$, we have $P(\gamma^{(a,b)})_{xy} = H(\gamma^{(a,b)})_{xy'}$ where $y' = n-1-y$. Hence by Lemma 5.2(iv), it is equivalent to show that

$$\sum_{x=y'}^{n} \frac{\big(\!\binom{x-y'+1}{b}\!\big)\big(\!\binom{y'+1}{a}\!\big)}{\big(\!\binom{x+1}{a+b+1}\!\big)}(-1)^x\binom{n-1}{x} = \frac{\big(\!\binom{n}{a}\!\big)}{\big(\!\binom{n}{a+b+1}\!\big)}(-1)^{n-1+y'}\binom{n-1}{y'}.$$

Observe that when a, $b \in \mathbb{N}$, we have $\binom{n-1}{x}/\left(\!\binom{x+1}{a+b+1}\!\right) = \binom{n-1}{x}/\binom{x+a+b+1}{a+b+1} = \binom{n+a+b}{x+a+b+1}/\binom{n+a+b}{a+b+1} = \binom{n+a+b}{n-1-x}/\left(\!\binom{n}{a+b+1}\!\right)$. Clearing denominators this becomes a polynomial identity, so it holds for all a, $b \in \mathbb{R}^{>-1}$. We use this to rewrite the left-hand side as

$$\frac{\left(\!\binom{y'+1}{a}\!\right)}{\left(\!\binom{n}{a+b+1}\!\right)} \sum_{w=0}^{n-1-y'} \left(\!\binom{n-w-y'}{b}\!\right) \binom{n+a+b}{w} (-1)^{n-1-w}.$$

By an instance of Lemma 5.1(iv), this is $\left(\!\binom{y'+1}{a}\!\right)(-1)^{n-1-y'}\binom{n+a-1}{n-1-y'}/\left(\!\binom{n}{a+b+1}\!\right)$. Now observe that

$$\left(\!\binom{y'+1}{a}\!\right)\binom{n+a}{n-1-y'} = \binom{y'+a}{y'}\binom{n+a-1}{n-1-y'} = \left(\!\binom{n}{a}\!\right)\binom{n-1}{y'}$$

which gives the relevant factors in the right-hand side. □

The corollary for right eigenvectors is worth noting.

Corollary 5.6. *Let a, $b \in \mathbb{R}^{>-1}$. Then*

$$g(n-1)_x \propto (-1)^x \binom{n-1}{x} / \left(\!\binom{n-x}{a}\!\right)\left(\!\binom{x+1}{a+b+1}\!\right).$$

Proof. This is immediate from Lemma 5.4 and the invariant distribution in (5.4). □

In particular, if $a = 0$ then from

$$\binom{n+b}{x+b+1}\left(\!\binom{x+1}{b+1}\!\right) = \binom{n+b}{x+b+1}\binom{x+b+1}{b+1} = \binom{n+b}{b+1}\binom{n-1}{y}$$

we see that $g(n-1)_x \propto (-1)^x\binom{n+b}{x+b+1}$. A slightly more lengthy argument shows that if $a = 1$ then $g(n-1)_x \propto (-1)^x\binom{n+b+2}{x+b+2}$. However there do not appear to be such 'division-free' formulae for higher a, or general $a \in \mathbb{R}^{>-1}$.

5.5. **The involutive random walk on subsets and proof of Corollary 1.9.** We conclude this section with a proof of Corollary 1.9. Let $0 < p < 1$. The $\gamma^{(-(1-p),-p)}$-weighted involutive walk on 2 has transition matrix

$$Q = \begin{pmatrix} \cdot & 1 \\ p & 1-p \end{pmatrix}.$$

By special cases of Theorem 1.4 and Theorem 1.5 the invariant distribution is $\left(\frac{p}{1+p}, \frac{1}{1+p}\right)$ and the eigenvalues are 1 and $-p$. Observe that Q is the transition matrix of the random walk in Corollary 1.9 when $m = 1$; $0 \in \{0,1\}$ corresponds to the empty set and $1 \in \{0,1\}$ to $\{1\}$.

Generally, if P is the matrix of an involutive random walk on n and P' is the matrix of an involutive random walk on n' then, indexing the entries of $P \otimes P'$ by $n \times n'$, we have $(P \otimes P')_{(x,x'),(z,z')} = P_{(x,z)}P_{(x',z')}$, and so $P \otimes P'$ is the transition matrix of the random walk on $n \times n'$ in which, starting from (x, x'), we choose (y, y') with $y \in [x]$, $y' \in [x']$ with probability $P_{(x,y^\star)}P_{(x',y'^\star)}$ and then step to $(y^\star, y'^\star)$. It is easily seen that if the random

walks on $n \times n'$ are irreducible, reversible, recurrent and ergodic then so is the random walk on $n \times n'$. In particular, the involutive random walk on subsets of $\{1, \dots, m\}$ in Corollary 1.9 has transition matrix $Q^{\otimes m}$. Its unique invariant distribution π is therefore $\left(\frac{p}{1+p}, \frac{1}{1+p}\right)^{\otimes m}$; by the identification at the end of the previous paragraph we have $\pi_X = \left(\frac{p}{1+p}\right)^{m-|X|}\left(\frac{1}{1+p}\right)^{|X|}$ for any $X \subseteq \{1, \dots, m\}$. Therefore $\pi_X = p^{m-|X|}/(1+p)^m$ as claimed. It is easily seen, either directly or by Proposition 5.5, that the left-eigenvector of Q with eigenvalue $-p$ is $(1, -1)$. Therefore the $(-p)^e$-eigenspace is spanned by all m-fold tensor products of $(p, 1)$ and $(1, -1)$ and has dimension $\binom{m}{e}$. This completes the proof of Corollary 1.9.

6. Transition matrices with the global anti-diagonal eigenvalue property

Given a sequence λ_0, λ_1, …, λ_{n-1} of real numbers, let H^λ be the matrix $B(n)\mathrm{Diag}(\lambda_0, \dots, \lambda_{n-1})B(n)^{-1}$. By the main result of [6], an $n \times n$ matrix has the global anti-diagonal property if and only if it is equal to some H^λ. Let $P^\lambda = H^\lambda J(n)$ be the anti-triangular matrix obtained from H^λ. We require the following description of H^λ.

Lemma 6.1. *Let $\lambda_0, \lambda_1, \dots, \lambda_{n-1} \geq \mathbb{R}^{>-1}$ and let $x, y \in n$.*

(i) *We have $H^\lambda_{xy} = \binom{x}{y}\sum_{e=0}^{x-y}(-1)^e\binom{x-y}{e}\lambda_{y+e}$.*

(ii) *If $x \geq y$ then*

$$H^\lambda_{x+1,y}\binom{x+1}{y}^{-1} = H^\lambda_{xy}\binom{x}{y}^{-1} - H^\lambda_{x+1,y+1}\binom{x+1}{y+1}^{-1}.$$

Proof. Let $S(n)$ be the signed Pascal's Triangle matrix with entries $S(n)_{xw} = (-1)^{x+w}\binom{x}{w}$. The entry in row x and column y of $B(n)\mathrm{Diag}(\lambda_0, \dots, \lambda_{n-1})S(n)$ is

$$\begin{aligned}\sum_{w=0}^{n-1}\binom{x}{w}\lambda_w(-1)^{w+y}\binom{w}{y} &= \sum_{w=y}^{n-1}\binom{x}{y}(-1)^{w+y}\binom{x-y}{w-y}\lambda_w \\ &= \binom{x}{y}\sum_{e=0}^{x-y}(-1)^e\binom{x-y}{e}\lambda_{y+e}.\end{aligned}$$

Taking $\lambda_d = 1$ for all $d \in n$ and using that $\sum_{e=0}^{m}(-1)^e\binom{m}{e} = [m = 0]$, in Iverson bracket notation, it follows that $S(n) = B(n)^{-1}$. Hence the entry of H^λ is as claimed in (i). For (ii), let $k = x - y \in \mathbb{N}$ and observe that by (i), the coefficients of λ_{y+e} in $H^\lambda_{xy}\binom{x}{y}^{-1}$, $H^\lambda_{x+1,y+1}\binom{x+1}{k}^{-1}$ and $H^\lambda_{x+1,y}\binom{x+1}{y}^{-1}$ are $(-1)^e\binom{k}{e}$, $(-1)^{e-1}\binom{k}{e-1}$ and $(-1)^e\binom{k+1}{e}$ respectively. Therefore (ii) follows from the basic identity $\binom{k}{e} + \binom{k}{e-1} = \binom{k+1}{e}$. □

6.1. **When is P^λ a transition matrix?** The introduction discusses the connection between the results in this subsection and the results on half-infinite matrices in [6]. Our approach, using the following lemma, is more elementary.

Lemma 6.2. *Let $\lambda_0, \lambda_1, \ldots, \lambda_{n-1} \in \mathbb{R}$. Then*

(i) *H^λ is non-negative if and only if its bottom row is non-negative;*

(ii) *if H^λ is non-negative and $H^\lambda_{xy} = 0$ then $H^\lambda_{x'y} = 0$ for all $x' \geq x$.*

Proof. As a technical tool, we define F^λ to be the lower-triangular $n \times n$ matrix with entries $F^\lambda_{xy} = \binom{x}{y}^{-1} H^\lambda_{xy}$ for $x \geq y$ and $F^\lambda_{xy} = 0$ for $x < y$. Observe that by Lemma 6.1(ii), the entries of F satisfy the recurrence

$$F^\lambda_{x+1,y} = F^\lambda_{xy} - F^\lambda_{x+1,y+1}.$$

Hence if $F^\lambda_{xy} < 0$ then either $F^\lambda_{x+1,y} < 0$ or $F^\lambda_{x+1,y+1} < 0$, and so a negative entry in row x of F^λ implies a negative entry in row $x+1$ of F^λ. Therefore F^λ is non-negative if and only if its bottom row is non-negative, and the same holds for H^λ. This proves (i). If H^λ is non-negative then by the displayed recurrence relation we have $F^\lambda_{x+1,z} \leq F^\lambda_{x,z}$, hence the columns of F are weakly decreasing when read from top to bottom. This implies (ii). □

Proposition 6.3. *Let $\lambda_0, \ldots, \lambda_{n-1} \in \mathbb{R}$.*

(i) *The matrix P^λ is the transition matrix of a Markov chain if and only if $\lambda_0 = 1$ and $\sum_{e=0}^{k} (-1)^e \binom{k}{e} \lambda_{n-1-k+e} \geq 0$ for all $k \in n$.*

(ii) *The matrix P^λ is the transition matrix of an involutive random walk if and only if $\lambda_0 = 1$, there exists $m \leq n$ such that $\lambda_m = \ldots = \lambda_{n-1} = 0$ and $\sum_{e=0}^{k} (-1)^e \binom{k}{e} \lambda_{m-1-k+e} > 0$ for all $k \in m$. Moreover, in this case, if x, $y \in n$ then $H^\lambda_{xy} = 0$ if $y \geq m$ and $H^\lambda_{xy} > 0$ if $y < m$ and $x > y$.*

Proof. By Lemma 6.1(i), the entry in row $n-1$ and column y of H^λ is $\sum_{e=0}^{n-1-y} (-1)^e \binom{n-1-y}{e} \lambda_{y+e}$. By Lemma 6.2(i), the matrix is non-negative if and only if all these entries are non-negative. Taking $y = n-1-k$ gives the inequalities above. Writing $w^{(t)}$ for the column vector with 1 in position y and $v^{(y)}$ for column t of $B(n)$ we have $B(n)w^{(y)} = v^{(y)}$, and so $B(n)^{-1}v^{(y)} = w^{(y)}$. Therefore $H^\lambda v^{(0)} = \lambda_0 v^{(0)}$. It follows that every row of H^λ has sum λ_0. Hence H^λ is a transition matrix if and only if, in addition, $\lambda_0 = 1$. This proves (i).

