A COMBINATORIAL PROOF OF THE MURNAGHAN–NAKAYAMA RULE

JASDEEP KOCHHAR AND MARK WILDON

Abstract. The Murnaghan–Nakayama rule is a combinatorial rule for the character values of symmetric groups. We give a new combinatorial proof by explicitly finding the trace of the representing matrices in the standard basis of Specht modules. This gives an essentially bijective proof of the rule. A key lemma is an extension of a straightening result proved by the second author to skew-tableaux. Our module theoretic methods also give short proofs of Pieri's rule and Young's rule.

1. Introduction

In this article we give a new combinatorial proof of the Murnaghan–Nakayama rule for the values of the ordinary character χ^λ of S_n canonically labelled by the partition λ of $n \in \mathbf{N}$. To state the rule, we require the following definitions.

Let $\ell(\lambda)$ denote the number of parts of λ. Given partitions μ and λ of m and $m+n$ respectively, we say that μ is a *subpartition* of λ, and write $\mu \subseteq \lambda$, if $\ell(\mu) \leq \ell(\lambda)$ and $\mu_i \leq \lambda_i$ for $1 \leq i \leq \ell(\mu)$. We define the *skew-diagram* $[\lambda/\mu]$ to be the set of *boxes*

$$\{(i,j) : 1 \leq i \leq t \text{ and } \mu_i < j \leq \lambda_i\},$$

and call λ/μ a *skew-partition.* We define *row* k (resp. *column* k) of λ/μ to be the subset of $[\lambda/\mu]$ of boxes whose first (resp. second) coordinate equals k. Let $\mathrm{ht}(\lambda/\mu)$ be one less than the number of non-empty rows of $[\lambda/\mu]$. We define a *border strip* to be a skew-partition whose skew-diagram is connected and which contains no four boxes forming the partition $(2,2)$.

Theorem 1.1 (Murnaghan–Nakayama rule)**.** *Let $m, n \in \mathbf{N}$. Let $\rho \in S_{m+n}$ be an n-cycle and let π be a permutation of the remaining m numbers. Then*

$$\chi^\lambda(\pi\rho) = \sum (-1)^{\mathrm{ht}(\lambda/\mu)} \chi^\mu(\pi),$$

where the sum is over all $\mu \subset \lambda$ such that $|\mu| = m$ and λ/μ is a border strip.

Before we continue, we provide an example of the Murnaghan–Nakayama rule, showing how it can be applied recursively to calculate single character values.

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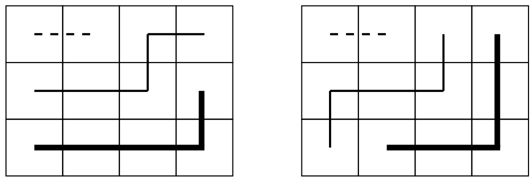


FIGURE 1. The border strips of size 5 (solid) and 2 (dashed) removed to compute $\chi^{(4,4,4)}\big((1,2)(3,4,5,6,7)(8,9,10,11,12)\big)$.

Example 1.2. Let $\sigma = (1,2)(3,4,5,6,7)(8,9,10,11,12) \in S_{12}$. We evaluate $\chi^{(4,4,4)}(\sigma)$. Taking $\rho = (8,9,10,11,12)$, we begin by removing border strips of size 5 from $(4,4,4)$. As shown in Figure 1 there are two such strips, namely $(4,4,4)/(4,3)$ and $(4,4,4)/(3,3,1)$, of heights 1 and 2, respectively. Therefore by the Murnaghan–Nakayama rule

$$\chi^{(4,4,4)}(\sigma) = (-\chi^{(4,3)} + \chi^{(3,3,1)})\big((1,2)(3,4,5,6,7)\big).$$

Two further applications of the Murnaghan–Nakayama rule to each summand now show that $\chi^{(4,4,4)}(\sigma) = (\chi^{(2)} + \chi^{(2)})\big((1,2)\big) = 1 + 1 = 2$.

As Stanley notes in [12, page 401], the Murnaghan–Nakayama rule was first proved by Littlewood and Richardson in [6, §11]. Their proof derives it, essentially as stated in Theorem 1.1, as a corollary of the older Frobenius formula [2, page 519, (6)] for the characters of symmetric groups. (For a modern statement of the Frobenius formula see [12, (7.77)] or [3, (4.10)].) Later Murnaghan [9, page 462, (13)] gave a similar but independent derivation of the rule. Murnaghan's paper was cited by Nakayama [10, page 183] who gave a more concise proof, still from the Frobenius formula. James gave a different proof in [4, Ch. 11] using the relatively deep Littlewood–Richardson rule. More recently, elegant involutive proofs have been given by Mendes and Remmel [8, Theorem 6.3] using Pieri's rule and Young's rule and by Loehr [7, §11] using his labelled abacus representation of antisymmetric functions.

1.1. **Outline of the proof.** Our starting point is Corollary 2.8 of Theorem 2.2 below, which states that $\chi^\lambda(\pi\rho) = \sum_\mu \chi^\mu(\pi)\chi^{\lambda/\mu}(\rho)$, where $\chi^{\lambda/\mu}$ is the ordinary character of the skew-Specht module $S^{\lambda/\mu}$ defined in §2.1. By this corollary, it suffices to show that if ρ is an n-cycle then

$$\chi^{\lambda/\mu}(\rho) = \begin{cases} (-1)^{\mathrm{ht}(\lambda/\mu)} & \text{if } \lambda/\mu \text{ is a border strip of size } n \\ 0 & \text{otherwise.} \end{cases} \tag{1.1}$$

We do this by explicitly computing the trace of the matrix representing the n-cycle ρ in the standard basis (see Theorem 2.1) of $S^{\lambda/\mu}$. In the critical case where λ/μ is a border strip, we show that there is a unique basis element giving a non-zero contribution to the trace. This gives a new and essentially bijective proof of the Murnaghan–Nakayama rule.

1.2. **Background to Theorem 2.2.** Theorem 2.2 is the main result in [5]. The proof in [5] constructs skew-Specht modules as ideals in the group algebra of S_n over a field. Our proof using polytabloids instead generalizes James' proof of the modular branching rule for Specht modules [4, Ch. 9]. In this way we obtain a stronger isomorphism for integral modules that replaces the lexicographic order used in [4] and [5] with the dominance order.

