Lie Algebras

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Small print: These notes are intended to give the logical structure of the first half of the course; proofs and further remarks will be given in lectures. I would very much appreciate being told of any corrections or possible improvements. These notes, together with some brief notes on the linear algebra prerequisites, and the problem sheets for the course, are available via my home page,

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1 Introduction

Challenge 1.1. *Let $n \geq 1$ and let x, y be square matrices with entries in $\mathbb{C}$. Suppose that $xy - yx$ commutes with x and y. Show that $xy - yx$ is nilpotent; that is, $(xy - yx)^r = 0$ for some $r > 0$.*

If you solve this problem, you'll have succeeding in finding the main idea needed to prove a major result in the theory of Lie algebras. (See Theorem 5.2.)

Convention 1.2. *Vector spaces in this course are always finite-dimensional. Throughout F will be a field.*

Definition 1.3. A *Lie algebra* over a field F is an F-vector space L together with a map

$$[-,-] : L \times L \to L$$

satisfying the following properties:

(1) $[-,-]$ is bilinear,
(2) $[x, x] = 0$ for all $x \in L$; *anticommutativity*,
(3) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all $x, y, z \in L$; the *Jacobi identity*.

One says $[x, y]$ is the *Lie bracket* or *commutator* of x and y.

Some motivation for this definition will be given later. For now we merely show that there are some interesting examples.

Example 1.4. *Let V be an F-vector space. Let $\mathsf{gl}(V)$ be the vector space of linear maps $V \to V$. Define $[-,-]$ on $\mathsf{gl}(V)$ by*

$$[x, y] = x \circ y - y \circ x$$

where $\circ$ is composition of maps. This Lie algebra is known as the general linear algebra.

Sometimes it is convenient to fix a basis and work with matrices rather than linear maps. If we do this, we get:

Example 1.4′. *Let $\mathsf{gl}_n(F)$ be the vector space of all $n \times n$ matrices with entries in F. Define the Lie bracket by*

$$[x, y] = xy - yx$$

where xy is the product of the matrices x and y. As a vector space $\mathsf{gl}_n(F)$ has as a basis the 'matrix units' e_{ij} for $1 \leq i, j \leq n$. When calculating with this basis, the formula

$$[e_{ij}, e_{kl}] = \delta_{jk} e_{il} - \delta_{il} e_{kj}$$

is often useful.

Example 1.5. *Let $F = \mathbb{R}$. Let $L = \mathbb{R}^3$ and define $[-,-] : \mathbb{R}^3 \to \mathbb{R}^3$ by $[x, y] = x \wedge y$, the cross-product of vectors.*

Example 1.6. *Let V be an F-vector space. Define the Lie bracket on V by $[x, y] = 0$ for all $x, y \in V$. Then V is a Lie algebra. Such Lie algebras are said to be* abelian.

Some motivation for Lie algebras (non-examinable)

Lie algebras were discovered by Sophus Lie[1] (1842–1899) while he was attempting to classify certain 'smooth' subgroups of general linear groups. The groups he considered are now called Lie groups. He found that by taking the tangent space at the identity element of such a group, one obtained a Lie algebra. Questions about the group could be reduced to questions about the Lie algebra, in which form they usually proved more tractable.

Example 1.7. *Let*

$$\mathrm{SL}_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = 1 \right\}.$$

We ask, given a 2×2 matrix x, when is $I + \varepsilon x \in \mathrm{SL}_2(\mathbb{R})$? If we neglect terms in ε^2 we get the statement

$$I + \varepsilon x \in \mathrm{SL}_2(\mathbb{R}) \iff \operatorname{tr} x = 0;$$

this defines the Lie algebra $\mathsf{sl}_2(\mathbb{R})$.

We refer to this kind of argument as an argument by 'naïve calculus'. The main disadvantage of the naïve approach is that it doesn't explain how the Lie bracket on $\mathsf{sl}_2(\mathbb{R})$ comes from the group multiplication in $\mathrm{SL}_2(\mathbb{R})$. For a short explanation of this, accessible to those who know a small amount about tangent spaces to manifolds, see Theorem 3.18 in Baker, *Matrix Groups*.

Example 1.8. *Let $\mathrm{GL}_n(\mathbb{R})$ be the group of invertible $n \times n$ matrices with entries in $\mathbb{R}$. Let S be an element of $\mathrm{GL}_n(\mathbb{R})$ and let*

$$G_S(\mathbb{R}) = \left\{ X \in \mathrm{GL}_n(\mathbb{R}) : X^t S X = S \right\}$$

where x^t is the transpose of the matrix x. Then $G_S(\mathbb{R})$ is a group. The associated Lie algebra is

$$\mathsf{gl}_S(\mathbb{R}) = \left\{ x \in \mathsf{gl}_n(\mathbb{R}) : x^t S + S x = 0 \right\}.$$

[1] From `www-groups.docs.st-and.ac.uk/~history/Mathematicians/Lie.html`: 'It was during the year 1867 that Lie had his first brilliant new mathematical idea. It came to him in the middle of the night and, filled with excitement, he rushed to see his friend Ernst Motzfeldt, woke him up and shouted:— "I have found it, it is quite simple!"'

History does not record Motzfeldt's reaction.

2 Fundamental definitions

Example 2.1. *Certain subspaces of $\mathsf{gl}_n(F)$ turn out to be Lie algebras in their own right.*

(i) *Let $\mathsf{sl}_n(F)$ be the vector subspace of $\mathsf{gl}_n(F)$ consisting of all matrices with trace 0. This is known as the* special linear algebra.

(ii) *Let $\mathsf{b}_n(F)$ be the vector subspace of $\mathsf{gl}_n(F)$ consisting of all upper-triangular matrices.*

(iii) *Let $\mathsf{n}_n(F)$ be the vector subspace of $\mathsf{b}_n(F)$ consisting of all strictly upper-triangular matrices.*

Example 2.1 suggests we should make the following definition.

Definition 2.2. Let L be a Lie algebra. A *Lie subalgebra* of L is a vector subspace $M \subseteq L$ such that $[M, M] \subseteq M$; that is, $[x, y] \in M$ for all $x, y \in M$.

A Lie subalgebra is a Lie algebra in its own right.

Definition 2.3. Let L be a Lie algebra. An *ideal* of L is a vector subspace $M \subseteq L$ such that $[L, M] \subseteq M$; that is, $[x, y] \in M$ for all $x \in L$, $y \in M$.

Definition 2.4. The *centre* of a Lie algebra L is

$$Z(L) = \{x \in L : [x, y] = 0 \text{ for all } y \in L\}.$$

If $[x, y] = 0$ we say that x and y *commute*, so the centre consists of those elements which commute with every element of L. So $L = Z(L)$ if and only if L is abelian.

Example 2.5. *An ideal is in particular a subalgebra. But a subalgebra need not be an ideal. For instance if $M = \mathsf{b}_2(\mathbb{C})$ and $L = \mathsf{gl}_2(\mathbb{C})$ then M is a subalgebra of L, but not an ideal.*

Whenever one has a collection of objects (here Lie algebras), one should expect to define maps between them. The interesting maps are those that are structure preserving.