Now suppose that H^λ is the transition matrix of the γ-weighted involutive walk. If all the entries of H^λ are strictly positive then the condition in (ii) holds taking $m = n$. Suppose that $H^\lambda_{xm} = 0$ for some x, $m \in n$ with $x \geq m$. Choose m minimal with this property. Then $\gamma_{[m,x]} = 0$ and by property (2) in Definition 1.1, we have $\gamma_{[m,m]} = 0$ and $H^\lambda_{mm} = 0$. By Lemma 6.2(ii), all entries in column m of H^λ are zero. It now follows from Lemma 6.1(ii) that $H^\lambda_{xy} = 0$ for all $x, y \in n$ with $x, y \geq m$. In particular, $\lambda_y = H^\lambda_{yy} = 0$ for all

$y \geq m$. Moreover, since m is minimal, $H^\lambda_{n-1,y} > 0$ for all $y < m$. Another application of Lemma 6.1(ii) shows that the condition in (ii) holds.

Suppose conversely that this condition holds. Since it is stronger than the condition in (i), H^λ is a transition matrix. In particular H^λ has non-negative entries, and so, by repeated applications of Lemma 6.1(ii), to row $n-2$, then row $n-3$, and so on, we have $H_{xy} = 0$ for all $x, y \in n$ with $x \geq y \geq m$. Moreover, if $H_{xy} = 0$ then by Lemma 6.2, $H_{n-1,y} = 0$, and so $y \geq m$. Therefore $H^\lambda_{xy} = 0$ for $x, y \in n$ with $x \geq y$ if and only if $y \geq m$. It easily follows that the function defined by $\gamma_{[y,x]} = H_{xy}$ satisfies property (2) in Definition 1.1. Since all the row sums of H^λ are 1, it also satisfies property (1). □

With the natural hypothesis that every state is accessible, Proposition 6.3(ii) has a simpler form. We repeat the statement from the introduction.

Corollary 1.7. *Let $\lambda_0, \ldots, \lambda_{n-1} \in \mathbb{R}$. The matrix P^λ is the matrix of a weighted involutive random walk in which all states are accessible if and only if $\lambda_0 = 1$ and $\sum_{e=0}^{k}(-1)^e\binom{k}{e}\lambda_{k^\star+e} > 0$ for all $k \in n$. Moreover, in this case $P^\lambda_{xz} > 0$ if and only if $x + z \geq n-1$.*

Proof. By Proposition 6.3(ii), if $H^\lambda_{n-1,y} = 0$ then the state y is inaccessible. Therefore $H^\lambda_{n-1,y} > 0$ for all y and the inequalities follow immediately from the proposition using that $n-1-k = k^\star$ and $P^\lambda_{xy} = H^\lambda_{xy^\star}$. □

We remark that since the entries immediately below the anti-diagonal of P^λ are $\lambda_0 - \lambda_1, 2(\lambda_1 - \lambda_2), \ldots, (n-1)(\lambda_{n-2} - \lambda_{n-1})$, the condition in Corollary 1.7 implies that $1 > \lambda_1 > \ldots > \lambda_{n-1}$. Similarly, from the entries below these, we get the convexity condition $(\lambda_{x-1} + \lambda_{x+1})/2 > \lambda_x$ for all $x \in n$ such that $1 \leq x < n-1$. The full condition is of course satisfied by the eigenvalues $\lambda(\gamma^{(a,b)})_d$ of the matrices $H(\gamma^{(a,b)})$ and it is worth noting the closed form for the sum. The proof of the following proposition sees two applications of the standard identity $\int_0^1 w^a(1-w)^b\,\mathrm{d}w = (a+b+1)^{-1}\binom{a+b}{a}^{-1}$, proved as part of Lemma 9.1 below.

Proposition 6.4. *Let a, $b \in \mathbb{N}$. Then*

$$\sum_{e=0}^{k}(-1)^k\binom{k}{e}\lambda(\gamma^{(a,b)})_{k^\star+e} = \frac{\binom{a+b+1}{a}}{\binom{a+b+n-1}{b+k}}\frac{b+1}{a+b+n}.$$

Proof. Using the formula $\lambda(\gamma^{(a,b)})_d = \binom{a+b}{d}/\binom{a+b+d+1}{a+b+1}$ mentioned after Theorem 1.5 we have

$$\sum_{e=0}^{k}(-1)^e\binom{k}{e}\lambda(\gamma^{(a,b)})_{n-1-k+e}$$
$$= \binom{a+b+1}{a}\sum_{e=0}^{k}(-1)^e\binom{k}{e}\frac{1}{\binom{a+b+n-1-k+e+1}{b+1}}$$

$$
\begin{aligned}
&= \binom{a+b+1}{a} \sum_{e=0}^{k} (-1)^e \binom{k}{e} \frac{1}{\binom{a+b+n-1-k+e}{b+k}} \frac{b+1}{a+b+n-k} \\
&= \binom{a+b+1}{a} (b+1) \sum_{e=0}^{k} (-1)^e \binom{k}{e} \int_0^1 w^{a+n-1-k+e} (1-w)^b \, \mathrm{d}w \\
&= \binom{a+b+1}{a} (b+1) \int_0^1 w^{a+n-1-k} (1-w)^{b+k} \, \mathrm{d}w \\
&= \binom{a+b+1}{a} (b+1) \frac{1}{\binom{a+n-1+b}{b+k} (a+n+b)}
\end{aligned}
$$

as required. □

We remark that essentially the same proof shows the result holds for all a, $b \in \mathbb{R}^{>-1}$, interpreting the binomial coefficients using the Γ-function. An alternative, even more elementary proof, that also gives this stronger result, simply uses induction on n; the details however are slightly fiddly, and it is less conceptual than the 'one-line' proof above.

6.2. **When is P^λ reversible?** By Theorem 1.5, the $\gamma^{(a,b)}$-, $\delta^{(a',b')}$- and $\gamma^{(c)}$-weighted involutive walks are reversible and their transition matrices have the global anti-diagonal eigenvalue property. This proves one direction of Theorem 1.8. The converse is the more surprising part.

Our starting hypothesis is that P is a reversible $n \times n$ transition matrix with the global anti-diagonal eigenvalue property in which every state is accessible. Suppose that P has anti-diagonal entries $P_{dd^\star} = (-1)^d \lambda_d$ for $d \in n$. By the main result of [6], $PJ(n)$ is equal to the matrix $H^\lambda = B(n)\mathrm{Diag}(\lambda_0, \ldots, \lambda_{n-1})B(n)^{-1}$ that we have been considering. By Proposition 1.6, $H(\gamma^{(a,b)})$, $H(\delta^{(a',b')})$ and $H(\gamma^{(c)})$ are each $B(n)DB(n)^{-1}$ where D is the diagonal matrix of the eigenvalues of each down-step matrix, again arranged in decreasing order. We therefore having the following key principle: if γ is the relevant weight $\gamma^{(a,b)}$, $\gamma^{(c)}$ or $\delta^{(a',b')}$ then

$$
(\star) \qquad P = P(\gamma) \iff \begin{array}{l} \text{for all } d \in n, \text{ the eigenvalue } \lambda_d \text{ of } PJ(n) \\ \text{is equal to the eigenvalue } \lambda_d(\gamma) \text{ of } H(\gamma). \end{array}
$$

Our proof of Theorem 1.8 is by induction on n, using the reversibility hypothesis to show that the three greatest eigenvalues $\lambda_0, \lambda_1, \lambda_2$ of H^λ determine all n eigenvalues $\lambda_0, \ldots, \lambda_{n-1}$, and hence that $(\star)$ holds. Throughout, we use the consequence of Proposition 6.3 that

$$
(\dagger) \qquad \lambda_0 = 1 \quad \text{and} \quad 1 \geq \lambda_1 \geq \ldots \geq \lambda_{n-1} > 0.
$$

The final inequality in $(\dagger)$ holds because $\lambda_{n-1} = P_{n-1,0}$, the 0 state can only be reached from $n-1$, and by hypothesis, all states are accessible.

While some more preliminaries are required, we immediately prove the base case, since this motivates several definitions we require. Throughout, we let $\mu = \lambda_1$ and $\nu = \lambda_2$.

Base case. By $(\star)$, it suffices to find a and b so that the the down-step matrix

$$H(\gamma^{(a,b)}) = \begin{pmatrix} 1 & \cdot & \cdot \\ \frac{a+1}{a+b+2} & \frac{b+1}{a+b+2} & \cdot \\ \frac{(a+1)(a+2)}{(a+b+2)(a+b+3)} & \frac{2(a+1)(b+1)}{(a+b+2)(a+b+3)} & \frac{(b+1)(b+2)}{(a+b+2)(a+b+3)} \end{pmatrix}$$

has eigenvalues 1, μ and ν. Therefore we require $(b+1)/(a+b+2) = \mu$ and $(b+1)(b+2)/(a+b+2)(a+b+3) = \nu$. Provided $\nu \neq \mu^2$, these equations have the unique solution specified by the functions

$$a(\mu,\nu) = \frac{\mu(\mu-\nu)}{\nu-\mu^2} - 1, \quad b(\mu,\nu) = \frac{(1-\mu)(\mu-\nu)}{\nu-\mu^2} - 1. \tag{6.1}$$

If $\nu > \mu^2$ then a, $b \in \mathbb{R}^{>-1}$ and by $(\star)$, $H^\lambda = H(\gamma^{(a,b)})$. If $\nu = \mu^2$ then either $\mu = \nu = 1$ and $P = J(3)$, or by $(\star)$, $H^\lambda = H(\gamma^{(c)})$ where $1/(c+1) = \mu$. If $\nu < \mu^2$ then a, $b < -1$ and setting $a' = -a$ and $b' = -b$, we have $H(\gamma^{(a,b)}) = H(\delta^{(a',b')})$, by the remark after (5.3), and hence, by $(\star)$, we have $H^\lambda = H(\delta^{(a',b')})$. In this final case, since by Lemma 6.1,

$$H^\lambda = \begin{pmatrix} 1 & \cdot & \cdot \\ 1-\mu & \mu & \cdot \\ 1-2\mu+\nu & 2(\mu-\nu) & 1 \end{pmatrix}$$

we must have $\nu \geq 2\mu - 1$. Observe that if $\tau \geq 0$ then

$$-b\big(\mu, 2\mu - 1 + \tau(1-\mu)\big) = 2 + \frac{\mu\tau}{(1-\mu)^2 - \tau}.$$

When $\tau = 0$ and so $\nu = 2\mu - 1$, we have $a' = \frac{1}{1-\mu}$ and $b' = 2$ and the 0 in the bottom left corner of H^λ means that the weight $\delta^{(\frac{1}{1-\mu},2)}$ is weak. Moreover, if $\mu = \frac{1}{2}$ then the state 0 is inaccessible, contrary to our hypothesis. Therefore $\frac{1}{1-\mu} > 2$ and $a' \in \mathbb{R}^{>2}$, as required in Theorem 1.8; note that $b' \not> n-1 = 2$, but instead $b' \in \mathbb{N}$, as permitted by this theorem. Otherwise $\nu > 2\mu - 1$ and by the displayed equation above, $b' > 2$.