1.3. **Outline.** In §2.1 and §2.2 we define λ/μ-polytabloids and state Theorem 2.1, that the set of standard λ/μ-polytabloids is a $\mathbf{Z}$-basis of $S^{\lambda/\mu}$. In §2.3 we prove Theorem 2.2 and deduce Corollary 2.8. In §3 we use Theorem 2.2 to give short module-theoretic proofs of Pieri's rule and Young's rule. In §4 we prove Lemma 4.3, which gives a necessary condition for a standard polytabloid to appear with a non-zero coefficient when a given λ/μ-polytabloid is written as a linear combination of standard polytabloids. This generalises Proposition 4.1 in [13] to skew-tableaux. In §5 we use Lemma 4.3 to give a bijective proof of (1.1) when λ/μ is a border strip. We then deal with the remaining case in §6 by a short argument using Pieri's rule and Young's rule.

2. Background

2.1. **Skew-tableaux and skew-Specht modules.** Fix m, $n \in \mathbf{N}$. Let λ be a partition of $m+n$ and let μ be a subpartition of λ of size m. We define a *λ/μ-tableau* t to be a bijective function $t : [\lambda/\mu] \to \{1, 2, \ldots, n\}$, and call t a *skew-tableau* of *shape* λ/μ. We call $(i,j)t$ the *entry of* t in position (i,j). Thus a λ/μ-tableau can be visualized as a filling of the boxes $[\lambda/\mu]$ with distinct entries from $\{1, \ldots, n\}$. We draw skew-diagrams with the largest part at the top of the page: thus the *top row* is row 1, and so on.

There is a natural action of S_n on the set of λ/μ-tableaux defined by $(i,j)(t\sigma) = \big((i,j)t\big)\sigma$ for $\sigma \in S_n$. Given a λ/μ-tableau t, let $R(t)$ (resp. $C(t)$) be the subgroup of S_n consisting of all permutations that setwise fix the entries in each row (resp. column) of t. We define an equivalence relation $\backsim$ on the set of λ/μ-tableaux by $t \backsim u$ if and only if there exists $\pi \in R(t)$ such that $u = t\pi$. The *λ/μ-tabloid* $\{t\}$ is the equivalence class of t. A short calculation shows that S_n acts on the set of λ/μ-tabloids by $\{t\}\sigma = \{t\sigma\}$.

Generalizing the usual definitions to skew-partitions, we say that a λ/μ-tableau is *row standard* if the entries in the rows are increasing when read from left to right, and *column standard* if the entries in the columns are increasing when read from top to bottom. A tableau t that is both row standard and column standard is a *standard* tableau.

Let $M^{\lambda/\mu}$ be the $\mathbf{Z}S_n$-permutation module spanned by the λ/μ-tabloids. We define the *λ/μ-polytabloid* $e(t) \in M^{\lambda/\mu}$ by

$$e(t) = \sum_{\sigma \in C(t)} \mathrm{sgn}(\sigma)\{t\}\sigma.$$

If t is a standard tableau then we say that $e(t)$ is a *standard polytabloid*. The *skew-Specht module* $S^{\lambda/\mu}$ is then the $\mathbf{Z}S_n$-module spanned by all λ/μ-polytabloids. Taking $\mu = \varnothing$ this is the Specht module S^λ, defined over $\mathbf{Z}$.

2.2. **Garnir relations and the Standard Basis Theorem.** If $\sigma \in S_n$ then an easy calculation shows that

$$e(t)\sigma = e(t\sigma). \tag{2.1}$$

Hence $S^{\lambda/\mu}$ is cyclic, generated by any λ/μ-polytabloid. Moreover given $\tau \in C(t)$ then

$$e(t)\tau = \mathrm{sgn}(\tau)e(t) \tag{2.2}$$

so $S^{\lambda/\mu}$ is spanned by the λ/μ-polytabloids $e(t)$ for t a column standard λ/μ-tableau. Let $\widetilde{t}$ be the unique column standard λ/μ-tableau whose columns agree setwise with t and let $\varepsilon_t \in \{+1, -1\}$ be defined by $e(\widetilde{t}) = \varepsilon_t e(t)$. We call $\widetilde{t}$ the *column straightening* of t.

Suppose that (i,j) and $(i,j+1)$ are boxes in $[\lambda/\mu]$. Given a λ/μ-tableau t, let $X = \{(i,j)t, (i+1,j)t, \ldots\}$ be the set of entries in column j of t weakly below box (i,j), and let $Y = \{\ldots, (i-1,j+1)t, (i,j+1)t\}$ be the set of entries in column $j+1$ of t weakly above box $(i,j+1)$. Let $C_{X,Y}$ be the set of all products of transpositions $(x_1,y_1)\ldots(x_k,y_k)$ for $x_1 < \ldots < x_k$ and $y_1 < \ldots < y_k$ where $\{x_1,\ldots,x_k\} \subseteq X$ and $\{y_1,\ldots,y_k\} \subseteq Y$ are non-empty k-sets. We define the *Garnir element for X and Y* by

$$G_{X,Y} = 1 + \sum_{\sigma \in C_{X,Y}} \mathrm{sgn}(\sigma)\sigma \in \mathbf{Z}S_{X\cup Y}. \tag{2.3}$$

Restated, replacing ideals in the group ring $\mathbf{Z}S_n$ with polytabloids, (3.8) in [1] implies that

$$e(t)G_{X,Y} = 0. \tag{2.4}$$

Similarly restated, Theorem 3.9 in [1] is as follows.

Theorem 2.1 (Standard Basis Theorem)**.**

(i) *Any λ/μ-polytabloid can be expressed as a $\mathbf{Z}$-linear combination of standard λ/μ-polytabloids by applications of column relations* (2.2) *and Garnir relations* (2.4).

(ii) *The $\mathbf{Z}S_n$-module $S^{\lambda/\mu}$ has the set of standard λ/μ-polytabloids as a $\mathbf{Z}$-basis.*

We remark that the proofs of Theorem 7.2 and 8.4 in [4], for Specht modules labelled by partitions, but defined using polytabloids, generalize easily to prove (2.4) and Theorem 2.1 exactly as stated above. We give a small example of Garnir relations in Example 2.9 below.

2.3. **A filtration for Specht modules.** Fix throughout this section m, $n \in \mathbf{N}$ and a partition λ of $m+n$. Let $S_{(m,n)} = S_{\{1,2,\ldots,m\}} \times S_{\{m+1,m+2,\ldots,m+n\}}$. We shall prove the following theorem.