Definition 2.6. Let L and M be Lie algebras. A linear map $\varphi : L \to M$ is a *Lie algebra homomorphism* if

$$\varphi([x, y]) = [\varphi(x), \varphi(y)] \quad \text{for all } x, y \in L.$$

A bijective Lie algebra homomorphism is an *isomorphism*.

Lemma 2.7. *Let $\varphi : L \to M$ be a Lie algebra homomorphism. Then $\ker \varphi$ is an ideal of L and $\operatorname{im} \varphi$ is a subalgebra of M.*

Example 2.8 (The adjoint homomorphism)**.** *Let L be a Lie algebra. Define $\operatorname{ad} : L \to \mathsf{gl}(L)$ by $\operatorname{ad} x = [x, -]$; that is, $(\operatorname{ad} x)(y) = [x, y]$ for $y \in L$. Then ad is a Lie algebra homomorphism.*

Suppose that L is a Lie algebra with vector space basis $x_1, \ldots, x_n$. By bilinearity, the Lie bracket on L is determined by the Lie brackets $[x_i, x_j]$ for $1 \leq i, j \leq n$. Define constants c_{ijk} for $1 \leq i, j, k \leq n$ by

$$[x_i, x_j] = \sum_{k=1}^{n} c_{ijk} x_k.$$

The c_{ijk} are the *structure constants of* L with respect to the basis $x_1, \ldots, x_n$.

Lemma 2.9. *The Lie algebras L and M are isomorphic if and only if they have bases affording the same structure constants.*

Structure constants depend on the choice of basis! If two Lie algebras [2] have different structure constants with respect to some chosen bases, they might still be isomorphic, since a different choice of bases might make the structure constants equal.

In practice, one specifies structure constants slightly informally.

Example 2.10. *Will determine the structure constants of $\mathsf{b}_2(\mathbb{C})$ with respect to some chosen basis.*

As expected, there are quotient Lie algebras and isomorphism theorems

Lemma 2.11. *Let L be a Lie algebra and let I be an ideal of L. The quotient vector space L/I is a Lie algebra with Lie bracket defined by*

$$[x + I, y + I] = [x, y] + I.$$

Theorem 2.12 (First isomorphism theorem)**.** *Let L and M be Lie algebras and let $\varphi : L \to M$ be a Lie algebra homomorphism. Then $\ker \varphi$ is an ideal of L and $L/\ker \varphi \cong \operatorname{im} \varphi$.*

For example, $L/Z(L) \cong \operatorname{im} \operatorname{ad} \subseteq \mathsf{gl}(L)$. This shows that, modulo its centre, any Lie algebra is isomorphic to a subalgebra of some general linear algebra.

Lemma 2.13 (Second isomorphism theorem)**.** *Let I and J be ideals of a Lie algebra L. Then $I + J$ and $I \cap J$ are ideals and $(I + J)/J \cong I/(I \cap J)$.*

[2] Dangerous bend ahead.

Some low-dimensional Lie algebras

To get a little more practice at working with Lie algebras we attempt to classify Lie algebras of small dimension. Depending on how time is going some of the results below may be omitted.

Any 1-dimensional Lie algebra is abelian, so up to isomorphism, there is just one 1-dimensional Lie algebra over any given field.

Theorem 2.14. *If L is a 2-dimensional non-abelian Lie algebra then L has a basis x, y such that $[x, y] = x$. Thus up to isomorphism there are exactly two 2-dimensional Lie algebras over any given field.*

Recall from the first problem sheet that if L is a Lie algebra, then the *derived algebra* L' is the linear span of the brackets $[a, b]$ for $a, b \in L$.

Theorem 2.15. *Suppose that L is a 3-dimensional Lie algebra such that L' is 1-dimensional and L' is not contained in $Z(L)$. Then L has a basis x, y, z such that z is central and $[x, y] = x$.*

Lemma 2.16. *Suppose that L is a 3-dimensional Lie algebra such that L' is 2-dimensional. Then L' is abelian. If $x \in L \backslash L'$ then* $\operatorname{ad} x$ *acts on L' as an invertible linear transformation.*

Theorem 2.17. *Suppose that L is a 3-dimensional complex Lie algebra such that $L' = L$. Then $L \cong \mathsf{sl}_2(\mathbb{C})$.*

Putting these results together with question 5 on sheet 1 and question 4 on sheet 2 will give a classification of all 3-dimensional Lie algebras.

For comparison, the classification of 4-dimensional complex Lie algebras, while known in principle, is still the subject of ongoing research. In one description there are 8 'one-off' Lie algebras and 6 infinite families.

3 Solvable Lie algebras

Lemma 3.1. *Suppose that I is an ideal of L. Then L/I is abelian if and only if I contains the derived algebra L'.*

Definition 3.2. Let L be a Lie algebra. The *derived series* of L is the series with terms

$$L^{(1)} = L' \quad \text{and} \quad L^{(k)} = [L^{(k-1)}, L^{(k-1)}] \text{ for } k \geq 2.$$

Then $L \supseteq L^{(1)} \supseteq L^{(2)} \supseteq \ldots$. We say L is *solvable* if $L^{(m)} = 0$ for some $m \geq 1$.

Example 3.3. *The 2-dimensional non-abelian Lie algebra is solvable over any field. So is the Heisenberg algebra. By question 2 on problem sheet 2,* $\mathsf{sl}_2(\mathbb{C})$ *is not solvable.*

If L is solvable, then the derived series of L provides us with an 'approximation' of L by a finite series of ideals with abelian quotients. This also works the other way around.

Lemma 3.4. *If L is a Lie algebra with ideals*

$$L = I_0 \supseteq I_1 \supseteq \ldots \supseteq I_{m-1} \supseteq I_m = 0$$

such that I_{k-1}/I_k is abelian for $1 \leq k \leq m$, then $L^{(k)} \subseteq I_k$ for $1 \leq k \leq m$. Hence L is solvable.

Thus the derived series of a solvable Lie algebra is its fastest descending series with abelian quotients.

Lemma 3.5. *Let L be a Lie algebra.*

(a) *Suppose L is solvable. Then every subalgebra of L is solvable and every homomorphic image of L is solvable.*
(b) *Suppose that L has an ideal I such that I and L/I are both solvable. Then L is solvable.*
(c) *If I and J are solvable ideals of L, then $I + J$ is a solvable ideal of L.*

Corollary 3.6. *Let L be a Lie algebra. There is a unique solvable ideal of L which contains every solvable ideal of L.*

This largest solvable ideal is said to be the *radical* of L and is denoted $\operatorname{rad} L$. The radical will turn out to be an essential tool in helping to describe finite-dimensional Lie algebras.

Definition 3.7. A non-zero Lie algebra L is said to be *semisimple* if it has no non-zero solvable ideals, or, equivalently, if $\operatorname{rad} L = 0$.

For example, $\mathsf{sl}_2(\mathbb{C})$ is semisimple. The reason for the word 'semisimple' will be revealed shortly.

Lemma 3.8. *If L is a Lie algebra, then the factor algebra $L/\operatorname{rad} L$ is semisimple.*

Definition 3.9. A Lie algebra L is *simple* if it has no ideals other than 0 and itself and it is not abelian.

The restriction that a simple Lie algebra may not be abelian removes only the 1-dimensional abelian Lie algebra. Without this restriction, this Lie algebra would be simple but not semisimple.