Inductive claim. To make the induction go through, we must prove the following stronger version of Theorem 1.8 that takes into account that the exceptional case $\nu = 2\mu - 1$ and $a' = \frac{1}{1-\mu}$ is the first in an infinite family. Observe that if $b' \in \mathbb{N}$ then $\delta^{(a',b')}_{[y,x]} = \binom{a'-1}{y}\binom{b'-1}{x-y}$ vanishes whenever $x - y \geq b'$. Thus the matrix $P(\delta^{(a',b')})$ has exactly m non-zero anti-diagonal bands. This was seen in $m = 2$ in the base case, and in Examples 2.7(3) and 2.8. To find these exceptional parameters, we must first express b' as a function of m and ν. Note that for each $\mu < 1$, the function $b'_\mu(\nu) = -b(\mu,\nu)$ is increasing on the interval $[0,\mu^2)$, with $b'_\mu(0) = 1 + \frac{1-\mu}{\mu}$ and $b'_\mu(\nu) \to \infty$ as $\nu \to \mu^2$. Therefore if $\mu > \frac{1}{2}$, for each $m \in \mathbb{N}$ with $m \geq 2$ there exists a unique ν such that $b'_\mu(\nu) = m$, say $\nu_m(\mu)$. Let $a'_m(\nu) = -a(\mu, \nu_m(\mu))$. By

inverting b'_μ and substituting one gets the explicit formulae

$$\nu_m(\mu) = \frac{\mu(m\mu - 1)}{m - 2 + \mu}, \quad a'_m(\mu) = \frac{(m-2)\mu + 1}{1 - \mu}.$$

Note that $\nu_2(\mu) = 2\mu - 1$ and $a'_2(\mu) = \frac{1}{1-\mu}$. Since b'_μ is an increasing function, so is the sequence $\nu_m(\mu)$. (This can also be verified from $\nu_{m+1}(\mu) - \nu_m(\mu) = (1-\mu)^2\mu/(m+\mu-2)(m+\mu-1)$.) Moreover, since $b'_\mu(\nu) \to \infty$ as $\nu \to \mu^2$, the sequence $\nu_m(\mu)$ converges to μ as $m \to \infty$. Instead fixing m, the relevant values of n are then given the lemma below.

Lemma 6.5. *The weak weight $\delta^{(a',m)}$ defines an involutive walk on n in which every state is accessible if and only if $n \leq \lceil a' \rceil$ and $m > \frac{1-\mu}{\mu}n + 2 - \frac{1}{\mu}$.*

Proof. We observed before Theorem 1.8 that $n \leq \lceil a' \rceil$ is a necessary and sufficient condition for every state in the $\delta^{(a',m)}$-weighted involutive walk to be accessible. This gives the first claim, and also that $n \leq \lceil \frac{(m-2)\mu+1}{1-\mu} \rceil$. Since $n \leq \lceil r \rceil$ if and only if $n < r + 1$ for any $n \in \mathbb{N}$ and $r \in \mathbb{R}$, an equivalent condition is $(1-\mu)n < (m-2)\mu + 1$ which rearranges to $m > \frac{1-\mu}{\mu}n + 2 - \frac{1}{\mu}$, as required. □

We can now state our precise result, knowing that all the transition matrices specified are well-defined and give involutive walks in which all states are accessible.

Proposition 6.6. *Let P^λ be the transition matrix of a reversible Markov chain on n where $n \geq 3$ in which every state is accessible. If $1 = \lambda_0$, $\mu = \lambda_1$, $\nu = \lambda_2$ then one of the following cases applies.*

(i) *If $\nu > \mu^2$ then $\mu > \nu$ and the unique a, $b \in \mathbb{R}^{>-1}$ such that $P^\lambda = P(\gamma^{(a,b)})$ are $a(\mu,\nu)$ and $b(\mu,\nu)$.*

(ii) *If $\nu = \mu^2$ then either $\nu = \mu = 1$ and $P^\lambda = J(n)$ or $P^\lambda = P(\gamma^{(c)})$ where $c = (1-\mu)/\mu$.*

(iii) *If $\nu_{n-1}(\mu) < \nu < \mu^2$ then the unique a', $b' \in \mathbb{R}^{>1}$ such that $P^\lambda = P(\delta^{(a',b')})$ are $-a(\mu,\nu)$ and $-b(\mu,\nu)$ and $n \leq \min(\lceil a' \rceil, \lceil b' \rceil)$.*

(iv) *Otherwise, $\mu > \frac{1}{2}$, $\nu = \nu_m(\mu)$ for a unique $m \in n$ such that $m > \frac{1-\mu}{\mu}n + 2 - \frac{1}{\mu}$ and the unique $a' \in \mathbb{R}^{>1}$ such that $P^\lambda = P(\delta^{(a',m)})$ is $a'_m(\nu)$. Moreover $n \leq \lceil a'_m(\nu) \rceil$.*

The example below shows the interesting behaviour in (iii) and (iv).

Example 6.7. Take $n = 10$. A special feature of the case $\mu = \frac{2}{3}$ used in Example 2.7 is that $a'_m(\mu) = 2m - 1$ is integral. Thus, by (iii) and (iv) either $\nu > \nu_9(\frac{2}{3}) = \frac{10}{23}$, or ν, a', b' (which is equal to the parameter m) are in the table below.

ν	$\frac{10}{23}$	$\frac{13}{30}$	$\frac{22}{51}$	$\frac{3}{7}$
$a' = a'(\frac{2}{3}, \nu)$	17	15	13	11
$b' = m$	9	8	7	6

By (iv), or equivalently Lemma 6.5, we require $m > \frac{1-\mu}{\mu}n+2-\frac{1}{\mu} = \frac{n+1}{2} = \frac{11}{2}$, so the table stops with $m = 6$. The next entry, were it permitted, would be $\nu = \frac{14}{45}$, $a' = 9$, $b' = 5$, but $9' \notin \mathbb{R}^{>10-1}$, and correspondingly 10 is not in the domain of the weight $\delta^{(9,5)}$. (It it was, then, as observed before Theorem 1.8, since $\delta^{(9,5)}_{[9,9]} = \binom{9-1}{9}\binom{5}{0} = 0$, the state $0 = 9^\star$ would be inaccessible.) Thus when $n = 10$, the reversible random walks classified in Proposition 6.6 with second largest eigenvalue $\frac{2}{3}$ are split into a continuously varying family when $\frac{4}{9} < \nu < \frac{2}{3}$, a one-off case when $\nu = \frac{4}{9}$, a continuously varying family when $\frac{10}{23} < \nu < \frac{4}{9}$, and final special cases for $\nu = \frac{10}{23}, \frac{13}{30}, \frac{22}{51}, \frac{3}{7}$, in which the transition matrix has respectively 9, 8, 7 and 6 non-zero anti-diagonal bands.

Lemmas on reversibility. Kolmogorov's Criterion (see for instance [2, §1.5]) states that a random walk with a unique invariant distribution is reversible if and only the product of transition probabilities around any cycle does not depend on the direction of travel. We state it as (ii) in the lemma below.

Lemma 6.8. *Let γ be a weight and suppose that every state in the γ-weighted involutive walk is accessible.*

(i) *For x, $z \in n$, we have $P(\gamma)_{xz} > 0$ if and only if $P(\gamma)_{zx} > 0$.*

(ii) *The walk is reversible if and only if*

$$P(\gamma)_{x_0x_1}P(\gamma)_{x_1x_2}\dots P(\gamma)_{x_{\ell-1}x_0} = P(\gamma)_{x_0x_{\ell-1}}\dots P(\gamma)_{x_2x_1}P(\gamma)_{x_1x_0}$$

for all distinct $x_0, x_1, \dots, x_{\ell-1} \in n$ with $\ell \geq 3$, such that $x_i + x_{i+1} \geq n-1$ for all $i \in n$, taking indices modulo ℓ.

Proof. By Theorem 1.3, the γ-weighted involutive walk is irreducible, aperiodic and recurrent with a unique strictly positive invariant distribution, π say. From the detailed balance equations $\pi_x P(\gamma)_{xz} = \pi_z P(\gamma)_{zx}$ we get $P(\gamma)_{xz} > 0$ if and only if $P(\gamma)_{zx} > 0$, proving (i). Moreover, since the invariant distribution is strictly positive, Kolmogorov's Criterion applies. Since $P(\gamma)_{xz} = 0$ unless $x + z \geq n-1$, it therefore suffices to verify the criterion for cycles of the form $x_0 \mapsto x_1 \mapsto x_2 \mapsto \dots \mapsto x_{\ell-1} \mapsto x_0$ where $x_i + x_{i+1} \geq n-1$ for all i. This is trivial for 2-cycles, so we may suppose that $\ell \geq 3$, giving (ii). □

Two special cases. It is most convenient to deal separately with the cases when $\mu = 1$ and when $\nu = 2\mu - 1 = \nu_2(\mu)$; these are the two cases where P^λ has just one or just two non-zero anti-diagonals. In each of these cases, the uniform proof in the final subsection cannot be applied, because there are no 4-cycles with non-zero probability, but instead there is a simple proof using that P^λ is non-negative. The proofs also introduce one recurring theme: the lower-triangular matrix H^λ depends on λ_{n-1} only in its bottom row.