Theorem 2.2 ([5, Theorem 3.1])**.** *The restricted Specht module $S^\lambda\!\downarrow_{S_{(m,n)}}$ has a filtration by $\mathbf{Z}S_{(m,n)}$-modules with factors isomorphic to $S^\mu \boxtimes S^{\lambda/\mu}$, where each subpartition μ of λ of size m occurs exactly once.*

The following preliminaries are required. Suppose that λ has first part c. Given a λ-tableau t we define the *m-shape* of t to be the composition $(\gamma_1, \ldots, \gamma_c)$ such that $\gamma_j = |\{x \in \text{column } j \text{ of } t : x \leq m\}|$. Let $\trianglerighteq$ denote the dominance order on compositions, defined by $\gamma \trianglerighteq \delta$ if and only if $\ell(\gamma) \leq \ell(\delta)$ and $\sum_{i=1}^k \gamma_i \geq \sum_{i=1}^k \delta_i$ whenever $1 \leq k \leq \ell(\gamma)$. For each composition γ such that $\ell(\gamma) \leq c$ we define

$$V^{\trianglerighteq\gamma} = \langle e(t) : t \text{ a column standard } \lambda\text{-tableau of } m\text{-shape } \delta \text{ where } \delta \trianglerighteq \gamma \rangle_{\mathbf{Z}}.$$

Note that the definition of the m-shape agrees with the notation $b(y)$ in the proof of [5, Theorem 3.1]. We require the following total ordering on the set of column standard λ-tableaux, defined implicitly in [4, page 30].

Definition 2.3. Let t and u be column standard λ-tableaux. We write $u > t$ if and only if the greatest entry appearing in a different column in u to t appears further right in u than t.

For instance, the $>$ order on column standard $(2,2)$-tableaux is

$$\begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline\end{array} > \begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline\end{array} > \begin{array}{|c|c|}\hline 2&1\\\hline 3&4\\\hline\end{array} > \begin{array}{|c|c|}\hline 1&2\\\hline 4&3\\\hline\end{array} > \begin{array}{|c|c|}\hline 2&1\\\hline 4&3\\\hline\end{array} > \begin{array}{|c|c|}\hline 3&1\\\hline 4&2\\\hline\end{array}.$$

Note that here, as in general, the greatest tableau under $>$ is standard. Several times below we use that if $x > y$ and x is to the left of y in the column standard tableau u then $\widetilde{u(x,y)} > u$.

Proposition 2.4. *Let u be a column standard λ-tableau of m-shape γ. Then $e(u)$ is equal to a $\mathbf{Z}$-linear combination of standard λ-polytabloids $e(t)$ where each t has m-shape μ' for some partition μ such that $\mu' \trianglerighteq \gamma$.*

Proof. If u is standard then γ is a partition, and there is nothing to prove. If u is not standard then there exists $(i,j) \in [\lambda]$ such that $(i,j)u > (i,j+1)u$. Let X and Y be as defined in (2.3). By (2.4) we have

$$0 = e(u) + \sum_{\sigma \in C_{X,Y}} \varepsilon_{u\sigma} \operatorname{sgn}(\sigma) e(\widetilde{u\sigma})$$

where $\widetilde{u\sigma}$ and $\varepsilon_{u\sigma} \in \{+1, -1\}$ are as defined at the start of §2.2. Let $\sigma \in C_{X,Y}$. Since the minimum of X exceeds the maximum of Y, we have $x > y$ for each transposition (x, y) in σ. Hence $\widetilde{u\sigma} > u$. Moreover, if there are exactly k transpositions (x, y) such that $x > m \geq y$ then $\delta_j = \gamma_j + k$, $\delta_{j+1} = \gamma_{j+1} - k$ and $\delta_{j'} = \gamma_j$ for $j' \neq j, j+1$. Hence $\delta \trianglerighteq \gamma$. The lemma now follows by induction on the $\geq$ and $\trianglerighteq$ orders. □

Corollary 2.5. *Let μ be a subpartition of λ of size m. Then $V^{\trianglerighteq \mu'}$ is a $\mathbf{Z}S_{(m,n)}$-submodule of S^λ with $\mathbf{Z}$-basis given by the standard λ-tableaux of m-shape ν' such that $\nu' \trianglerighteq \mu'$.*

Proof. Since the standard λ-polytabloids are linearly independent by Theorem 2.1(ii), it follows immediately from Proposition 2.4 that $V^{\trianglerighteq \mu'}$ has a $\mathbf{Z}$-basis as claimed. If $\pi \in S_{(m,n)}$ and s is a standard λ-tableau of m-shape ν' then $s\pi$ also has m-shape ν', as does $\widetilde{s\pi}$. By (2.2) and Proposition 2.4, $e(s\pi) = \pm e(\widetilde{s\pi}) \in V^{\trianglerighteq \nu'} \subseteq V^{\trianglerighteq \mu'}$. Hence $V^{\trianglerighteq \mu'}$ is a $\mathbf{Z}S_{(m,n)}$-module. □

Given a μ-tableau u with (as usual) entries $\{1, \ldots, m\}$ and a λ/μ-tableau v with entries $\{m+1, \ldots, m+n\}$, let $u \cup v$ denote the λ-tableau defined by

$$(i, j)(u \cup v) = \begin{cases} (i, j)u & \text{if } (i, j) \in [\mu] \\ (i, j)v & \text{if } (i, j) \in [\lambda/\mu]. \end{cases}$$

Clearly every λ-tableau of m-shape μ' is of this form. We shall show that the action of $S_{(m,n)}$ on standard λ-polytabloids is compatible with this factorization. We require the following lemma and proposition, which are illustrated in Example 2.9 below.