We shall prove later (in §7) that a complex Lie algebra is semisimple if and only if it is a direct sum of simple Lie algebras. So to understand an arbitrary complex Lie algebra we need to

(i) understand the structure of solvable Lie algebras over $\mathbb{C}$, and

(ii) classify the simple Lie algebras over $\mathbb{C}$.

4 Engel's theorem and nilpotent Lie algebras

Our first aim is to prove the following.

Theorem 4.1 (Engel's Theorem)**.** *Let V be a vector space. Suppose that L is a Lie subalgebra of $\mathsf{gl}(V)$ such that every element of L is a nilpotent linear transformation of V. Then there is a basis of V in which every element of L is represented by a strictly upper triangular matrix.*

To prove Engel's Theorem, we adapt the strategy used to prove the analogous result for a single nilpotent linear transformation. Thus the key step is to find a vector v such $x(v) = 0$ for all $x \in L$.

Lemma 4.2. *Suppose that L is a Lie subalgebra of $\mathsf{gl}(V)$, where V is non-zero, such that every element of L is a nilpotent linear transformation. Then there is some non-zero $v \in V$ such that $x(v) = 0$ for all $x \in L$.*

There is another way to look at Engel's theorem, which does not depend on the Lie algebra L being given to us as subalgebra of some general linear algebra.

Definition 4.3. The *lower central series* of a Lie algebra L is the series with terms

$$L^1 = L' \quad \text{and} \quad L^k = [L, L^{k-1}] \text{ for } k \geq 2.$$

Then $L \supseteq L^1 \supseteq L^2 \supseteq \ldots$. We say L is *nilpotent* if $L^m = 0$ for some $m \geq 1$.

Example 4.4. (i) *For $n \geq 1$, the Lie algebra of strictly upper triangular matrices $\mathsf{n}_n(F)$ is nilpotent. In particular, the Heisenberg Lie algebra is nilpotent.*

(ii) *A nilpotent Lie algebra is solvable. But a solvable Lie algebra need not be nilpotent. For example $\mathsf{b}_2(\mathbb{C})$ is solvable but not nilpotent.*

Lemma 4.5. *If L is a Lie algebra then L is nilpotent if and only if $L/Z(L)$ is nilpotent.*

Theorem 4.6 (Engel's Theorem, second version)**.** *A Lie algebra L is nilpotent if and only if for all $x \in L$ the linear map* $\operatorname{ad} x : L \to L$ *is nilpotent.*

WARNING: It is very tempting to assume that a Lie subalgebra L of $\mathsf{gl}(V)$ is nilpotent if and only if there is a basis of V in which the elements of L are all represented by strictly upper-triangular matrices. The 'if' direction is true, because a subalgebra of a nilpotent Lie algebra is nilpotent. However, the 'only if' direction is false.

5 Lie's theorem

Lemma 5.1. *Let V be an vector space and let $L \subseteq \mathsf{gl}(V)$ be a Lie algebra. Suppose that there is a basis of V with respect to which every element of L is represented by an upper-triangular matrix. Then L is solvable.*

Thus a necessary condition for L to be triangularisable is that L should be solvable. Lie's theorem says that, over $\mathbb{C}$, this necessary condition is also sufficient.

Theorem 5.2 (Lie's Theorem)**.** *Let V be a complex vector space and let $L \subseteq \mathsf{gl}(V)$ be a Lie algebra. Suppose that L is solvable. Then there is a basis of V in which every element of L is represented by an upper-triangular matrix.*

As for Engel's Theorem, the critical step is to find a common eigenvector for all the elements of L.

Theorem 5.3. *Let V be a complex vector space and let $L \subseteq \mathsf{gl}(V)$ be a solvable Lie algebra. There exists a non-zero vector $v \in V$ such that v is a common eigenvector for all the elements of L.*

Lie's theorem is the definitive structure theorem on solvable complex Lie algebras. Some corollaries:

Corollary 5.4. *Let V be a complex vector space and let $L \subseteq \mathsf{gl}(V)$ be a solvable Lie algebra.*

(i) *If $x \in L'$ then x is a nilpotent endomorphism of V.*

(ii) *There is a a chain of L-invariant subspaces of V,*

$$0 = V_0 \subset V_1 \subset V_2 \ldots \subset V_{n-1} \subset V_n = V$$

such that $\dim V_k = k$.

Corollary 5.5. *Let L be a complex Lie algebra. Then L is solvable if and only if L' is nilpotent.*

Some further applications of Lie's Theorem will be made in §6.

6 Some representation theory

Definition 6.1. Let L be a Lie algebra over a field F. A *representation* of L is a Lie algebra homomorphism

$$\varphi : L \to \mathsf{gl}(V)$$

where V is a finite-dimensional vector space over F. If φ is injective the representation is said to be *faithful*.

It is usual just to say V is a representation of L. This is a useful shorthand, but it puts the emphasis in the wrong place; on its own V is just a vector space — the important part is the map φ.

Example 6.2. *Let L be a a Lie algebra over F.*

(i) *The map* $\mathrm{ad} : L \to \mathsf{gl}(L)$ *is a representation of L.*

(ii) *Suppose L is given as a subalgebra of some $\mathsf{gl}(V)$. Then the inclusion map $L \to \mathsf{gl}(V)$ is a representation of L, the* natural *representation.*

(iii) *Let $V = F$ and define $\varphi : L \to \mathsf{gl}(V)$ by $\varphi(x) = 0$ for all $x \in L$. This is the* trivial *representation.*

(iv) *By question* 3(c) *on sheet 1, the Lie algebra $\mathbb{R}^3_\wedge$ has a faithful* 3-*dimensional representation.*

Definition 6.3. Let $\varphi : L \to \mathsf{gl}(V)$ be a representation of a Lie algebra L. A *subrepresentation* of V is a subspace W of V such that $\varphi(x)(W) \subseteq W$ for all $x \in L$. If V is non-zero and has no non-zero proper subrepresentations then V is said to be *irreducible*, or *simple*.

If $\varphi : L \to \mathsf{gl}(V)$ is a representation of a Lie algebra L and W is a subrepresentation of V, then the quotient vector space V/W becomes a representation of L via the map $\bar{\varphi} : L \to \mathsf{gl}(V/W)$ defined by

$$\bar{\varphi}(x)(v + W) = \varphi(x)(v) + W.$$

In the language of representation theory, Theorem 5.3 becomes:

Theorem 6.4. *Let L be a solvable complex Lie algebra and let $\varphi : L \to \mathsf{gl}(V)$ be a representation of L. Then V is irreducible if and only if $\dim V = 1$.*

Note that the 'if' direction is trivial since a 1-dimensional vector space has no non-trivial proper subspaces. Since it got slightly rushed in lectures, here is a slightly more careful proof of the equivalence of the 'only if' part of Theorem 6.4 with Theorem 5.3.

Proof. Suppose that Theorem 5.3 holds and let $\varphi : L \to \mathsf{gl}(V)$ be a representation of the solvable complex Lie algebra L. The image, $\varphi(L) \subseteq \mathsf{gl}(V)$, is a solvable complex Lie algebra, so by Theorem 5.3, there is a vector $v \in V$ such that $\varphi(x)(v) \in \langle v \rangle$ for all $x \in L$. Thus $\langle v \rangle$ is a 1-dimensional subrepresentation of V. Hence if V is irreducible, we must have $V = \langle v \rangle$ and $\dim V = 1$.