Lemma 6.9. *Let $n \geq 3$. If $\mu = 1$ then H^λ is the $n \times n$ identity matrix and* (ii) *in Proposition 6.6 holds.*

Proof. By Lemma 6.1, $H^\lambda_{2,0} = 1 - 2\mu + \nu = -1 + \nu$. Hence $\nu = 1$ and if $n = 3$ then H^λ is the identity matrix. Suppose inductively the lemma holds for $n-1$. Therefore the $n \times n$ matrix H^λ agrees with the identity matrix in its first $n-1$ rows and $\lambda_0 = \ldots = \lambda_{n-2} = 1$. By Lemma 6.1(i), we have $H^\lambda_{n-1,n-1} = \lambda_{n-1}$, $H^\lambda_{n-1,n-2} = (n-1)(1-\lambda_{n-1})$ and $H^\lambda_{n-1,n-3} = \binom{n-1}{2}(1-2+\lambda_{n-1})$. Hence $\lambda_{n-1} \leq 1$ and $\lambda_{n-1} \geq 1$, so by (†), $\lambda_{n-1} = 1$, as required. □

Lemma 6.10. *Let $n \geq 3$. If $\nu = 2\mu - 1$ then $H^\lambda = H(\delta^{(\frac{1}{1-\mu},2)})$ and* (iv) *in Proposition 6.6 holds with $m = 2$.*

Proof. It follows from (1.4) (this was seen in Example 2.8) that the matrix $H(\delta^{(\frac{1}{1-\mu},2)})$ has entries

$$H(\delta^{(\frac{1}{1-\mu},2)})_{xy} = \begin{cases} x\mu - (x-1) & \text{if } x = y \\ x(1-\mu) & \text{if } x = y+1 \\ 0 & \text{otherwise.} \end{cases} \tag{6.2}$$

When $n = 3$, this has the same eigenvalues, namely 1, μ, $2\mu - 1$ as H^λ, and so the matrices are equal by ($\star$). Suppose inductively this holds for $n-1$. Therefore the $n \times n$ matrix P^λ agrees with $P(\delta^{(\frac{1}{1-\mu},2)})$ in its first $n-1$ rows. In particular, $\lambda_x = x\mu - (x-1)$ for $x \in n-1$. By Lemma 6.1, $P^\lambda_{x,n-1} = P(\delta^{(\frac{1}{1-\mu},2)})_{x,0} = 0$ if $1 < x < n-1$. Hence, by Lemma 6.8(i), $P^\lambda_{n-1,n-3} = P^\lambda_{n-3,n-1} = 0$. But by Lemma 6.1, $P^\lambda_{n-1,n-3} = \binom{n-1}{n-3}(\lambda_{n-3} - 2\lambda_{n-2} + \lambda_{n-1})$. Therefore

$$\big((n-3)\mu - (n-4)\big) - 2\big((n-2)\mu - (n-3)\big) + \lambda_{n-1} = 0$$

which simplifies to $-(n-1)\mu + (n-2) + \lambda_{n-1} = 0$. Hence $\lambda_{n-1} = (n-1)\mu - (n-2)$ which is $\lambda(\gamma^{(\frac{1}{1-\delta},1)})_{n-1}$ by (6.2). Therefore by ($\star$), $P^\lambda = P(\delta^{(\frac{1}{1-\mu},2)})$. Therefore $a' = \frac{1}{1-\mu} = a'_2(\nu)$ Moreover, by the general Lemma 6.5, $2 > \frac{1-\mu}{\mu}n + 2 - \frac{1}{\mu}$. This simplifies to $n < \frac{1}{1-\mu}$, hence $n \leq \lceil a'_2(\nu) \rceil$ giving the final requirement for (iv). □

Inductive step. Finally we are ready to prove Proposition 6.6. Let $n \geq 4$ and suppose inductively that the Proposition holds in the case $n-1$. Therefore one of the cases in the proposition applies to the $k \times k$ top-left submatrix of $H^\lambda = P^\lambda J(n)$ for all $k \in n-1$ with $k \geq 3$. In particular, taking $k = 3$, the base case implies that $\nu \geq 2\mu - 1$. If $\nu = 2\mu - 1$ then we are done by Lemma 6.10. Lemma 6.9 deals with the case $\mu = 1$. Therefore we may assume $1 > \mu > \nu > 2\mu - 1$. Now taking $k = n-1$ we get a (possibly weak) weight γ such that H^λ agrees with $H(\gamma)$ except perhaps in its bottom row. Moreover, the parameters for γ (i.e. a and b when $\gamma = \gamma^{(a,b)}$, c when $\gamma = \gamma^{(c)}$ and $a', b' = \delta^{(a',b')}$) are unique. Consider the 4-cycle

$$n-1 \mapsto 1 \mapsto n-2 \mapsto 2 \mapsto n-1.$$

(If $n = 4$ then instead take the 3-cycle $3 \mapsto 1 \mapsto 2 \mapsto 3$ obtained by deleting the repeated vertex.) By Kolmogorov's Criterion (Lemma 6.8(ii)), and the hypothesis that P^λ is reversible, we have

$$P^\lambda_{n-1,1}P^\lambda_{1,n-2}P^\lambda_{n-2,2}P^\lambda_{2,n-1} = P^\lambda_{n-1,2}P^\lambda_{2,n-2}P^\lambda_{n-2,1}P^\lambda_{1,n-1}. \tag{6.3}$$

(This holds when $n = 4$ since we have simply introduced an extra term of $P^\lambda_{2,2}$.) By the inductive assumption $P^\lambda_{xz} = P(\gamma)_{xz}$ for all $x \in n-1$. Moreover by Theorem 1.5 each γ-weighted involutive walk is reversible, and so (6.3) holds when every P^λ_{xz} is replaced with $P(\gamma)_{xz}$. Let q be the common value. Note that $q \neq 0$, since γ is not the exceptional weak weight $\delta^{(\frac{1}{1-\mu},2)}$, and so the matrix $P(\gamma)$ has at least 3 non-zero anti-diagonals. This also tells us that $P(\gamma)_{n-1,1}, P(\gamma)_{n-1,2} \neq 0$. Thus

$$\begin{aligned} P^\lambda_{1,n-2}P^\lambda_{n-2,2}P^\lambda_{2,n-1} &= q/P(\gamma)_{n-1,1} \\ P^\lambda_{2,n-2}P^\lambda_{n-2,1}P^\lambda_{1,n-1} &= q/P(\gamma)_{n-1,2}. \end{aligned}$$

By Lemma 6.1 we have $P^\lambda_{n-1,1} = \binom{n-1}{1}(\lambda_{n-2}-\lambda_{n-1})$ and $P^\lambda_{n-1,2} = \binom{n-1}{2}(\lambda_{n-3}-2\lambda_{n-2}+\lambda_{n-1})$. Again the analogous equation holds for $P(\gamma)$. Moreover, by the induction hypothesis $\lambda_x = H^\lambda_{xx} = H(\gamma)_x x = \lambda_x(\gamma)$ for $x \in n-1$. Therefore we obtain another pair of equations

$$\begin{aligned} P^\lambda_{n-1,1} &= P(\gamma)_{n-1,1} - \binom{n-1}{1}\big(\lambda_{n-1} - \lambda_{n-1}(\gamma)\big) \\ P^\lambda_{n-1,2} &= P(\gamma)_{n-1,2} + \binom{n-1}{2}\big(\lambda_{n-1} - \lambda_{n-1}(\gamma)\big). \end{aligned}$$

Writing Δ for $\lambda_{n-1} - \lambda_{n-1}(\gamma)$ and making all the indicated substitutions in (6.3) we obtain

$$\big(P(\gamma)_{n-1,1} - (n-1)\Delta\big)\frac{q}{P(\gamma)_{n-1,1}} = \big(P(\gamma)_{n-1,2} + \binom{n-1}{2}\Delta\big)\frac{q}{P(\gamma)_{n-1,2}}.$$

Using that $P(\gamma)_{n-1,z} = \gamma_{[z^\star,n-1]}/N(\gamma)_{n-1}$ and cancelling all common terms, including $N(\gamma)_{n-1}$, we obtain

$$-(n-1)\frac{\Delta}{\gamma_{[n-2,n-1]}} = \binom{n-1}{2}\frac{\Delta}{\gamma_{[n-3,n-1]}}$$

which simplifies to

$$\big((n-1)\gamma_{[n-3,n-1]} + \binom{n-1}{2}\gamma_{[n-2,n-1]}\big)\Delta = 0. \tag{6.4}$$

Suppose that γ is the weight $\gamma^{(c)}$ then $\gamma^{(c)}_{[y,n-1]} = \binom{n-1}{y}c^{n-1-y}$. and so (6.4) becomes $(n-1)\binom{n-1}{2}c(c-1)\Delta = 0$. Since $c > 1$, we have $\Delta = 0$. Hence $\lambda_{n-1} = \lambda(\gamma^{(c)})_{n-1}$ and by $(\star)$ we get (i). If instead γ is $\gamma^{(a,b)}$ then $\gamma^{(a,b)}_{[y,n-1]} = \binom{a+y}{y}\binom{b+n-1-y}{n-1-y}$ and so (6.4) becomes

$$\Big((n-1)\binom{a+n-3}{n-3}\binom{b+2}{2} + \binom{n-1}{2}\binom{a+n-2}{n-2}\binom{b+1}{1}\Big)\Delta = 0.$$

Using $\binom{m+1}{c+1} = \binom{m}{c}\frac{m+1}{c+1}$ this simplifies to

$$\tfrac{1}{2}\Big((n-1)\binom{a+n-3}{n-3}\binom{b+1}{1}\Big(\frac{b+2}{2}+\frac{n-2}{2}\frac{a+n-2}{n-2}\Big)\Delta = 0,$$

and hence to $(a+b-n)\Delta = 0$. Since a, $b \in \mathbb{R}^{>-1}$ and $n \geq 4$, again we get $\Delta = 0$, and again by $(\star)$ we get (ii). The final case is when (iii) or (iv) holds in Proposition 6.6. Now γ is $\delta^{(a',b')}$ and so $\delta^{(a',b')}_{[y,n-1]} = \binom{a'-1}{y}\binom{b'-1}{n-1-y}$. Substituting in (6.4) we get

$$\Big((n-1)\binom{a'-1}{n-3}\binom{b'-1}{2}+\binom{n-1}{2}\binom{a'-1}{n-2}\binom{b'-1}{1}\Big)\Delta = 0$$

which simplifies by $\binom{m}{c+1} = \binom{m}{c}\frac{m-c}{c+1}$ to

$$(n-1)\binom{a'-1}{n-3}\binom{b'-1}{1}\Big(\frac{b'-2}{2}+\frac{n-2}{2}\frac{a'-n}{n-2}\Big)\Delta = 0$$

and hence to $(a'+b'-n-2)\Delta = 0$. If the inductive case is (iii) then $n-1 \leq \min(\lceil a'\rceil, \lceil b'\rceil)$, and from $n-1 \leq a'-1$, $n-1 \leq b'-1$ we get $2n \leq a'+b'$, hence $a'+b'-2 \geq 2n-2 = n+(n-2) > n$ and so $a'+b'-n-2 \neq 0$. Otherwise (iv) holds, $a' = a'_m(\nu)$ for some $m > (n-2)\frac{1-\mu}{\mu}+1$, $b' = m$ and $n-1 \leq \lceil a'\rceil$. Since $\nu > 2\mu-1 = \nu_2(\mu)$ and the sequence $\nu_m(\mu)$ is strictly increasing, we have $m \geq 3$. If $a'+\tau = \lceil a'\rceil$ then

$$a'+b'-2-n = \lceil a'\rceil - \tau + b' - 2 - n \geq \lceil a'\rceil - \tau + 3 - 2 - n \geq 1-\tau > 0$$

so again $a'+b'-n-2 > 0$. Therefore $\Delta = 0$ and by $(\star)$ we get $P^\lambda = P(\delta^{(a',b')})$.