Lemma 2.6. *Let μ be a subpartition of λ of size m. Let u be a column standard μ-tableau and let v be a λ/μ-tableau. Let $(i, j) \in [\mu]$ be a box such that*

$$m \geq (i, j)u > (i, j+1)u.$$

Let $r = \mu'_j$ so (r, j) is the lowest box in column j of u, and define

$$X = \{(i, j)u, (i+1, j)u, \ldots, (r, j)u, (r+1, j)v, \ldots\},$$
$$Y = \{\ldots, (i-1, j+1)u, (i, j+1)u\},$$
$$X^\star = \{(i, j)u, (i+1, j)u, \ldots, (r, j)u\}.$$

Let $C_{X^\star,Y} = \{\sigma \in C_{X,Y} : x\sigma = x \text{ for all } x \in X \backslash X^\star\}$. Then

$$0 = e(u \cup v) + \sum_{\sigma^\star \in C_{X^\star,Y}} \operatorname{sgn}(\sigma^\star) e(u \cup v)\sigma^\star + \sum_{\sigma \in C_{X,Y} \backslash C_{X^\star,Y}} \operatorname{sgn}(\sigma) e(u \cup v)\sigma$$

where

(i) *for each $\sigma^\star$, we have $e(u \cup v)\sigma^\star = e(u\sigma^\star \cup v)$ and $\widetilde{u\sigma^\star} > u$;*

(ii) *for each σ, $e(u \cup v)\sigma$ is a $\mathbf{Z}$-linear combination of polytabloids $e(s)$ for standard tableaux s of m-shape ν' where $\nu' \triangleright \mu'$.*

Proof. Since $G_{X,Y} = 1+\sum_{\sigma^\star \in C_{X^\star,Y}} \operatorname{sgn}(\sigma^\star)\sigma^\star + \sum_{\sigma \in C_{X,Y}\backslash C_{X^\star,Y}} \operatorname{sgn}(\sigma)\sigma$, the displayed equation follows from (2.4). Since $C_{X^\star,Y} \subseteq S_{\{1,\ldots,m\}}$, (i) follows from the observation after Definition 2.3. Take $\sigma \in C_{X,Y}\backslash C_{X^\star,Y}$ and let $w = (u \cup v)\sigma$. Since σ involves a transposition (x, y) with $x > m \geq y$, the statistic k in the proof of Proposition 2.4 is non-zero. Hence the m-shape of $e(\widetilde{w})$ is δ for some composition δ with $\delta \rhd \mu'$. The statement of Proposition 2.4 now implies that $e(\widetilde{w})$ is a $\mathbf{Z}$-linear combination of standard polytabloids $e(s)$ for s of m-shape ν' where $\nu' \trianglerighteq \delta$. Hence $\nu' \rhd \mu'$, as required for (ii). □

Proposition 2.7. *Let μ be a subpartition of λ of size m. Let u be a column standard μ-tableau and let t be a standard λ/μ-tableau. If $e(u) = \sum_S \alpha_S e(S)$ where the sum is over all standard μ-tableaux S and $\alpha_S \in \mathbf{Z}$ for each S then*

$$e(u \cup t) \in \sum_S \alpha_s e(S \cup t) + \sum_{\nu' \rhd \mu'} V^{\trianglerighteq \nu'}.$$

Proof. If u is standard the result is obvious. If not, there exists a box $(i, j) \in [\mu]$ such that $m \geq (i, j)u > (i + 1, j)u$. Let $X^\star$ and Y be as in Lemma 2.6. By Lemma 2.6(ii) we have

$$e(u \cup t) \in -\sum_{\sigma^\star \in C_{X^\star,Y}} \operatorname{sgn}(\sigma^\star)e(u \cup t)\sigma^\star + \sum_{\nu' \rhd \mu'} V^{\trianglerighteq \nu'}.$$

Using Lemma 2.6(i), the result now follows by induction on the $\geq$ order. □

We also need the analogous lemma in which $(i, j)u > (i, j+1)u > m$, $Y^\star = \{(r, j+1)u, \ldots, (i, j+1)u\}$ where now $r = \mu'_{j+1} + 1$, and the relevant sets of coset representatives are $C_{X,Y^\star}$ and $C_{X,Y}\backslash C_{X,Y^\star}$. It implies the analogous proposition in which $e(t \cup v)$ is straightened, where t is a standard μ-tableau and v is a column standard λ/μ-tableau. The proofs are entirely analogous.

Proof of Theorem 2.2. We start by proving that there is an isomorphism ϕ of $\mathbf{Z}S_{(m,n)}$-modules

$$\frac{V^{\trianglerighteq \mu'}}{\sum_{\nu' \rhd \mu'} V^{\trianglerighteq \nu'}} \overset{\phi}{\cong} S^\mu \boxtimes S^{\lambda/\mu}.$$

By Corollary 2.5, setting $e(s \cup t)\phi = e(s) \otimes e(t)$ defines ϕ as a $\mathbf{Z}$-linear isomorphism. To show that ϕ is a $\mathbf{Z}S_{(m,n)}$-module homomorphism, it suffices to consider the actions of $S_{\{1,\ldots,m\}}$ and $S_{\{m+1,\ldots,m+n\}}$ separately. Let $\pi \in S_{\{1,\ldots,m\}}$ and let $s \cup t$ be a standard λ-tableau. Observe that $\widetilde{(s \cup t)\pi} = \widetilde{s\pi} \cup t$ and $\varepsilon_{(s \cup t)\pi} = \varepsilon_{s\pi}$. Suppose that $e(\widetilde{s\pi}) = \sum_S \alpha_S e(S)$ where the sum is over all standard μ-tableaux S. On the one hand

$$\big(e(s) \otimes e(t)\big)\pi = -\varepsilon_{s\pi} \sum_S \alpha_S e(S) \otimes e(t).$$

On the other hand, by Proposition 2.7 we have

$$e(s \cup t)\pi \in -\varepsilon_{s\pi} \sum_{S} \alpha_S e(S \cup t) + \sum_{\nu' \rhd \mu'} V^{\trianglerighteq \nu'}.$$

The argument is entirely analogous for the action of $S_{\{m+1,\ldots,m+n\}}$.

We now write $\geq$ for the lexicographic order of compositions. We define $V^{\geq \mu'}$ in a similar way to $V^{\trianglerighteq \mu'}$, replacing the condition $\delta \trianglerighteq \mu'$ with $\delta \geq \mu'$. We define $V^{>\mu'}$ in an analogous way. As $\nu' \trianglerighteq \mu'$ implies that $\nu' \geq \mu'$, Corollary 2.5 implies that $V^{\geq \mu'}$ is also a $\mathbf{Z}S_{(m,n)}$-module. Moreover, there is an isomorphism

$$\frac{V^{\geq \mu'}}{V^{>\mu'}} \cong S^{\mu} \boxtimes S^{\lambda/\mu},$$

and so the modules $V^{\geq \mu'}$, where μ ranges over all subpartitions of λ of size m, give the required filtration. □

Corollary 2.8. *Let $\rho \in S_{m+n}$ be an n-cycle and let π be a permutation of the remaining m numbers. Then*

$$\chi^{\lambda}(\pi\rho) = \sum_{\mu} \chi^{\mu}(\pi)\chi^{\lambda/\mu}(\rho)$$

where the sum is over all subpartitions μ of λ of size m.