Conversely, assume that Theorem 6.4 is true. Let V be a complex vector space and let $L \subseteq \mathsf{gl}(V)$ be a solvable Lie algebra. So L has a natural representation on V, via the inclusion map $L \to \mathsf{gl}(V)$. Let U be a non-zero L-invariant subspace of V of the smallest possible dimension. Then U is an irreducible subrepresentation of V, so by Theorem 6.4, $\dim U = 1$. Suppose $U = \langle u \rangle$. As U is closed under the action of L, $x(u) \in \langle u \rangle$ for all $x \in L$. Thus u is a common eigenvector for the elements of L. □

We can now prove Theorem 6.4 and hence Lie's Theorem. We need the following two lemmas.

Lemma 6.5. *Let $\varphi : L \to \mathsf{gl}(V)$ be a representation of a Lie algebra L. If I is an ideal of L then*

$$\varphi(I)V = \langle \varphi(a)v : a \in I, v \in V \rangle$$

is a subrepresentation of V.

Lemma 6.6. *Let $\varphi : M \to \mathsf{gl}(V)$ be an irreducible representation of a Lie algebra M. If I an ideal of M such that the maps $\varphi(x) : V \to V$ are nilpotent for all $x \in I$ then $\varphi(I) = 0$.*

The proof of Theorem 6.4 also needs the solution to Problem 1.1 in the form stated below.

Lemma 5.0. *Let $x, y : V \to V$ be linear maps on a complex vector space V. If $xy - yx$ commutes with x then $xy - yx$ is nilpotent.*

For brevity, if $\varphi : L \to \mathsf{gl}(V)$ is a representation we may write $x \cdot v$ rather than $\varphi(x)(v)$. In this form, a representation V is known as an *L-module*. Alternatively, one can define an L-module directly as follows.

Definition 6.7. Suppose that L is a Lie algebra over F. A *Lie module* for L, or alternatively an *L-module*, is an F-vector space V together with a bilinear map

$$L \times V \to V \quad (x, v) \mapsto x \cdot v$$

satisfying the condition

$$[x, y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v) \quad \text{for all } x, y \in L \text{ and } v \in V.$$

Given an L-module V one recovers the associated the associated representation $\varphi : L \to \mathsf{gl}(L)$ by setting $\varphi(x)(v) = x \cdot v$ for $x \in L$ and $v \in V$.

Definition 6.8. Let L be a Lie algebra and let U and V be L-modules. A linear map $\theta : U \to V$ is an *L-module homomorphism* if

$$\theta(x \cdot u) = x \cdot \theta(u) \quad \text{for all } u \in U \text{ and } x \in L.$$

As usual, an L-module isomorphism is a bijective L-module homomorphism.

Example 6.9. (1) *Let $\langle x \rangle$ be a 1-dimensional Lie algebra over F. Given any vector space V over F and any element $t \in \mathsf{gl}(V)$, one may define a representation of $\langle x \rangle$ by mapping x to t. The representations where x acts as the linear maps t and t' are isomorphic if and only if t and t' are similar matrices.*

(2) *Let $L = \langle x, y \rangle$ be the 2-dimensional non-abelian Lie algebra over F, with Lie bracket defined by $[x, y] = x$. For any $\alpha \in F$, the adjoint representation of L is isomorphic to the representation φ_α defined by*

$$\varphi(x) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \varphi(y) = \begin{pmatrix} -1 & \alpha \\ 0 & 0 \end{pmatrix}.$$

Lemma 6.10. *Let $\theta : U \to V$ be a homomorphism of modules for a Lie algebra L.*

(i) $\ker \theta$ *is an L-submodule of U and* $\operatorname{im} \theta$ *is an L-submodule of V.*

(ii) *The quotient map $U \to U/\ker \theta$ is an L-module homomorphism*

(iii) *The vector space isomorphism $U/\ker \theta \to \operatorname{im} \theta$ which maps $u + \ker \theta \in U/\ker \theta$ to $\theta(u)$ is an L-module homomorphism.*

By this lemma, all the usual isomorphism theorems hold for L-modules. The Jordan-Hölder theorem also holds, with essentially the same proof as for representations of associative algebras.

Lemma 6.11 (Schur's Lemma)**.** *Let L be a complex Lie algebra and let U and V be irreducible L-modules.*

(i) *An L-module homomorphism $\theta : U \to V$ is either 0 or an isomorphism.*

(ii) *A linear map $\theta : V \to V$ is an L-module homomorphism if and only if $\theta = \lambda 1_V$ for some $\lambda \in \mathbb{C}$.*

Representations of $\mathsf{sl}_2(\mathbb{C})$

Recall that $\mathsf{sl}_2(\mathbb{C})$ has basis e, f, h where

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \; h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and that the Lie bracket is determined by $[e, f] = h$, $[h, e] = 2e$, $[h, f] = -2f$.

Consider the vector space $\mathbb{C}[X, Y]$ of polynomials in two indeterminants X, Y with complex coefficients. Let V_d be the subspace of homogeneous polynomials of degree d, so V_0 is the vector space of constant polynomials, and for $d \geq 1$, V_d has basis $X^d, X^{d-1}Y, \dots, Y^d$. This means that V_d has dimension $d + 1$.

Lemma 6.12. *Let $\varphi_d : \mathsf{sl}_2(\mathbb{C}) \to \mathsf{gl}(V_d)$ be defined by*

$$\varphi_d(e) = X\frac{\partial}{\partial Y}, \; \varphi_d(f) = Y\frac{\partial}{\partial X}, \; \varphi_d(h) = X\frac{\partial}{\partial X} - Y\frac{\partial}{\partial Y}.$$

and linear extension. Then φ_d is a representation of $\mathsf{sl}_2(\mathbb{C})$.

Theorem 6.13. *The representations V_d are irreducible.*

In fact the V_d give all the irreducible (finite-dimensional) representations of $\mathsf{sl}_2(\mathbb{C})$. To prove this it will be convenient to use the language of modules. The following lemma is critical.

Lemma 6.14. *Let V be an $\mathsf{sl}_2(\mathbb{C})$-module.*

(i) *Let v be an h-eigenvector with $h \cdot v = \lambda v$. If $e \cdot v$ is non-zero then it is an h-eigenvector with eigenvalue $\lambda + 2$. If $f \cdot v$ is non-zero then it is an h-eigenvector with eigenvalue $\lambda - 2$.*

(ii) *There is a h-eigenvector $v \in V$ such that $e \cdot v = 0$. Such a vector v is said to be a* highest-weight vector*.*

We can now prove the statement following Theorem 6.13. In fact it is no harder to prove something slightly stronger.

Theorem 6.15. *Let V be an $\mathsf{sl}_2(\mathbb{C})$-module. Let $v \in V$ be a highest weight vector with $h \cdot v = \lambda v$. Then $\lambda \in \mathbb{N}_0$ and the submodule of V generated by v is isomorphic to V_λ.*

Definition 6.16. A module V for a Lie algebra L is said to be *completely reducible* if V can be written as a direct sum of irreducible L-modules.