Since every state is accessible, Lemma 6.5 implies that $n \leq \lceil a'\rceil$. If the inductive case was (iv) then (iv) still holds: $\mu > \frac{1}{2}$, $\nu = \nu_m(\mu)$ and $a' = a'_m(\mu)$ are inductive assumptions, and $m > \frac{1-\mu}{\mu}n + 2 - \frac{1}{\mu}$ and $n \leq \lceil a'_m(\nu)\rceil$ follow from Lemma 6.5. Suppose the inductive was (iii), so $\nu_{n-2}(\mu) < \nu < \mu^2$. If we have the stronger inequality $\nu_{n-1}(\mu) < \nu$ then (iii) still holds. In the remaining case (iii) holds with $\nu_{n-2}(\mu) < \nu \leq \nu_{n-1}(\mu)$. Since b'_μ is an increasing function of ν, we have $n-2 < b' \leq n-1$. If $n-2 < b' < n-1$ then $\delta^{(a',b')}_{[0,n-1]} = \binom{b'-1}{n-1} < 0$ contradicts that $P(\delta^{(a',b')})$ is equal to the stochastic matrix P^λ. Therefore $b' = n-1$, $\nu = \nu_{n-1}(\mu)$ and $a' = a'_{n-1}(\mu)$. (This is the case where $P(\delta^{(a'_{n-1}(\mu),n-1)})$ has $n-1$ non-zero anti-diagonal bands and a zero in its bottom right corner.) By Lemma 6.5, $n-1 > \frac{1-\mu}{\mu}n + 2 - \frac{1}{\mu}$. If $\mu \leq \frac{1}{2}$ then the right-hand side is at least n, a contradiction. Therefore $\mu > \frac{1}{2}$. Finally, by Lemma 6.5, $n \leq \lceil a'_{n-1}(\mu)$. Therefore (iv) holds.

7. A family of involutive walks with chosen eigenvalues

Given a strictly decreasing sequence $\lambda_1, \lambda_2, \ldots$ of strictly positive real numbers with $\lambda_1 < 1$, we define a weight $\varepsilon^{(\lambda)}$ by

$$\varepsilon^{(\lambda)}_{[y,x]} = \begin{cases} \lambda_y & \text{if } x = y \\ \lambda_y - \lambda_{y+1} & \text{if } x > y. \end{cases}$$

where $\lambda_0 = 1$.

Lemma 7.1. *The transition matrix of $\varepsilon^{(\lambda)}$-weighted involutive walk on n satisfies*

$$P(\varepsilon^{(\lambda)})_{xz} = \begin{cases} 0 & \text{if } x+z < n-1 \\ \varepsilon_{z^\star} & \text{if } x+z = n-1 \\ \varepsilon_{z^\star} - \varepsilon_{z^\star+1} & \text{if } x+z \geq n \end{cases}$$

for all $x, z \in n$.

Proof. The telescoping sum shows that $N(\varepsilon^{(\lambda)})_x = 1$ for all x. Therefore the anti-diagonal entries in the transition matrix $P(\varepsilon^{(\lambda)})$ are $P(\varepsilon^{(\lambda)})_{xx^\star} = \lambda_x$ and the entries strictly below the anti-diagonal are $P(\varepsilon^{(\lambda)})_{xz} = \lambda_{z^\star} - \lambda_{z^\star+1}$. □

For example, when $n = 4$, we have

$$P(\varepsilon^{(\lambda)}) = \begin{pmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \lambda_1 & 1-\lambda_1 \\ \cdot & \lambda_2 & \lambda_1-\lambda_2 & 1-\lambda_1 \\ \lambda_3 & \lambda_2-\lambda_3 & \lambda_1-\lambda_2 & 1-\lambda_1 \end{pmatrix}.$$

The special case when $\lambda_x = 1/r^x$ was seen in Example 2.9.

7.1. **Reversibility.** We use Lemma 7.1 to determine when the $\varepsilon^{(\lambda)}$-weighted involutive walk is reversible. We also require Kolmogorov's Criterion, as stated in Lemma 6.8(ii).

Theorem 7.2. *The $\varepsilon^{(\lambda)}$-weighted involutive walk is reversible if and only if*

$$\lambda_1\lambda_{n-1} = \lambda_2\lambda_{n-2} = \ldots = \lambda_{n-1}\lambda_1.$$

Proof. Suppose that the $\varepsilon^{(\lambda)}$-weighted involutive walk is reversible. Let $1 \leq x < (n-1)/2$. Consider the 3-cycle $n-1 \mapsto x \mapsto x^\star \mapsto n-1$ and its reverse $n-1 \mapsto x^\star \mapsto x \mapsto n-1$. Since $x + x^\star = n-1$, the positions $(x, x^\star)$ and $(x^\star, x)$ are on the anti-diagonal of $P(\gamma)$, while the other two relevant positions are strictly below the anti-diagonal. By Lemma 7.1 and Kolmogorov's Criterion we have

$$(\lambda_{x^\star} - \lambda_{x^\star+1})\lambda_x(1-\lambda_1) = (\lambda_x - \lambda_{x+1})\lambda_{x^\star}(1-\lambda_1).$$

Simplifying, this becomes $\lambda_x\lambda_{x^\star+1} = \lambda_{x+1}\lambda_{x^\star}$ as required.

Conversely, suppose that this condition holds whenever $1 \leq x < n-1$. Let $x_0 \mapsto x_1 \mapsto \ldots \mapsto x_{\ell-1} \mapsto x_0$ be a cycle (with distinct vertices). Denote this cycle by C and let C' denote the reversed cycle $x_0 \mapsto x_{\ell-1} \mapsto \ldots \mapsto x_1 \mapsto x_i$. Throughout, all indices are to be regarded modulo p. Using Lemma 6.8(ii), we may assume that $\ell \geq 3$ and $x_{i-1} + x_i \geq n-1$ for each i; it then suffices to show that the product of transition probabilities is the same for C and C'. Let $I = \{i : x_{i-1} + x_i = n-1\}$ be the set of indices i of those steps $x_{i-1} \mapsto x_i$ that contribute $\lambda_{x_i^\star}$ (rather than $\lambda_{x_i^\star} - \lambda_{x_i^\star+1}$) to the product for C. Now i' appears in the analogous set for C', of those indices i' such

that the step $x_{i'+1} \mapsto x_{i'}$ contributes $\lambda_{x^\star_{i'}}$ (rather than $\lambda_{x^\star_{i'}} - \lambda_{x^\star_{i'}+1}$) to the product of C', if and only if $x_{i'} + x_{i'+1} = n-1$, so if and only if $i'-1 \in I$. Let $I - 1 = \{i-1 : i \in I\}$ be the set of such indices i'. Observe that if $i \in I \cap (I-1)$ then the step $x_{i-1} \mapsto x_i$ in C is $x^\star_i \mapsto x_i$, and the step $x_{i+1} \mapsto x_i$ in C' is also $x^\star_i \mapsto x_i$. Therefore C has a subcycle of length 2, contrary to our assumption that the vertices are distinct. Hence I and $I-1$ are disjoint. If $i \notin I \cup (I-1)$ then the step to x_i contributes $\lambda_{x^\star_i} - \lambda_{x^\star_i+1}$ to both products. Hence the two products are equal if and only if

$$\prod_{i\in I} \lambda_{x^\star_i} \prod_{i \in I-1} (\lambda_{x^\star_i} - \lambda_{x^\star_i+1}) = \prod_{i\in I} (\lambda_{x^\star_i} - \lambda_{x^\star_i+1}) \prod_{i\in I-1} \lambda_{x^\star_i}.$$

Equivalently

$$\prod_{i\in I} \lambda_{x^\star_i} (\lambda_{x^\star_{i-1}} - \lambda_{x^\star_{i-1}+1}) = \prod_{i\in I} (\lambda_{x^\star_i} - \lambda_{x^\star_i+1}) \lambda_{x^\star_{i-1}}.$$

If $i \in I$ then $x_{i-1} + x_i = n-1$, and so $x^\star_{i-1} = x_i$. Therefore a final equivalent form is

$$\prod_{i\in I} \lambda_{x^\star_i} (\lambda_{x_i} - \lambda_{x_i+1}) = \prod_{i\in I} (\lambda_{x^\star_i} - \lambda_{x^\star_i+1}) \lambda_{x_i}.$$

This holds term-by-term, since $\lambda_{x^\star_i} \lambda_{x_i+1} = \lambda_{x^\star_i+1} \lambda_{x_i}$. □

Example 2.9 shows that if $\lambda_x = r^x$ then the detailed balance equations have the explicit solution $\pi_x = (r^{x+1} - r^x)/(r^n - 1)$ and, as expected from the theorem just proved, the involutive random walk is reversible. In general the invariant distribution is π where

$$\pi_x = \begin{cases} \dfrac{\lambda_{n-1}(1-\lambda_1)}{1-\lambda_1\lambda_{n-1}} & \text{if } x = 0 \\[2ex] \dfrac{(\lambda_{x^\star} - \lambda_{x^\star+1})(1-\lambda_{x+1}) + (\lambda_x - \lambda_{x+1})(1-\lambda_{x^\star})\lambda_{x^\star+1}}{(1-\lambda_x\lambda_{x^\star+1})(1-\lambda_{x+1}\lambda_{x^\star})} & \text{if } 0 < x < n-1 \\[2ex] \dfrac{1-\lambda_1}{1-\lambda_1\lambda_{n-1}} & \text{if } x = n-1. \end{cases}$$

Our only proof is an explicit calculation that can readily be checked by computer algebra. We omit further details.

7.2. **Spectrum.** In contrast to the invariant distribution, the other left-eigenvectors of $P(\varepsilon^{(\lambda)})$ are quite easy to write down. For $x \in n$, let $u^{(x)} \in \mathbb{R}^n$ be the row vector with 1 in position x.