Proof. By taking a suitable conjugate of $\pi\rho$ we may assume that $\pi \in S_{\{1,\ldots,m\}}$ and $\rho \in S_{\{m+1,\ldots,m+n\}}$. Taking characters in Theorem 2.2 gives

$$\chi^{\lambda}\big\downarrow_{S_{(m,n)}} = \sum_{\mu} \chi^{\mu} \times \chi^{\lambda/\mu} \tag{2.5}$$

where the sum is over all subpartitions μ of λ of size m. Now evaluate both sides at $\pi\rho$. □

We end this section by considering the following example, which makes explicit the statements of Lemma 2.6 and Proposition 2.7.

Example 2.9. Let u, t and so $u \cup t$ be the skew-tableaux shown below.

$$u = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 & 3 \\ \hline \end{array}\,, \quad t = \begin{array}{ccc} & & \boxed{5} \\ & & \boxed{7} \\ \boxed{6} & \boxed{8} & \end{array}\,, \quad u \cup t = \begin{array}{ccc} \boxed{1} & \boxed{2} & \boxed{5} \\ \boxed{4} & \boxed{3} & \boxed{7} \\ \boxed{6} & \boxed{8} & \end{array}\,.$$

As $4 = (2,1)(u \cup t) > (2,2)(u \cup t) = 3$, to straighten $u \cup t$ we define $X = \{4, 6\}$ and $Y = \{2, 3\}$. The relation $e(u \cup t)G_{X,Y} = 0$ gives

$$\begin{aligned} e(u \cup t) = &- e\left(\begin{array}{ccc} \boxed{1} & \boxed{3} & \boxed{5} \\ \boxed{2} & \boxed{4} & \boxed{7} \\ \boxed{6} & \boxed{8} & \end{array}\right) + e\left(\begin{array}{ccc} \boxed{1} & \boxed{2} & \boxed{5} \\ \boxed{3} & \boxed{4} & \boxed{7} \\ \boxed{6} & \boxed{8} & \end{array}\right) \\ &+ e\left(\begin{array}{ccc} \boxed{1} & \boxed{3} & \boxed{5} \\ \boxed{2} & \boxed{6} & \boxed{7} \\ \boxed{4} & \boxed{8} & \end{array}\right) - e\left(\begin{array}{ccc} \boxed{1} & \boxed{2} & \boxed{5} \\ \boxed{3} & \boxed{6} & \boxed{7} \\ \boxed{4} & \boxed{8} & \end{array}\right) - e\left(\begin{array}{ccc} \boxed{1} & \boxed{4} & \boxed{5} \\ \boxed{2} & \boxed{6} & \boxed{7} \\ \boxed{3} & \boxed{8} & \end{array}\right). \end{aligned}$$

In the notation of Lemma 2.6, we have $X^\star = \{4\}$. The standard polytabloids in the top and bottom lines come from the permutations in $C_{X^\star,Y}$ and $C_{X,Y}\backslash C_{X^\star,Y}$, respectively. Furthermore, the 4-shape of each polytabloid in the top line is $(2,2)$ and in the bottom line is $(3,1)$. Therefore

$$e(t) \in -e\left(\begin{array}{|c|c|c|}\hline 1&3&5\\ \hline 2&4&7\\ \hline 6&8& \\ \cline{1-2}\end{array}\right) + e\left(\begin{array}{|c|c|c|}\hline 1&2&5\\ \hline 3&4&7\\ \hline 6&8& \\ \cline{1-2}\end{array}\right) + V^{\unrhd(3,1)},$$

as expected from Proposition 2.7.

3. Pieri's rule and Young's rule

A skew-partition λ/μ is a *vertical* (resp. *horizontal*) *strip* if $[\lambda/\mu]$ has at most one box in each row (resp. column).

Theorem 3.1 (Pieri's rule)**.** *Let λ be a partition of $m+n$. If μ is a subpartition of λ of size m then*

$$\langle \chi^\lambda\big\downarrow_{S_m\times S_n}, \chi^\mu \times \mathrm{sgn}_{S_n}\rangle = \begin{cases} 1 & \textit{if } \lambda/\mu \textit{ is a vertical strip} \\ 0 & \textit{otherwise.}\end{cases}$$

Proof. By Maschke's Theorem and (2.5), applied to a suitable conjugate of $S_m \times S_n$, it suffices to prove that the multiplicity of sgn_{S_n} as a direct summand of $S^{\lambda/\mu} \otimes_{\mathbf{Z}} \mathbf{C}$ is 1 if λ/μ is a vertical strip and otherwise 0. For this we use the corresponding idempotent $E = \frac{1}{n!}\sum_{\tau\in S_n} \tau\,\mathrm{sgn}(\tau) \in \mathbf{C}S_n$.

If λ/μ is not a vertical strip then it contains boxes (i,j), $(i,j+1)$ in the same row. If t is a λ/μ-tableau then $\{t\}(1-(x,y)) = 0$ where $x = (i,j)t$ and $y = (i+1,j)t$. Since $E = \frac{1}{n!}\big(1-(x,y)\big)\sum_\pi \pi\,\mathrm{sgn}(\pi)$, where the sum is over a set of right coset representatives for the cosets of $\langle (x,y)\rangle$ in S_n, it follows that $M^{\lambda/\mu}E = 0$. Hence $S^{\lambda/\mu}E = 0$ as required.

Suppose that λ/μ is a vertical strip. Let t be a λ/μ-tableau. Let $Y_1,\ldots,Y_c$ be the sets of entries in each column of t. Let $G = S_{Y_1}\times\cdots\times S_{Y_c}$ and let $\pi_1,\ldots,\pi_d$ be a set of right coset representatives for the cosets of G in S_n. Observe that

$$\big\{(Y_1\pi_j,\ldots,Y_c\pi_j) : 1 \le j \le d\big\}$$

is the complete set of set compositions of $\{1,\ldots,n\}$ into parts of sizes $|Y_1|,\ldots,|Y_c|$. Let $M = |Y_1|!\ldots|Y_c|!$. By (2.2), $e(t)\tau = \mathrm{sgn}(\tau)e(t)$ for each $\tau \in G$. The observation now implies that

$$e(t)E = \frac{M}{n!}\sum_{i=1}^{d}\mathrm{sgn}(\pi_i)e(t\pi_i)$$

is non-zero and depends on t only up to a sign. Hence the multiplicity of sgn_{S_n} in $S^{\lambda/\mu}$ is 1. This completes the proof. □

For example, the unique submodule of $S^{(2,1,1)/(1)} \otimes_{\mathbf{Z}} \mathbf{C}$ affording sgn_{S_3} is spanned by $e(t)E = \frac{1}{3}e(t) - \frac{1}{3}e\big(t(1,2)\big) + \frac{1}{3}e\big(t(1,3,2)\big)$ where

$$t = \begin{array}{|c|c|} \cline{2-2} \multicolumn{1}{c|}{} & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}\,, \quad t(1,2) = \begin{array}{|c|c|} \cline{2-2} \multicolumn{1}{c|}{} & 2 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1} 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}\,, \quad t(1,3,2) = \begin{array}{|c|c|} \cline{2-2} \multicolumn{1}{c|}{} & 3 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1} 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}\,.$$

The following lemma is also used in §6.