Fact 6.17. *Modules for $\mathsf{sl}_2(\mathbb{C})$ are completely reducible.*

This fact is a special case of Weyl's Theorem, which states that if L is *any* complex semisimple Lie algebra then representations of L are completely reducible. An important corollary of Fact 6.17 is that an $\mathsf{sl}_2(\mathbb{C})$-module is determined, up to isomorphism, by the set of eigenvalues of h.

7 Semisimple Lie algebras

The aim of this section is to prove that if L is a complex semisimple Lie algebra then L is a direct sum of simple Lie algebras.

We first prove a criterion for a complex Lie algebra to be solvable.

Lemma 7.1. *Let V be a complex vector space and let $L \subseteq \mathsf{gl}(V)$ be a Lie algebra. If L is solvable then $\mathrm{tr}(xy) = 0$ for all $x \in L$ and $y \in L'$.*

In fact this necessary condition is also sufficient. The proof needs a small result from linear algebra.

Lemma 7.2. *Let V be a complex vector space and let $t : V \to V$ be a linear map. Suppose that t has minimal polynomial*

$$f(X) = (X - \lambda_1)^{a_1} \dots (X - \lambda_r)^{a_r},$$

where the λ_i are pairwise distinct. Let the corresponding primary decomposition of V as a direct sum of generalised eigenspaces be

$$V = V_1 \oplus \dots \oplus V_r,$$

where $V_i = \ker(t - \lambda_i 1_V)^{a_i}$. Then, given any $\mu_1, \dots, \mu_r \in \mathbb{C}$, there is a polynomial $p(X)$ such that

$$p(t) = \mu_1 1_{V_1} + \mu_2 1_{V_2} \dots + \mu_r 1_{V_r}.$$

The proof of this lemma is not likely to be examined and will probably not be dwelt on in lectures.

Proof. Suppose we could find a polynomial $p(X) \in \mathbb{C}[X]$ such that

$$p(X) \equiv \mu_i \bmod (X - \lambda_i)^{a_i}.$$

Take $v \in V_i = \ker(t - \lambda_i 1_V)^{a_i}$. By our supposition, $f(X) = \mu_i + a(X)(X - \lambda_i)^{a_i}$ for some polynomial $a(X)$. Hence

$$p(t)v = \mu_i 1_{V_i} v + a(X)(t - \lambda_i)^{a_i} v = \mu_i v,$$

as required.

The polynomials $(X - \lambda_1)^{a_1} \dots, (X - \lambda_r)^{a_r}$ are coprime. We may therefore apply the Chinese Remainder Theorem, which states that in these circumstances the map

$$\mathbb{C}[X] \to \bigoplus_{i=1}^{r} \frac{\mathbb{C}[X]}{(X - \lambda_i)^{a_i}}$$

$$p(X) \mapsto (p(X) \bmod (X - \lambda_1)^{a_1}, \dots, f(X) \bmod (X - \lambda_r)^{a_r})$$

is surjective, to obtain a suitable $f(X)$. $\square$

Theorem 7.3. *Let V be a complex vector space and let $L \subseteq \mathsf{gl}(V)$ be a Lie algebra. If $\operatorname{tr}(xy) = 0$ for all $x \in L$ and $y \in L'$ then L is solvable.*

To apply this theorem to an abstract Lie algebra we use the adjoint representation.

Definition 7.4. Let L be a complex Lie algebra. The *Killing form* on L is the symmetric bilinear form defined by

$$\kappa(x, y) = \operatorname{tr}(\operatorname{ad} x \circ \operatorname{ad} y) \quad \text{for } x, y \in L.$$

Theorem 7.5 (Cartan's First Criterion)**.** *The complex Lie algebra L is solvable if and only if $\kappa(x, y) = 0$ for all $x \in L$, $y \in L'$.*

Example 7.6. (1) *Will determine the Killing form of the 2-dimensional non-abelian Lie algebra.*

(2) *Suppose that I is an ideal of a Lie algebra L. Write κ_I for the Killing form on I, considered as a Lie algebra in its own right. Then $\kappa_I(x, y) = \kappa(x, y)$ for all $x, y \in I$.*

Lemma 7.7. *Let I be an ideal of a Lie algebra L. Set*

$$I^\perp = \{y \in L : \kappa(x, y) = 0 \text{ for all } x \in I\}.$$

Then $I^\perp$ is an ideal of L and $I \cap I^\perp$ is solvable.

Theorem 7.8 (Cartan's Second Criterion)**.** *The complex Lie algebra L is semisimple if and only if its Killing form κ is non-degenerate.*

Corollary 7.9. *If L is a complex semisimple Lie algebra then L has ideals $I_1 \ldots, I_r$ such that each I_j is a simple Lie algebra and $L = I_1 \oplus \ldots \oplus I_r$.*

8 The root space decomposition

In this section we explore the structure of complex semisimple Lie algebras.

Definition 8.1. Let L be a complex Lie algebra. If $x \in L$ is such that $\operatorname{ad} x : L \to L$ is diagonalisable, we say that x is *semisimple*. A subalgebra H of L is said to be a *Cartan subalgebra* if (i) all the elements of H are semisimple *and* (ii) H is abelian, and H is maximal with these properties.

This definition is convenient for our purposes and fairly easy to work with. It is not the one originally given by Cartan (see page 80 of Humphrey's book), but it can be shown to be equivalent to it when L is semisimple. Perhaps surprisingly, the requirement that H be abelian is redundant: see question 1 on problem sheet 7.

Example 8.2. *Let $L = \mathsf{sl}_3(\mathbb{C})$ and let H be the 2-dimensional Lie subalgebra of L consisting of diagonal matrices. Suppose $h \in H$ has diagonal entries $a_1, \ldots, a_n$. Then for $i \neq j$,*

$$[h, e_{ij}] = (a_i - a_j)e_{ij}.$$

Thus the e_{ij} for $i \neq j$ are simultaneous eigenvectors for the elements of $\operatorname{ad} H$*. So in the basis*

$$\{e_{ij} : i \neq j\} \cup \{e_{ii} - e_{i+1\,i+1} : 1 \leq i < n\}$$

of L, the maps $\operatorname{ad} h$ *for $h \in H$ are represented by a diagonal matrices. This shows that H consists of semisimple elements.*

We can record the eigenvalues of $\operatorname{ad} H$ *on the simultaneous eigenvector e_{ij} by a function $\lambda_{ij} : H \to \mathbb{C}$, defined by*

$$[h, e_{ij}] = \lambda_{ij}(h)e_{ij}.$$

Here $\lambda_{ij}(h) = a_i - a_j$. So in fact, $\lambda_{ij} \in H^\star$, the dual space of linear maps from H to $\mathbb{C}$. Let $L_{\lambda_{ij}}$ be the space of all $\operatorname{ad} H$ *eigenvectors where the eigenvalues of elements of* $\operatorname{ad} H$ *are given by λ_{ij}; that is,*

$$L_{\lambda_{ij}} = \{x \in L : [h, x] = \lambda_{ij}(h)x \textit{ for all } h \in H\}.$$

Here one can check that $L_{\lambda_{ij}} = \langle e_{ij} \rangle$. So, as a vector space (or more precisely, as a representation of H), L decomposes as

$$L = H \oplus \bigoplus_{i \neq j} L_{\lambda_{ij}}$$

It follows from this decomposition that H is not contained in any larger abelian subalgebra of H, so H is a Cartan subalgebra of L.