Lemma 7.3. *If* $0 < x < n-1$ *then*

$$\big(u^{(x)} - u^{(x-1)}\big) P(\varepsilon^{(\lambda)}) = -\lambda_x \big(u^{(x^\star+1)} - u^{(x^\star)}\big).$$

Proof. Since $u^{(x)} P(\varepsilon^{(\lambda)}) = \sum_{y=0}^{x-1} (\lambda_y - \lambda_{y+1}) u^{(y^\star)} + \lambda_x u^{(x^\star)}$, we have $\big(e(x) - e(x-1)\big) P(\varepsilon^{(\lambda)}) = -\lambda_x u^{(x-1)^\star} + \lambda_x u^{x^\star}$ for each $x > 0$. Since $(x-1)^\star = x^\star + 1$, this simplifies to $-\lambda_x \big(u^{(x^\star+1)} - u^{(x^\star)}$ as required. □

We now find the spectrum and hence the eigenvalues of $P(\varepsilon^{(\lambda)})$ by applying this lemma to $P(\varepsilon^{(\lambda)})$ and $P(\varepsilon^{(\lambda)})^2$.

Theorem 7.4. *The matrix $P(\varepsilon^{(\lambda)})$ is diagonalizable with eigenvalues* 1 *and* $\pm\sqrt{\lambda_x\lambda_{x^\star+1}}$ *for* $0 < x \leq n/2$. *If* $x < n/2$ *then the* $\pm\sqrt{\lambda_x\lambda_{x^\star+1}}$ *eigenspace contains*

$$\sqrt{\lambda_{x^\star+1}}\big(u^{(x)} - u^{(x-1)}\big) \pm \sqrt{\lambda_x}\big(u^{(x^\star+1)} - u^{(x^\star)}\big).$$

When $n = 2m$, *there is an eigenvalue* $-\lambda_m$ *and the* $-\lambda_m$*-eigenspace contains* $u^{(m)} - u^{(m-1)}$.

Proof. Let $v^{(x)} = u^{(x)} - u^{(x-1)}$. By the previous lemma, $v^{(x)}P(\varepsilon^{(\lambda)}) = -\lambda_x v^{(x^\star+1)}$. By two applications of the lemma we have

$$v^{(x)}P(\varepsilon^{(\lambda)})^2 = -\lambda_x v^{(x^\star+1)} = \lambda_x\lambda_{x^\star+1}v^{(x)}$$

where the second equality uses $(x^\star + 1)^\star = x - 1$. Thus $v^{(x)}$ is a left-eigenvector of $P(\varepsilon^{(\lambda)})^2$ with eigenvalue $\lambda_x\lambda_{x^\star+1}$. The $n - 1$ eigenvectors given by taking $0 < x < n - 1$ span the subspace of $\mathbb{R}^n$ of vectors with sum 0, and so are linearly independent, and complementary to the subspace spanned by the 1-eigenvector given by the invariant distribution. Therefore $P(\varepsilon^{(\lambda)})^2$ is diagonalizable with eigenvalues 1 and all $\lambda_x\lambda_{x^\star+1}$ for $0 < x < n$. Since $\lambda_x\lambda_{x^\star+1}$ is unchanged under the involution swapping x and $x^\star+1$, and either $x > n/2$ or $x^\star > n/2$, taking square-roots shows that the eigenvalues of $P(\varepsilon^{(\lambda)})$ are as claimed. A further calculation now gives

$$\begin{aligned}
&\big(\sqrt{\lambda_{x^\star+1}}v^{(x)} \pm \sqrt{\lambda_x}v^{(x^\star+1)}\big)P(\varepsilon^{(\lambda)}) \\
&\qquad = -\sqrt{\lambda_{x^\star+1}}\lambda_x v^{(x^\star+1)} \mp \sqrt{\lambda_x}\lambda_{x^\star+1}v^{(x)} \\
&\qquad = \mp\sqrt{\lambda_{x^\star+1}\lambda_x}\big(\sqrt{\lambda_{x^\star+1}}v^{(x)} \pm \sqrt{\lambda_x}v^{(x^\star+1)}\big)
\end{aligned}$$

and so the eigenvectors of $P(\varepsilon^{(\lambda)})$ are as claimed. □

We note that the eigenvalues in Theorem 7.4 need not be distinct: indeed in Example 2.9 we saw that if $\lambda_x = r^x$ for each x, then the eigenvalue $-r^{-n/2}$ has multiplicity $\lfloor n/2 \rfloor$. The reversibility of the walks in this example is an instance of the corollary below.

Corollary 7.5. *The* $\varepsilon^{(\lambda)}$*-weighted involutive walk is reversible if and only if it has exactly* 3 *distinct eigenvalues.*

Proof. By Theorem 7.2, the $\varepsilon^{(\lambda)}$-weighted involutive walk is reversible if and only if $\lambda_x\lambda_{x^\star+1}$ is a constant, α say. By Theorem 7.4, this is the case if and only if the eigenvalues of $P(\varepsilon^{(\lambda)})$ are 1 and $\pm\sqrt{\alpha}$. □

8. The involutive walk on $[0, 1]$

We now turn to involutive walks defined on the real interval $[0, 1] = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$ with its usual order and unique anti-involution $x^\star = 1 - x$. Our chosen analogue of Definition 1.1 is as follows.

Definition 8.1. A *real weight* is a function γ on the set of non-empty intervals of $[0,1]$ taking values in $\mathbb{R}^{\geq 0}$ such that, for each $x \in [0,1]$, the function $y \mapsto \gamma_{[y,x]}$ is continuous and non-zero almost everywhere in $[0,x]$, and $\int_0^x \gamma_{[y,x]}\,\mathrm{d}y < \infty$.

Thus a real weight defines a probability distribution on each interval $[0,x]$ having probability density function $\gamma_{[y,x]}/N(\gamma)_x$ where $N(\gamma)_x = \int_0^x \gamma_{[y,x]}\,\mathrm{d}y$. The γ-weighted involutive walk on $[0,1]$ is defined exactly as in Definition 1.2. We also define *atomic* and *$\star$-symmetric* for real weights as before.

8.1. **Invariant distributions.**

Proposition 8.2. *Let α be an atomic real weight and let β be a $\star$-symmetric real weight. Suppose that $\int_0^1 \alpha_{1-x} N(\alpha\beta)_x\,\mathrm{d}x < \infty$. Then the $\alpha\beta$-weighted involutive walk on $\mathcal{P}$ is reversible with respect to an invariant distribution proportional to $\alpha_{1-x}N(\alpha\beta)_x$.*

Proof. It is routine to solve the detailed balance equation using essentially the same arguments seen in Lemma 3.2 and Lemma 3.3. □

The integrability assumption in the lemma is essential.

Example 8.3. Let $a \in \mathbb{R}$ with $a > 0$. Let $\alpha_{[y,x]} = 1/y^a(1-y)^{a+1}$ and let $\beta_{[y,x]} = y^a(1-x)^a$. Observe that α is atomic and β is $\star$-symmetric. Let $\gamma = \alpha\beta$, so $\gamma_{[y,x]} = (1-x)^a/(1-y)^{a+1}$. We have

$$N(\gamma)_x = \int_0^x \frac{(1-x)^a}{(1-y)^{a+1}}\,\mathrm{d}y = \frac{1}{a}\big(1-(1-x)^a\big)$$

hence γ is a real weight. However

$$\alpha_{1-x}N(\gamma)_x = \frac{1}{ax^a(1-x)}\Big(\frac{1}{(1-x)^a} - 1\Big)$$

is not integrable on any interval containing 1, and not integrable on any interval containing 0 whenever $a \geq 1$.

From now on we assume that the hypotheses for Proposition 8.2 hold. Let $\vartheta = \alpha_{1-x}N(\alpha)_x$ be the function proportional to the invariant distribution given by in this proposition. Note that, by our assumption, ϑ_x is non-zero almost everywhere and $\sqrt{\vartheta_x} \in L^2([0,1])$.

8.2. **Hilbert spaces.** In this section we describe a setting in which the analogues of left- and right-multiplication by the transition matrix for a finite involutive walk correspond to compact linear operators on Hilbert space. Our §5.4 and [1] suggest the correct inner-product spaces in which to work. Let

$$K(x,z) = [x+z \geq 1]\frac{\sqrt{\alpha_{1-z}}\sqrt{\alpha_{1-x}}\beta_{[1-z,x]}}{\sqrt{N(\alpha\beta)_z}\sqrt{N(\alpha\beta)_x}}. \tag{8.1}$$

When $K(x,z) \in L^2\big([0,1]^2\big)$. the integral operator $M : L^2([0,1]) \to L^2([0,1])$ defined by $(Mf)(x) = \int_0^1 K(x,z)f(z)\,\mathrm{d}z$ is compact. (See for instance [7, Theorem 8.8].) Since $K(x,z) = K(z,x)$ for all $x,z \in [0,1]$, M is also self-adjoint.

Definition 8.4. Let $\mathcal{H}_L = \{f(x)/\sqrt{\vartheta_x} : f \in L^2([0,1])\}$ with inner product $\langle\ ,\ \rangle_{\mathcal{H}_L}$ defined by $\langle f,g\rangle = \int_0^1 \vartheta_x f(x)g(x)\,\mathrm{d}x$. Let $\mathcal{H}_R = \{f(x)\sqrt{\vartheta_x} : f \in L^2([0,1])\}$ with inner product $\langle\ ,\ \rangle_{\mathcal{H}_R}$ defined by $\langle f,g\rangle = \int_0^1 f(x)g(x)/\vartheta_x\,\mathrm{d}x$.

The maps sending $f \in L^2([0,1])$ to $f/\sqrt{\vartheta_x} \in \mathcal{H}_L$ and $f\sqrt{\vartheta_x} \in \mathcal{H}_R$ are isometric isomorphisms from $L^2([0,1])$ to $\mathcal{H}_L$ and $\mathcal{H}_R$, respectively. (The first map is well-defined because the zeros of ϑ_x form a null-set.) Therefore $\mathcal{H}_L$ and $\mathcal{H}_R$ are Hilbert spaces. Moreover the constant function is in $\mathcal{H}_L$ and, by our integrability assumption, ϑ_x is in $\mathcal{H}_R$. We may now define $L_P : \mathcal{H}_L \to \mathcal{H}_L$ and $R_P : \mathcal{H}_R \to \mathcal{H}_R$ by $(L_P g)(x) = M(g(z)\sqrt{\vartheta_z})/\sqrt{\vartheta_x}$ for $g \in \mathcal{H}_L$ and $R_P(g)(z) = M(g(x)/\sqrt{\vartheta_x})\sqrt{\vartheta_z}$ for $g \in \mathcal{H}_R$. (Note we deliberately use different variables: for left-multiplication we integrate over the final state z; for right-multiplication we integrate over the initial state x.) Equivalently,

$$(L_P g)(x) = \int_{1-x}^{1} \frac{\alpha_{1-z}\beta_{[1-z,x]}}{N(\alpha\beta)_x} g(z)\,\mathrm{d}z \tag{8.2}$$

$$(R_P g)(z) = \int_{1-z}^{1} \frac{\alpha_{1-z}\beta_{[1-z,x]}}{N(\alpha\beta)_x} g(x)\,\mathrm{d}x. \tag{8.3}$$

Since L_P and R_P are conjugate by isometric isomorphisms to the self-adjoint compact operator M, they are also self-adjoint and compact, and have the same eigenvalues.