Lemma 3.2. *Let λ be a partition of $m+n$ and let μ be a subpartition of λ of size m. If ψ is a character of S_n then*

$$\langle \chi^{\lambda/\mu}, \psi \rangle_{S_n} = \big\langle \chi^\lambda, \chi^\mu \times \psi \big\uparrow_{S_m \times S_n}^{S_{m+n}} \big\rangle_{S_{m+n}}.$$

Proof. By Frobenius reciprocity and Corollary 2.5,

$$\begin{aligned} \langle \chi^\lambda, \chi^\mu \times \psi \big\uparrow_{S_m \times S_n}^{S_{m+n}} \rangle &= \langle \chi^\lambda \big\downarrow_{S_m \times S_n}^{S_{m+n}}, \chi^\mu \times \psi \rangle \\ &= \langle \sum_\nu \chi^\nu \times \chi^{\lambda/\nu}, \chi^\mu \times \psi \rangle \end{aligned}$$

where the sum runs over all partitions ν of m such that $\nu \subset \lambda$. The only non-zero summand is $\langle \chi^\mu \times \chi^{\lambda/\mu}, \chi^\mu \times \psi \rangle = \langle \chi^{\lambda/\mu}, \psi \rangle$. □

Using Lemma 3.2 we immediately obtain the more usual statement of Pieri's rule that if ν is a partition of n then $(\chi^\nu \times \mathrm{sgn}_{S_\ell}) \big\uparrow_{S_n \times S_\ell}^{S_{n+\ell}} = \sum_\kappa \chi^\kappa$ where the sum is over all partitions κ of $n+\ell$ such that κ/ν is a vertical strip. Multiplying by the sign character using the basic result that $\chi^\nu \times \mathrm{sgn}_{S_n} = \chi^{\nu'}$ (see for instance [4, (6.6)]) then gives Young's rule: $(\chi^\nu \times 1_{S_\ell}) \big\uparrow_{S_n \times S_\ell}^{S_{n+\ell}} = \sum_\kappa \chi^\kappa$ where the sum is over all partitions κ of $n+\ell$ such that κ/ν is a horizontal strip.

Remark 3.3. A similarly explicit proof of Young's rule can be given, using a similar argument to the proof of Theorem 3.1. To reduce to horizontal strips, observe that if t is a standard λ/μ-tableau with boxes (i,j) and $(i+1,j)$ then $e(t)\big(1 + (x,y)\big) = 0$ where $x = (i,j)t$ and $y = (i+1,j)t$.

4. The dominance lemma for skew-tableaux

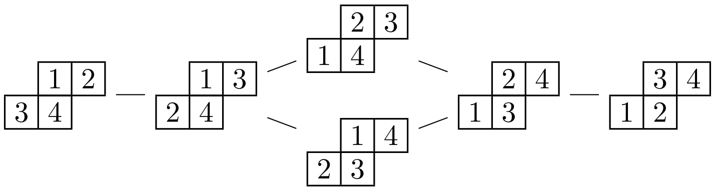
The dominance order for tabloids is defined in [4, Definition 3.11], or, in a way more convenient for us, in [11, Definition 2.5.4]. We extend it to compare row standard skew-tableaux of shape a fixed skew-partition.

Definition 4.1. Let t be a row standard λ/μ-tableau where $|\lambda/\mu| = n$. We define $\mathrm{sh}_{\le y}(t)$ to be the composition β such that

$$\beta_i = \big|\{x : x \in \text{row } i \text{ of } t,\ x \le y\}\big|$$

for $1 \le i \le \ell(\lambda)$. If s is another row standard λ/μ-tableau, then we say that *s dominates t*, and write $s \trianglerighteq t$, if $\mathrm{sh}_{\le y}(s) \trianglerighteq \mathrm{sh}_{\le y}(t)$ for all $y \in \{1, \ldots, n\}$, where on the right-hand side $\trianglerighteq$ denotes the dominance order of compositions defined in §2.3.

Example 4.2. The $\trianglerighteq$ order on the row standard $(3,2)/(1)$-tableaux is shown below.



Given a λ/μ-tableau t, we define its *row straightening* $\bar{t}$ to be the unique row standard λ/μ-tableau whose rows agree setwise with t. We extend the dominance order to λ/μ-tabloids by setting $\{s\} \trianglerighteq \{t\}$ if and only if $\bar{s} \trianglerighteq \bar{t}$.

Lemma 4.3 (Dominance Lemma)**.** *If t is a column standard λ/μ-tableau then $\bar{t}$ is standard and*

$$e(t) = e(\bar{t}) + w,$$

where w is a $\mathbf{Z}$-linear combination of standard polytabloids $e(s)$ such that $s \lhd \bar{t}$.

Preliminaries for the proof of the Dominance Lemma. We first show that $\bar{t}$ is standard. Suppose, for a contradiction, that there exist boxes (i,j) and $(i+1,j) \in [\lambda/\mu]$ such that $(i,j)\bar{t} > (i+1,j)\bar{t}$. Define

$$R = \{(i,k)\bar{t} : j \leq k \leq \lambda_i\}$$
$$S = \{(i+1,k)\bar{t} : \mu_{i+1} < k \leq j\}.$$

Since

$$(i+1, \mu_{i+1}+1)\bar{t} < \ldots < (i+1,j)\bar{t} < (i,j)\bar{t} < \ldots < (i,\lambda_i)\bar{t}$$

we have $x > y$ for each $x \in R$ and $y \in S$. But since $|R| + |S| = \lambda_i - \mu_{i+1} + 1$, the pigeonhole principle implies that there exist $x \in R$ and $y \in S$ lying in the same column of the column standard skew-tableau t, a contradiction.