We now show that the way $\mathsf{sl}_3(\mathbb{C})$ decomposes is typical of the general behaviour.

Lemma 8.3. *Let L be a complex Lie algebra and let H be an abelian Lie subalgebra of L consisting of semisimple elements. For $\alpha \in H^\star$ set*

$$L_\alpha = \{x \in L : [h, x] = \alpha(h)x \text{ for all } h \in H\}.$$

Then each L_α is an $\operatorname{ad} H$*-invariant vector subspace of L and there is a finite subset $\Phi \subseteq H^\star \backslash \{0\}$ such that*

$$L = L_0 \oplus \bigoplus_{\alpha \in \Phi} L_\alpha.$$

Definition 8.4. Keep the hypotheses of the previous lemma. If $\alpha \in H^\star$ is non-zero and $L_\alpha \neq 0$ then α is a *root* of H. The subspace L_α is the *root space* associated to α. The decomposition of L given by the lemma is known as the *root space decomposition* of L with respect to H.

The next example illustrates why the root space decomposition is most useful when H is a Cartan subalgebra of L.

Example 8.5. *Let $L = \mathsf{sl}_3(\mathbb{C})$. Let K be the span of the single element $h = e_{11} - e_{22}$. The root space decomposition of L with respect to K is just the direct sum decomposition of L into eigenspaces for $\operatorname{ad} h$, considered in question 2 of problem sheet 6. This decomposition may be refined by extending K to the Cartan subalgebra H of Example 8.2.*

Basic properties of the root space decomposition

Until the end of this section, let L be a complex semisimple Lie algebra, let H be a Cartan subalgebra of L and let Φ be the set of roots of H. Let κ be the Killing form on L.

Lemma 8.6. *Suppose that $\alpha, \beta \in H^\star$. Then*

(i) $[L_\alpha, L_\beta] \subseteq L_{\alpha+\beta}$.

(ii) *If $\alpha + \beta \neq 0$, then $\kappa(L_\alpha, L_\beta) = 0$.*

(iii) *The restriction of κ to L_0 is non-degenerate.*

Since H is abelian, H is contained in L_0. For semisimple Lie algebra something much better is true.

Theorem 8.7. *Let L be a complex semisimple Lie algebra. If H is a Cartan subalgebra of L then $H = L_0$. An equivalent formulation is that H is self-centralising; that is,*

$$H = \{x \in L : [H, x] = 0\}.$$

There will not be time to prove this theorem in lectures. (One proof is given on page 36 of Humphreys.)

Subalgebras isomorphic to $\mathsf{sl}_2(\mathbb{C})$

We now associate to each root $\alpha \in \Phi$ a Lie subalgebra of L isomorphic to $\mathsf{sl}_2(\mathbb{C})$.

Theorem 8.8. *Suppose that $\alpha \in \Phi$ and that x is a non-zero element in L_α. Then $-\alpha$ is a root and there exists $y \in L_{-\alpha}$ such that $\langle x, y, [x,y]\rangle$ is a Lie subalgebra of L isomorphic to $\mathsf{sl}_2(\mathbb{C})$.*

We denote this subalgebra by $\mathsf{sl}(\alpha)$. We fix once and for all a standard basis of each $\mathsf{sl}(\alpha)$ so that $\mathsf{sl}(\alpha) = \langle e_\alpha, f_\alpha, h_\alpha\rangle$ where $e_\alpha \in L_\alpha$, $f_\alpha \in L_{-\alpha}$, $h_\alpha = [e_\alpha, f_\alpha]$ and $\alpha(h_\alpha) = 2$.

Lemma 8.9. *Given $h \in H$, let θ_h be the map $\theta_h \in H^\star$ defined by*

$$\theta_h(k) = \kappa(h, k) \text{ for all } k \in H.$$

The map $h \mapsto \theta_h$ is a vector space isomorphism between H and $H^\star$.

In particular, associated to each root $\alpha \in \Phi$ there is a unique element $t_\alpha \in H$ such that

$$\kappa(t_\alpha, k) = \alpha(k) \text{ for all } k \in H.$$

Lemma 8.10. *Let $\alpha \in \Phi$. If $x \in L_\alpha$ and $y \in L_{-\alpha}$, then $[x,y] = \kappa(x,y)t_\alpha$. In particular, $h_\alpha = [e_\alpha, f_\alpha] \in \langle\{t_\alpha\}\rangle$.*

We are now in a position to apply the results of §6 on the representation theory of $\mathsf{sl}_2(\mathbb{C})$.

Lemma 8.11. *Let $\alpha \in \Phi$ be a root. We may regard L as an $\mathsf{sl}(\alpha)$-module via restriction of the adjoint representation.*

(i) *The eigenvalues of $\mathsf{sl}(\alpha)$ acting on L are integers.*

(ii) *Let β be a root and let*

$$U = \bigoplus_c L_{\beta + c\alpha},$$

where the sum is over all $c \in \mathbb{C}$ such that $\beta + c\alpha \in \Phi$. Then U is an $\mathsf{sl}(\alpha)$-submodule of L.

The $\mathsf{sl}(\alpha)$-submodule considered in this lemma is the *α-root string through β*. The dimension of U is the *length* of the root string.

Theorem 8.12. *Let $\alpha \in \Phi$. The root spaces $L_{\pm\alpha}$ are 1-dimensional. Moreover, the only multiples of α which lie in Φ are $\pm\alpha$.*

Theorem 8.13. *Suppose that $\alpha, \beta \in \Phi$ and $\beta \neq \pm\alpha$.*

(i) *$\beta(h_\alpha) \in \mathbb{Z}$.*

(ii) *There are integers $r, q \geq 0$ such that $\beta + k\alpha \in \Phi$ if and only if $k \in \mathbb{Z}$ and $-r \leq k \leq q$. Moreover, $r - q = \beta(h_\alpha)$.*

(iii) *If $\alpha + \beta \in \Phi$, then $[e_\alpha, e_\beta]$ is a non-zero scalar multiple of $e_{\alpha+\beta}$.*

(iv) *$\beta - \beta(h_\alpha)\alpha \in \Phi$.*

This essentially determines all the structure constants of L.

Cartan subalgebras as inner product spaces

Lemma 8.14. *If $h \in H$ and $h \neq 0$, then there exists a root $\alpha \in \Phi$ such that $\alpha(h) \neq 0$. The set Φ of roots spans $H^\star$.*

Recall that if $\theta \in H^\star$ then there is an element $t_\theta \in H$ is chosen so that $\kappa(t_\theta, x) = \theta(x)$ for all $x \in L$. By Lemma 8.10, if α is a root then t_α lies in the span of h_α. The next lemma gives a stronger result.