We can now give a convenient sufficient condition for the real involutive walk to converge. Say that R_P is *strictly positive* if whenever $\vartheta \in \mathcal{H}_L$ is non-negative, its image $R_P\vartheta$ is strictly positive. Let $\pi_x = C\vartheta_x$ be the invariant probability distribution given by Proposition 8.2.

Proposition 8.5. *Suppose that R_P is strictly positive. Then the invariant distribution π_x is unique and the $\alpha\beta$-weighted involutive walk started at any probability distribution $\vartheta \in \mathcal{H}_L$ converges to $\pi_x \in \mathcal{H}_L$.*

Proof. The invariant distribution π_x is a 1-eigenvector for R_P. The proposition therefore follows from the Krein–Rutman Theorem (see [3, Theorem 6.3]), that this eigenvector is the unique (up to scalars) eigenvector for the largest eigenvalue (in modulus) of the compact self-adjoint operator R_P. □

8.3. **Trigonometric example.** As an illustrative example we take $\alpha_x = \sin \pi x$ for $x \in [0,1]$ and $\beta_{[y,x]} = 1$ for each non-empty interval $[y,x]$. We have $N(\alpha)_x = \int_0^x \sin \pi y\,\mathrm{d}y = \frac{1}{\pi}(1 - \cos \pi x)$. Note that the hypothesis for Proposition 8.2, that $\sqrt{\vartheta_x} \in L^2[(0,1)]$ holds. Since $\alpha_{1-x} = \alpha_x$, this

proposition implies that the α-weighted involutive walk is reversible with invariant distribution proportional to ϑ_x where $\vartheta_x = \frac{1}{\pi}(\sin \pi x)(1-\cos \pi x) = \frac{1}{\pi}\sin \pi x - \frac{1}{2\pi}\sin 2\pi x$. Integrating over $[0,1]$ the first summand gives $\frac{2}{\pi^2}$ and the second 0, hence $\pi_x = \frac{\pi}{2}(\sin \pi x)(1-\cos \pi x)$ is an invariant distribution. The kernel K in (8.1) satisfies

$$K(x,z)^2 = [x+z \geq 1]\frac{\sin \pi(1-z)\sin \pi(1-x)}{(1-\cos \pi x)(1-\cos \pi z)}.$$

Using $\sin \pi(1-z) = \sin \pi z$, $\sin \pi(1-x) = \sin \pi x$ and double angle formulae, this simplifies to $K(x,z)^2 = [x+z \geq 1]\cot \frac{\pi}{2}z \cot \frac{\pi}{2}x$. We have

$$\int_{x+z\geq 1} \cot \tfrac{\pi}{2}z \cot \tfrac{\pi}{2}x \,\mathrm{d}z\,\mathrm{d}x = \int_0^1\int_0^1 x \tan \tfrac{\pi}{2}xu \cot \tfrac{\pi}{2}x \,\mathrm{d}u\,\mathrm{d}x$$

by the substitution $1-z = xu$. Since $x\tan \frac{\pi}{2}xu \cot \frac{\pi}{2}x \leq x \tan \frac{\pi}{2}x \cot \frac{\pi}{2}x = x$ we see that the integrand is bounded. (In fact it is even continuous, except at $(1,1)$.) Therefore $K \in L^2\big([0,1]^2\big)$ and the general theory in the previous subsection applies.

Using $\alpha_{1-x} = \alpha_x$, we see from (8.2) that the analogue of left-multiplication by the transition matrix is the linear operator

$$(L_P g)(x) = \pi \int_0^x \frac{\sin \pi z}{1-\cos \pi x} g(z)\,\mathrm{d}z,$$

acting on the Hilbert space $\mathcal{H}_L = \{f(x)/\sqrt{\vartheta_x} : f \in L^2([0,1])\}$. Similarly the analogue of right-multiplication is the linear operator

$$(R_P g)(x) = \pi \int_0^z \frac{\sin \pi z}{1-\cos \pi x} g(x)\,\mathrm{d}x,$$

acting on the Hilbert space $\mathcal{H}_R = \{f(x)\sqrt{\vartheta_x} : f \in L_2[0,1]\}$. It is easily seen that each operator is strictly positive. Therefore we may apply Proposition 8.5 to conclude that π_x is the unique invariant distribution for the α-weighted involutive walk on $[0,1]$, and that the process converges, in the Hilbert space sense, to π_x.

We now find its spectrum. In §4 and §5 we used right-eigenvectors, corresponding to left-multiplication by the transition matrix, and again here it is most convenient to work with $L_P : \mathcal{H}_L \to \mathcal{H}_L$. For $n \in \mathbb{N}_0$, let V_n be the n-dimensional subspace of $\mathcal{H}_L$ spanned by $1, \cos \pi x, \cos 2\pi x, \ldots, \cos \pi(n-1)x$ and let $V_\infty = \bigcup_{n=0}^\infty V_n$. The image of multiplication by $\sqrt{\pi_x}$ on $L^2([0,1])$ is dense in $L^2([0,1])$. (For example, it contains any step function supported on a closed subinterval of $(0,1)$.) Since $1, \cos \pi x, \cos 2\pi x, \ldots$ is an orthonormal Hilbert space basis of $L^2([0,1])$, the set $\sqrt{\pi_x}, \sqrt{\pi_x}\cos \pi x, \sqrt{\pi_x}\cos 2\pi x, \ldots$ is dense in $L^2([0,1])$. Hence, dividing by $\sqrt{\pi_x}$, we see that V_∞ is dense in $\mathcal{H}_L$.

Lemma 8.6. *For any $n \in \mathbb{N}_0$ we have*

$$L_P(\cos^n \pi x) \in \tfrac{(-1)^n}{n+1}\cos^n \pi x + V_n.$$

Proof. We have

$$\begin{aligned}
L_P(\cos^n \pi x) &= \frac{\pi}{1-\cos \pi x} \int_{1-x}^{1} \sin \pi(1-z) \cos^n \pi z \, \mathrm{d}z \\
&= \frac{\pi}{1-\cos \pi x} \int_{1-x}^{1} \sin \pi z \cos^n \pi z \, \mathrm{d}z \\
&= \frac{\pi}{1-\cos \pi x} \frac{1}{\pi(n+1)} (\cos^{n+1} \pi z)\Big|_{1-x}^{1} \\
&= \frac{1}{n+1} \frac{(-1)^{n+1} - \cos^{n+1} \pi(1-x)}{1-\cos \pi x} \\
&= \frac{(-1)^n}{n+1} \frac{1-\cos^{n+1} \pi x}{1-\cos \pi x} \\
&= (-1)^n \frac{1+\cos \pi x + \cdots + \cos^n \pi x}{n+1}
\end{aligned}$$

as required. □

Hence L_P has eigenvalues $(-1)^{n-1}/n$ for $n \in \mathbb{N}$, each with a unique (up to scalars) eigenfunction. The convergence of the involutive walk is determined by the second largest eigenvalue (in modulus), namely $-1/2$.

Remark 8.7. We remark that a similar method to Lemma 8.6 gives part of the spectrum of L_P whenever $\alpha_{1-x} = \alpha_x$ for all $x \in [0,1]$ and so $\alpha_{1-z} N(\alpha)_z^n$ is the derivative of $\frac{1}{n+1} N(\alpha)_z^{n+1}$. But one cannot expect in general that the eigenfunctions obtained in this way will be complete for the Hilbert space $\mathcal{H}_L$.

9. The polynomially-weighted involutive walk on the interval

Fix, throughout this section, a, $b \in \mathbb{N}_0$. Observe that the weight $\gamma^{(a,b)} = \binom{y+a}{y}\binom{b+x-y}{x-y}$ defined in (1.2) is asymptotically proportional to $y^a(x-y)^b$ when y and $x-y$ are both large. Let $\kappa^{(a,b)}$ be the real weight defined by $\kappa^{(a,b)}_{[y,x]} = y^a(x-y)^b$. Let $N(\kappa^{(a,b)})_x = \int_0^x \kappa^{(a,b)}_{[y,x]} \, \mathrm{d}y$ be the relevant normalization factor. In this section we use the theory from §8 to show that the $\kappa^{(a,b)}$-weighted involutive walk, is, in a precise sense, the continuous limit of the (a,b)-involutive walk. In particular, it has a discrete spectrum and its eigenvalues are given by letting n tend to infinity in Theorem 1.5.

We begin with the analogue of Lemmas 5.1 and 5.2. Let $P(\kappa^{(a,b)})_{xz}$ be the probability density of a step from x to z. Part (i) is a standard result related to the B-function; we include a proof to make the paper more self-contained.

Lemma 9.1. *We have*

(i) $\int_0^1 w^a(1-w)^b \, \mathrm{d}w = (a+b+1)^{-1}\binom{a+b}{a}^{-1}$;

(ii) $N(\kappa^{(a,b)})_x = x^{a+b+1}(a+b+1)^{-1}\binom{a+b}{b}^{-1}$;

(iii) $P(\kappa^{(a,b)})_{xz} = [x+z \geq 1](a+b+1)\dbinom{a+b}{b}\dfrac{(1-z)^a(x+z-1)^b}{x^{a+b+1}}$.

Proof. By integrating $(w+(1-w)t)^c = \sum_{b=0}^{c} \binom{c}{b} w^{c-b}(1-w)^b t^b$ over w we get

$$\frac{x^{c+1}(1-t^{c+1})}{(c+1)(1-t)} = \sum_{b=0}^{c} \binom{c}{b} t^b \int_0^1 w^{c-b}(1-w)^b \,\mathrm{d}y.$$

Since the left-hand side is $\frac{x^{c+1}}{c+1}(1+t+\cdots+t^c)$, (i) follows by setting $c=a+b$ and comparing coefficients of t^b. Part (ii) follows from (i) using the change of variables $xw = y$ to write $\int_0^x y^a(x-y)^b \,\mathrm{d}y = x^{a+b}\int_0^1 w^a(1-w)^b x \,\mathrm{d}w$. Now, stepping from x, we may step to z if and only if $1-z \le x$; hence $P(\kappa^{(a,b)})_{xz} = [1-z \le x]\kappa^{(a,b)}_{[1-z,x]}/N(\kappa^{(a,b)})_x$. Part (iii) now follows from the definition of $\kappa^{(a,b)}_{[1-z,x]}$ and (ii). □

Set $\pi_x = C(1-x)^a x^{a+b+1}$ where $C = (a+b+2)\binom{a+b+2}{a}$ is the normalization factor given by Lemma 9.1.