The following two lemmas generalise Lemmas 3.15 and 8.3 in [4] to skew-tableaux.

Lemma 4.4. *Let t be a λ/μ-tableau. Let x, $y \in \{1,\ldots,n\}$ be such that $x < y$. If x is strictly higher than y in t then $\overline{t(x,y)} \lhd \bar{t}$.*

Proof. Let x be in row k of t and let y be in row ℓ of t. By hypothesis, $k < \ell$. Let $z \in \{1,\ldots,n\}$. If $x \leq z < y$ then

$$\mathrm{sh}_{\leq z}(\overline{t(x,y)})_k = \mathrm{sh}_{\leq z}(\bar{t})_k - 1$$
$$\mathrm{sh}_{\leq z}(\overline{t(x,y)})_\ell = \mathrm{sh}_{\leq z}(\bar{t})_\ell + 1.$$

Whenever $i \notin \{k,\ell\}$ or $z < x$ or $y \leq z$ we have $\mathrm{sh}_{\leq z}(\overline{t(x,y)})_i = \mathrm{sh}_{\leq z}(\bar{t})_i$. It easily follows from these equations and the definition of the dominance order for compositions that $\overline{t(x,y)} \lhd \bar{t}$. □

Lemma 4.5. *Let t be a column standard λ/μ-tableau. Then $e(t) = \{t\} + w$, where w is a $\mathbf{Z}$-linear combination of λ/μ-tabloids $\{s\}$ such that $\{s\} \lhd \{t\}$.*

Proof. The proof of Lemma 8.3 in [4] still holds, replacing Lemma 3.15 in [4] with our Lemma 4.4. □

Proof of Lemma 4.3. Let $e(t) = \sum_s \alpha_s e(s)$ where the sum is over all standard λ/μ-tableaux and $\alpha_s \in \mathbf{Z}$ for each s. Let u be a standard tableau maximal in the dominance order such that $\alpha_u \neq 0$. Applying Lemma 4.5 to $e(u)$ gives

$$e(u) = \{u\} + w^{\lhd\{u\}},$$

where $w^{\lhd\{u\}}$ is a $\mathbf{Z}$-linear combination of λ/μ-tabloids each dominated by $\{u\}$. By Lemma 4.5 and the maximality of u, there is no other standard λ/μ-tableau s with $\alpha_s \neq 0$ such that $e(s)$ has $\{u\}$ as a summand. Therefore the coefficient of $\{u\}$ in $e(t)$ is α_u. Applying Lemma 4.5, now to $e(t)$, gives

$$e(t) = \{t\} + w^{\lhd\{t\}},$$

where $w^{\lhd\{t\}}$ is a $\mathbf{Z}$-linear combination of λ/μ-tabloids each dominated by $\{t\}$. In particular $\{t\} \unrhd \{u\}$, and so we have that $\bar{t} = u$ by the maximality of u. Hence

$$e(t) = \alpha_{\bar{t}} e(\bar{t}) + w,$$

where w is a $\mathbf{Z}$-linear combination of standard polytabloids $e(v)$ for standard tableaux v such that $v \lhd \bar{t}$. It follows that $\{t\}$ cannot be a summand of w in the equation immediately above. Since the coefficient of $\{t\}$ in $e(t)$ is 1, we have $\alpha_{\bar{t}} = 1$. □

We isolate the following corollary of Lemma 4.3.

Corollary 4.6. *Let s be a standard λ/μ-tableau, and let u be a column standard λ/μ-tableau. Suppose that there exists $x \in \{1, 2, \ldots, n\}$ such that the boxes containing $1, 2, \ldots, x-1$ are the same in s and u, and x is lower in u than s. If*

$$e(u) = \sum \alpha_v e(v),$$

where the sum is over all standard λ-tableaux v, then $\alpha_s = 0$.

Proof. By assumption, $\mathrm{sh}_{\leq z}(s) = \mathrm{sh}_{\leq z}(\overline{u})$ if $1 \leq z < x$. As x is in a lower row in u than in s, we have $\mathrm{sh}_{\leq x}(\overline{u}) \not\trianglerighteq \mathrm{sh}_{\leq x}(s)$. Now apply Lemma 4.3. □

5. The Murnaghan–Nakayama rule for border strips

In this section we give a bijective proof that $\chi^{\lambda/\mu}(\rho) = (-1)^{\mathrm{ht}(\lambda/\mu)}$ when λ/μ is a border strip of size n and ρ is the n-cycle $(1, 2, \ldots, n)$. This deals with one of the two cases in (1.1). Our proof shows that the matrix representing ρ in the standard basis of $S^{\lambda/\mu}$ has a unique non-zero entry on its diagonal. The relevant standard tableau is defined as follows.

Definition 5.1. Let λ/μ be a border strip of size n. Say that a box $(i,j) \in [\lambda/\mu]$ is *columnar* if $(i+1,j) \in [\lambda/\mu]$. We define the standard λ/μ-tableau $s_{\lambda/\mu}$ as follows:

(i) assign the numbers $\{1,\ldots,z\}$ in ascending order to the z columnar boxes of λ/μ, starting with 1 in row 1 and finishing with z in the row above the bottom row;

(ii) then assign the numbers $\{z+1,\ldots,n\}$ in ascending order to the $n-z$ non-columnar boxes, starting with $z+1$ in column 1 and finishing with n in the rightmost column.

For example, $s_{(5,3,3)/(2,2)}$, $s_{(5,3,2)/(2,1)}$ and $s_{(5,1,1)/\varnothing}$ are respectively

$$\begin{array}{ccccc} & & 1 & 6 & 7 \\ & & 2 & & \\ 3 & 4 & 5 & & \end{array}, \qquad \begin{array}{ccccc} & & 1 & 6 & 7 \\ & 2 & 5 & & \\ 3 & 4 & & & \end{array} \quad \text{and} \quad \begin{array}{ccccc} 1 & 4 & 5 & 6 & 7 \\ 2 & & & & \\ 3 & & & & \end{array}$$

where 1 and 2 are the entries in columnar boxes in each case. We remark that there are no columnar boxes if and only if λ/μ is a horizontal strip, as defined in §3.