Lemma 8.15. *Let $\alpha \in \Phi$.*

(i) $t_\alpha = \dfrac{h_\alpha}{\kappa(e_\alpha, f_\alpha)}$ *and* $h_\alpha = \dfrac{2t_\alpha}{\kappa(t_\alpha, t_\alpha)}$;

(ii) $\kappa(t_\alpha, t_\alpha)\kappa(h_\alpha, h_\alpha) = 4$.

We may define a bilinear form $(-,-)$ on $H^\star$ by

$$(\theta, \varphi) = \kappa(t_\theta, t_\varphi) \quad \text{for } \theta, \varphi \in H^\star$$

The following lemma gives a more convenient way to work with $(-,-)$.

Lemma 8.16. *Let α, β be roots. Then $\dfrac{2(\beta, \alpha)}{(\alpha, \alpha)} = \beta(h_\alpha)$, the eigenvalue of h_α acting on the root space L_β.*

There will probably not be time to prove the following theorem in lectures. (No new ideas are required, but the calculations are a little fiddly. See §8.5 of Humphreys for a proof.)

Theorem 8.17. *Let $\alpha_1, \dots, \alpha_\ell$ be a basis of $H^\star$ consisting of roots. The* real *subspace spanned by the α_i contains all the roots, and so does not depend on the choice of basis. Call this space E. The restriction of $(-,-)$ to E is a real valued inner product.*

The next theorem summarises some important results we have proved.

Theorem 8.18. *The roots Φ have the following properties.*

(i) *Φ spans E.*

(ii) *If $\alpha \in \Phi$ then $-\alpha \in \Phi$ but no other scalar multiple of Φ is a root.*

(iii) *If $\alpha, \beta \in \Phi$ then $\beta - \dfrac{2(\beta, \alpha)}{(\alpha, \alpha)}\alpha \in \Phi$*

(iv) *If $\alpha, \beta \in \Phi$ then $\dfrac{2(\beta, \alpha)}{(\alpha, \alpha)} \in \mathbb{Z}$*

Example 8.19. *Let $L = \mathsf{sl}_3(\mathbb{C})$. With the notation of Example 8.2, let $\alpha = \lambda_1 - \lambda_2 \in H^\star$ and let $\beta = \lambda_2 - \lambda_3 \in H^\star$. Then $(\alpha, \alpha) = (\beta, \beta)$ and the angle between α and β is $2\pi/3$. As a set,*

$$\Phi = \{\pm\alpha, \pm\beta, \pm(\alpha + \beta)\}.$$

9 The classification of complex semisimple Lie algebras

The aim of this section is to sketch a proof of the following classification theorem. Some parts will be treated informally, and for this reason none of the material in this section will be examinable.

Theorem 9.1. *Let L be a complex simple Lie algebra. Then either L is isomorphic to one of the* classical Lie algebras*, $\mathsf{sl}_{\ell+1}(\mathbb{C})$ for $\ell \geq 1$, $\mathsf{so}_{2\ell+1}(\mathbb{C})$ for $\ell \geq 2$, $\mathsf{sp}_{2\ell}$ for $\ell \geq 3$ and $\mathsf{so}_{2\ell}(\mathbb{C})$ for $\ell \geq 4$, or L is one of the five* exceptional Lie algebras*, g_2, f_4, e_6, e_7, e_8.*

The definitions of the orthgonal and symplectic Lie algebras were given in the first lecture. Recall that $\mathsf{so}_{2\ell}(\mathbb{C}), so_{2\ell+1}(\mathbb{C}), \mathsf{sp}_{2\ell}(\mathbb{C})$ are defined to be $\mathsf{gl}_S(\mathbb{C})$ where S is

$$\begin{pmatrix} 0 & I_\ell \\ I_\ell & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_\ell \\ 0 & I_\ell & 0 \end{pmatrix}, \begin{pmatrix} 0 & I_\ell \\ -I_\ell & 0 \end{pmatrix}$$

respectively, and $\mathsf{gl}_S(\mathbb{C}) = \{x \in \mathsf{gl}_n(\mathbb{R}) : x^t S + Sx = 0\}$; see Example 1.8.

The smallest exceptional Lie algebra is g_2, the 14-dimensional Lie algebra of derivations of the algebra of octonions. (The other exceptional Lie algebra are too complicated to even try to define them here.) Highly recomended for further reading is the article by John Baez, *Octonions*; see his website.

Root systems

Let E be a real vector space with an inner product written $(-,-)$. Given a non-zero vector $v \in E$, let s_v be the reflection in the hyperplane normal to v. Thus s_v sends v to $-v$ and fixes all elements y such that $(y, v) = 0$.

Lemma 9.2. *For each $x \in E$,*

$$s_v(x) = x - \frac{2(x, v)}{(v, v)} v.$$

The reflection s_v preserves the inner product $(-,-)$; that is, $(s_v(x), s_v(y)) = (x, y)$ for all $x, y \in E$.

Convention 9.3. *Set* $\langle x, v\rangle := \dfrac{2(x,v)}{(v,v)}$

Note that the map $\langle -,-\rangle : E \times E \to \mathbb{R}$ is only linear with respect to its first variable.

Definition 9.4. A subset R of a real vector space E is a *root system* if it satisfies the following axioms.

(R1) R is finite, it spans E, and it does not contain 0.

(R2) If $\alpha \in R$, then the only scalar multiples of α in R are $\pm\alpha$.

(R3) If $\alpha \in R$, then the reflection s_α permutes the elements of R.

(R4) If $\alpha, \beta \in R$, then $\langle \beta, \alpha\rangle \in \mathbb{Z}$.

The elements of R are called *roots*.

By Theorem 8.18, if Φ is the set of roots of a complex semisimple Lie algebra L with respect to a Cartan subalgebra H, and E is the real inner product space given by restricting the Killing form on $H^\star$ to the real span of the roots Φ, then Φ is a root system in E.

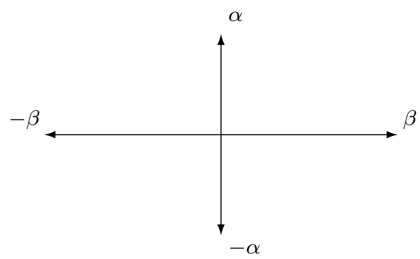
By question 5 on sheet 7, if L is simple then it is not possible to partition Φ into non-empty subsets Φ_1, Φ_2 such that $(\alpha, \beta) = 0$ for all $\alpha \in \Phi_1, \beta \in \Phi_2$; such root systems are said to *irreducible*. Any root system decomposes as direct sum of irreducible root systems.

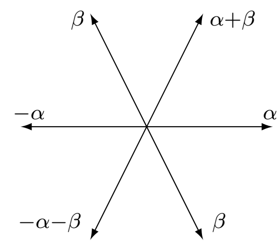
The following lemma gives the first indication that the axioms for root systems are quite restrictive.