Lemma 9.2. *The (a,b)-interval involutive walk has unique invariant probability density function π_x and is reversible.*

Proof. This is immediate from Proposition 8.2. □

We now find the spectrum of the involutive walk. The Hilbert space $\mathcal{H}_L$ defined in §8.2 is as follows.

Definition 9.3. Let $\mathcal{H}_L = \{f(x)/\sqrt{\pi_x} : f \in \mathrm{L}^2([0,1])\}$, with inner product $\langle\ ,\ \rangle_{\mathcal{H}_L}\mathcal{H}_L$ defined by $\langle f,g\rangle = \int_0^1 (1-x)^a x^{a+b+1} f(x)g(x)\,\mathrm{d}x$.

For $e \in \mathbb{N}$, let V_e be the subset of $\mathcal{H}_L$ of polynomials of degree strictly less than e, and let $V_\infty = \bigcup_{e\in\mathbb{N}} V_e$. Thus each V_e is a closed linear subspace of $\mathcal{H}_L$ of dimension e and by a similar argument to §8.3, V_∞ is a dense subspace of $\mathcal{H}_L$. By (8.2) and (8.3), the analogue of left-multiplication by the transition matrix is

$$(L_P f)(x) = (a+b+1)\binom{a+b}{a}\int_{1-x}^1 \frac{(1-z)^a(x+z-1)^b}{x^{a+b+1}} f(z)\,\mathrm{d}z. \tag{9.1}$$

We also need the analogous operator for left-multiplication by the transition matrix for down-steps, namely

$$(L_H f)(x) = (a+b+1)\binom{a+b}{a}\int_0^x \frac{y^a(x-y)^b}{x^{a+b+1}} f(y)\,\mathrm{d}y. \tag{9.2}$$

We saw in §8.2 that L_P has range contained in $\mathcal{H}_L$. Changing variables from y to $1-z$ in the definition of $L_H f$ shows that $L_P = L_H J$, where $J : \mathcal{H}_L \to \mathcal{H}_L$ is the self-adjoint involution defined by $(Jf)(x) = f(1-x)$. Hence the same holds for L_H. (Note that J is the analogue of the matrices $J(n)$.) Since $\mathcal{H}_L$ has V_∞ as a dense subspace, these results also follow from the following analogue of Lemma 5.3. Recall from Theorem 1.5 that the eigenvalues of the discrete involutive walk on $\{0,\dots,n-1\}$ are $(-1)^d\lambda(\gamma^{(a,b)})_d$ for $0 \le d < n$, where $\lambda(\gamma^{(a,b)})_d = \binom{a+d}{d}/\binom{a+b+d+1}{d}$.

Lemma 9.4. *Let $d \in \mathbb{N}_0$. We have $L_H(x^d) = \lambda(\gamma^{(a,b)})_d x^d$.*

Proof. By the substitution $y = xw$ in (9.2) and then Lemma 9.1(i) we get

$$\begin{aligned} L_H(x^d) &= (a+b+1)\binom{a+b}{a}\int_0^x \frac{y^a(x-y)^b}{x^{a+b+1}} y^d \,\mathrm{d}y \\ &= (a+b+1)\binom{a+b}{a} x^d \int_0^1 w^{a+d}(1-w)^b \,\mathrm{d}w \\ &= \frac{(a+b+1)\binom{a+b}{a}}{(a+b+d+1)\binom{a+b+d}{a+d}} x^d \end{aligned}$$

Now use $\binom{a+b+d}{a+d}/\binom{a+b}{a} = \binom{a+b+d}{d}/\binom{a+d}{a}$ and $\frac{a+b+d+1}{a+b+1}\binom{a+b+d}{d} = \binom{a+b+d+1}{a+b+1}$ to see that the coefficient is $\left(\!\binom{d+1}{a}\!\right)/\left(\!\binom{d+1}{a+b+1}\!\right)$. □

We now prove the analogue of Lemma 4.2.

Lemma 9.5. *Let $d \in \mathbb{N}$. Then*

$$L_P(x^d) \in (-1)^d \lambda(\gamma^{(a,b)})_d x^d + \mathcal{H}_{d-1}.$$

Proof. Since $L_P = L_H J$ and $(L_H J)(x^d) = L_H((1-x)^d) \in (-1)^d x^d + \mathcal{H}_d$, this is immediate from Lemma 9.4. □

We can now give a complete description of the spectrum and eigenfunctions for L_P. Let $g_0(x), g_1(x), g_2(x), \ldots$ be the functions obtained by Gram–Schmidt orthonormalization applied to 1, x, $x^2, \ldots$, regarded as elements of the Hilbert space $\mathcal{H}_L$.

Theorem 9.6. *The operator $L_P : \mathcal{H}_L \to \mathcal{H}_L$ is compact and self-adjoint. It has eigenvalues $(-1)^d \lambda(\gamma^{(a,b)})_d$ for $d \in \mathbb{N}_0$. The normalized eigenfunction for $(-1)^d \lambda(\gamma^{(a,b)})_d$ is g_d. The eigenfunctions $g_0, g_1, \ldots$ are an orthonormal Hilbert space basis for H and $L_P(f) = \sum_{d=0}^{\infty} (-1)^d \lambda(\gamma^{(a,b)})_d \langle g_d, f\rangle g_d$ for each $f \in \mathcal{H}$.*

Proof. Let $f, g \in \mathcal{H}_L$. Let $D = (a+b+1)\binom{a+b}{a}$. Since

$$\begin{aligned} \langle f, L_P g\rangle_H &= D\int_0^1 (1-x)^a x^{a+b+1} f(x) \int_{1-x}^1 \frac{(1-z)^a (x+z-1)^{a+b+1}}{x^{a+b+1}} g(z)\,\mathrm{d}z \\ &= D\iint_{x+z\geq 1} (1-x)^a (1-z)^a (x+z-1)^{a+b+1} f(x) g(z)\,\mathrm{d}x\,\mathrm{d}z \end{aligned}$$

is symmetric with respect to swapping f and g and x and z, we have $\langle f, L_P g\rangle_{\mathcal{H}_L} = \langle L_P f, g\rangle_{\mathcal{H}_L}$. By induction on d, it follows from Lemma 9.5 that for each $d \in \mathbb{N}_0$, there is a unique polynomial $g \in V_\infty$ of degree d such that $\langle g, g\rangle_{\mathcal{H}_L} = 1$ and $L_P g = (-1)^d \lambda(\gamma^{(a,b)})_d g$. Since the eigenfunctions for distinct eigenvalues of L_P are orthogonal, we have $g = g_d$. Since the g_d form a Hilbert space basis for $\mathcal{H}_L$, we have the claimed spectral decomposition of L_P. We saw in §5.3 that $\lambda(\gamma^{(a,b)})_d \to 0$ as $d \to \infty$. Therefore L_P is compact. □

Dually, we defined in §8.2 the Hilbert space $\mathcal{H}_R = \{f(x)\sqrt{\pi_x} : f \in \mathrm{L}^2([0,1])\}$, with inner product $\langle f, g\rangle_{\mathcal{H}_R} = \int_0^1 \frac{f(x)g(x)}{(1-x)^a x^{a+b+1}}\,\mathrm{d}x$. The map S which sends $f \in \mathcal{H}$ to $(1-x)^a x^{a+b+1} f(x) \in \mathcal{H}_R$ is an isometric isomorphism from $\mathcal{H}$ to $\mathcal{H}_R$. Let $R_P : \mathcal{H}_R \to \mathcal{H}_R$ be the linear operator corresponding to right-multiplication by the transition matrix of the interval involutive walk, defined by

$$(R_P f)(x) = (a+b+1)\binom{a+b}{a}\int_{1-x}^1 \frac{(1-x)^a(x+z-1)^b}{z^{a+b+1}} f(z)\,\mathrm{d}z.$$

Set $D = (a+b+1)\binom{a+b}{a}$ and recall that $\pi_x = C(1-x)^a x^{a+b+1}$. Hence if $g \in \mathcal{H}$ then

$$\begin{aligned}\big((R_P S)g\big)(x) &= D\int_{1-x}^1 \frac{(1-x)^a(x+z-1)^b}{z^{a+b+1}} C(1-z)^a z^{a+b+1} g(z)\,\mathrm{d}z \\ &= CD(1-x)^a \int_{1-x}^1 (1-z)^a(x+z-1)^b g(z)\,\mathrm{d}z \\ &= C(1-x)^a x^{a+b+1}(L_P g)(x) \\ &= \big((SL_P)g\big)(x).\end{aligned}$$

Set $g'_d(x) = (1-x)^a x^{a+b+1} g_d$ for $d \in \mathbb{N}_0$.

Theorem 9.7. *The operator $R_P : \mathcal{H}_R \to \mathcal{H}_R$ is compact and self-adjoint. It has eigenvalues $(-1)^d\lambda(\gamma^{(a,b)})_d$ for $d \in \mathbb{N}_0$. The normalized eigenfunction for $(-1)^d\lambda(\gamma^{(a,b)})_d$ is g'_d. The eigenfunctions $g'_0, g'_1, \ldots$ are an orthonormal Hilbert space basis for H' and $R_P(f) = \sum_{d=0}^\infty (-1)^d\lambda(\gamma^{(a,b)})_d\langle g'_d, f\rangle g'_d$ for each $f \in \mathcal{H}_R$.*

Proof. This is immediate from $R_P S = SL_P$ and Theorem 9.6. □

In particular, we deduce that the invariant distribution π_x is unique. Moreover, starting at any probability distribution in $\mathcal{H}_R$, the interval involutive walk converges to π_x. As in the discrete case, the rate of convergence controlled by the second largest eigenvalue $-\lambda(\gamma^{(a,b)})_1 = -(a+1)/(a+b+2)$.

Remark 9.8. For $f, g \in \mathcal{H}$ we have

$$\begin{aligned}\langle f, g\rangle_{\mathcal{H}} &= \int_0^1 (1-x)^a x^{a+b+1} f(x)g(x)\,\mathrm{d}x \\ &= 2^{-(2a+b+1)}\int_{-1}^1 (1-y)^a(1+y)^{a+b+1} f(\frac{1+y}{2})g(\frac{1+y}{2})\,\mathrm{d}y.\end{aligned}$$

It follows that the $g_d(\frac{1+y}{2})$ are orthonormal with respect to the inner product $\langle F, G\rangle = \int_{-1}^1 (1-y)^a(1+y)^{a+b+1}F(y)G(y)\,\mathrm{d}y$. The families of orthogonal polynomials for weights of the form $(1-y)^A(1+y)^B$ are known as the *Jacobi polynomials*. Thus $g_d(\frac{1+y}{2})$, is up to a scalar, the Jacobi polynomial with parameters a and $a+b+1$. Conversely, if $J(y)$ is this Jacobi polynomial then $g_d(x) = J(2x-1)$, again up to a scalar.

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