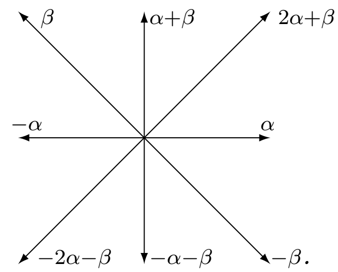
Lemma 9.5. *Suppose that R is a root system in the real inner-product space E. Let $\alpha, \beta \in R$ with $\beta \neq \pm\alpha$. Then*

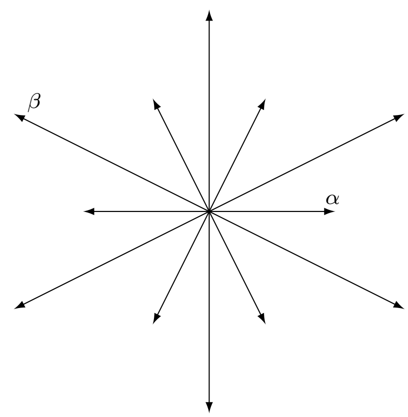
$$\langle \alpha, \beta\rangle \langle \beta, \alpha\rangle \in \{0, 1, 2, 3\}.$$

Example 9.6. *Suppose that R is a root system in the* 1*-dimensional inner product space E. Then $R = \{\pm\alpha\}$ for some non-zero $\alpha \in E$. There are essentially four different* 2*-dimensional root systems, $A_1 \times A_1$, A_2, B_2 and G_2, with diagrams as below. Of these, all but $A_1 \times A_1$ are irreducible.*









This example motivates the definition of isomorphism for root systems.

Definition 9.7. Let R and R' be root systems in the real inner-product spaces E and E', respectively. We say that R and R' are *isomorphic* if there is a vector space isomorphism $\varphi : E \to E'$ such that

(a) $\varphi(R) = R'$, and
(b) for any two roots $\alpha, \beta \in R$, $\langle \alpha, \beta \rangle = \langle \varphi(\alpha), \varphi(\beta) \rangle$.

Suppose that L is a complex semisimple Lie algebra with roots Φ with respect to some Cartan subalgebra. Theorem 8.13 says that given the values $\langle \beta, \alpha \rangle$ for all $\beta, \alpha \in \Phi$ we can determine most of the structure of L. In fact, with more work, one can show that these values determine L up to isomorphism.

Fact 9.8. *Let L and M be complex semisimple Lie algebras. Then L and M are isomorphic if and only if their root systems are isomorphic.*

So to classify complex semisimple Lie algebras up to isomorphism, it is enough to classify root systems up to isomorphism. To do this we need some further ideas.

Bases and Dynkin diagrams

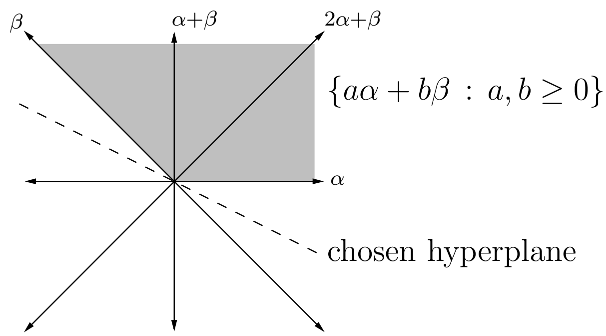
Definition 9.9. Let R be a root system in the real inner product space E. A subset B of R is a *base* for the root system R if

(B1) B is a vector space basis for E, and

(B2) every $\beta \in R$ can be written as $\beta = \sum_{\alpha \in B} k_\alpha \alpha$ with $k_\alpha \in \mathbb{Z}$, where all the non-zero coefficients k_α have the same sign.

The elements of B are said to be *simple roots*.

It is quite easy to show that every root system has a base. The idea of the proof is to fix a hyperplane in E and choose as elements of the base the elements of Φ that are nearest to E. For instance, α, β is a base of the root system of type B_2.



Definition 9.10. Let R be a root system and let $\alpha_1, \dots, \alpha_\ell$ be a base for R. The *Cartan matrix* of R is the $\ell \times \ell$-matrix with entries $C_{ij} = \langle \alpha_i, \alpha_j \rangle$.

Given a base for a root system it is possible to reconstruct all the roots by repeatedly taking reflections in the simple roots. This can be used to show the following.

Fact 9.11. *Let R and R' be root systems in the real inner product spaces E and E'. Then R is isomorphic to R' if and only if R and R' have bases affording the same Cartan matrices*

Dynkin introduced a convenient way to summarise the information in a Cartan matrix.

Definition 9.12. Let R be a root system with simple roots $\alpha_1, \dots, \alpha_\ell$. Let Δ be the graph with vertices labelled by the simple roots. Between the vertices labelled by simple roots α, β draw $d_{\alpha\beta}$ edges where

$$d_{\alpha\beta} = \langle \alpha, \beta \rangle \langle \beta, \alpha \rangle \in \{0, 1, 2, 3\}.$$

If $d_{\alpha\beta} > 0$, which happens whenever α and β have different lengths and are not orthogonal, draw an arrow on the edge pointing from the longer root to the shorter root. The graph Δ is the *Dynkin diagram of R*.

Lemma 9.13. *A root system is irreducible if and only if its Dynkin diagram is connected.*

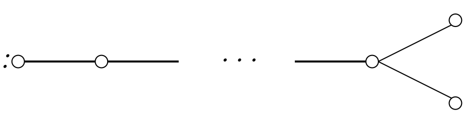
So to classify root systems up to isomorphism it is enough to classify the connected Dynkin diagrams. Unfortunately there won't be time to do this in lectures. A nice account is given in §2.5 of *Lectures on Lie groups and Lie algebras* by Carter, Segal, Macdonald. The result is as follows:

Theorem 9.14. *Given an irreducible root system R, the unlabelled Dynkin diagram associated to R is either a member of one of the four families*

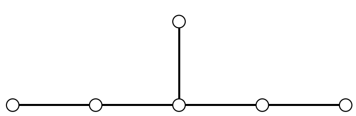
A_ℓ for $\ell \geq 1$: 

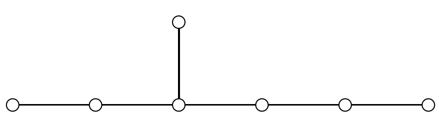
B_ℓ for $\ell \geq 2$: 

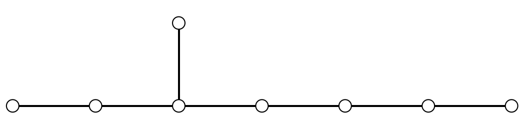
C_ℓ for $\ell \geq 3$: 

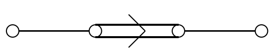
D_ℓ for $\ell \geq 4$: 

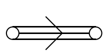
where each of the diagrams above has ℓ vertices, or one of the five exceptional diagrams

E_6: 

E_7: 

E_8: 

F_4: 

G_2: 

Note that there are no repetitions in this list. For example, we have not included C_2 in the list, as it is the same diagram as B_2, and so the associated root systems are isomorphic. This corresponds to an isomorphism of Lie algebras, $\mathsf{sp}_4(\mathbb{C}) \cong \mathsf{so}_5(\mathbb{C})$.

Finally one can check that the root systems of $\mathsf{sl}_\ell(\mathbb{C})$, $\mathsf{so}_{2\ell+1}(\mathbb{C})$, $\mathsf{sp}_{2\ell}(\mathbb{C})$ and $\mathsf{so}_{2\ell}(\mathbb{C})$ have types A, B, C and D respectively. So, apart from the five exceptional Lie algebras, any complex simple Lie algebra is isomorphic to a member of one of these families.