As useful pieces of notation, we define x^- and x^+ for $x \in \{1,\ldots,n\}$ by $x^- = x-1$ and

$$x^+ = \begin{cases} x+1 & \text{if } 1 \leq x < n \\ 1 & \text{if } x = n. \end{cases}$$

Thus $x\rho = x^+$ for all $x \in \{1,\ldots,n\}$ and $1^- = 0$. Given a λ/μ-tableau t, we define t^+ by $(i,j)t^+ = \big((i,j)t\big)^+$. By (2.1), $e(t\rho) = e(t^+)$. A standard λ/μ-tableau s such that $e(s)$ has a non-zero coefficient in the unique expression of $e(s^+)$ as a $\mathbf{Z}$-linear combination of standard polytabloids is said to be *trace-contributing*. Since $\chi^{\lambda/\mu}(\rho)$ is the trace of the matrix representing ρ in the standard basis, it suffices to prove the following proposition.

Proposition 5.2. *Let λ/μ be a border strip. The unique trace-contributing λ/μ-tableau is $s_{\lambda/\mu}$. The coefficient of $e(s_{\lambda/\mu})$ in $e(s^+_{\lambda/\mu})$ is $(-1)^{\mathrm{ht}(\lambda/\mu)}$.*

The proof of Proposition 5.2 is by induction on the number of top corner boxes of λ/μ, as defined in Definition 5.3 below. The necessary preliminaries are collected below. We then prove the base case, when $\lambda/\mu = (n-\ell, 1^\ell)$ for some $\ell \in \mathbf{N}_0$; this gives a good flavour of the general argument. In the remainder of this section we give the inductive step.

We assume, without loss of generality, that $\mu_1 < \lambda_1$ and $\mu_{\ell(\lambda)} = 0$, so the non-empty rows of λ/μ are $1,\ldots,\ell(\lambda)$ and column 1 of λ/μ is non-empty.

5.1. **Preliminaries for the proof of Proposition 5.2.** For $Z \subseteq \{1,\ldots,n\}$ and t a row standard λ/μ-tableau we define $\mathrm{sh}_Z(t)$ to be the composition β such that

$$\beta_i = \big|\{x : x \in \text{row } i \text{ of } t,\, x \in Z\}\big|$$

for $1 \le i \le \ell(\lambda)$. Set $\mathrm{sh}_{<y}(t) = \mathrm{sh}_{\{1,\ldots,y^-\}}(t)$. We also use $\mathrm{sh}_{\le y}(t)$, as already defined in Definition 4.1.

Definition 5.3. Let λ/μ be a border strip. We say that column j of λ/μ is *singleton* if it contains a unique box. We define a *top corner box* to be a box $(i, j) \in [\lambda/\mu]$ such that $(i, j-1), (i-1, j) \notin [\lambda/\mu]$ and a *bottom corner box* to be a box $(i, j) \in [\lambda/\mu]$ such that $(i+1, j), (i, j+1) \notin [\lambda/\mu]$.

Lemma 5.4. *Let λ/μ be a border strip and let t be a λ/μ-tableau. If columns j and $j+1$ of λ/μ are singleton, with their unique box in row i, then $e(t) = e(t)(x, y)$ where $x = (i, j)t$ and $y = (i, j+1)t$.*

Proof. This follows immediately from the Garnir relation (2.4), taking $X = \{x\}$ and $Y = \{y\}$. □

In fact, all the Garnir relations that we use can be reduced to single transpositions. Let x and y be entries in adjacent columns of a column standard tableau, with x left of y and $x > y$. We say that (x, y) is a *Garnir swap* if at least one column is not singleton, and otherwise that (x, y) is a *horizontal swap*.

Lemma 5.5. *Let s be a trace-contributing border strip tableau. Then s can be obtained from $\widetilde{s^+}$ by iterated horizontal swaps, Garnir swaps and column straightenings. If in such a sequence 1 moves, then 1 moves either left or down.*

Proof. The first claim is immediate from Theorem 2.1(i). The second follows from Corollary 4.6 taking $x = 1$. □

Given $X \subseteq \{1, 2, \ldots, n\}$, we define $X^+ = \{x^+ : x \in X\}$. The following combinatorial result on the map $x \mapsto x^+$ is used several times to restrict the possible entries of trace-contributing tableaux.

Lemma 5.6. *Let X be a set of natural numbers disjoint from b and c. We have $\{b^+\} \cup X^+ = X \cup \{c\}$ if and only if $b = \min X$, $c = \max X^+$ and $X = \{b^+, \ldots, c^-\}$.*

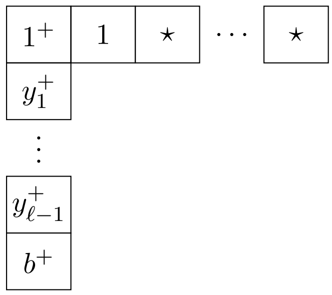
Proof. Since $\min X \notin X^+$ we have $\min X = b^+$. Similarly, since $\max X^+ \notin X$ we have $\max X^+ = c$. Suppose for a contradiction that X is a proper subset of $\{b^+, \ldots, c^-\}$. Setting

$$d = \min(\{b^+, \ldots, c^-\} \backslash X)$$

we see that since $b^+ = \min X \in X$, we have $d > b^+$. The minimality of d implies that $d^- \in X$ and so $d \in X^+$; since $d < c$ and $\{b^+\} \cup X^+ = X \cup \{c\}$, we have $d \in X$, a contradiction. The converse is obvious. □

Finally, as a notational convention, when we specify a set, we always list the elements in increasing order. In diagrams the symbol $\star$ marks an entry we have no need to specify more explicitly.

5.2. **Base case: one top corner box.** In this case $\mu = \varnothing$ and $\lambda = (n - \ell, 1^\ell)$ for some $\ell \in \mathbf{N}_0$. If $\ell = 0$ then there is a unique standard (n)-tableau and the result is clear. Suppose that $\ell > 0$ and let s be a standard $(n-\ell, 1^\ell)$-tableau with entries $\{1, y_1, \ldots, y_{\ell-1}, b\}$ in column 1. (By our notational convention, $1 < y_1 < \ldots < y_{\ell-1} < b$.) If $b = n$ then $\widetilde{s^+}$ is standard with first column entries $\{1, 1^+, y_1^+, \ldots, y_{r-1}^+\}$. Hence, assuming that s is trace-contributing, we have $b < n$. After a sequence of horizontal swaps applied to $\widetilde{s^+}$ we obtain the tableau shown below.



A Garnir swap of 1 with 1^+ or any y_i^+ gives, after column straightening and a sequence of horizontal swaps, a standard tableau having b^+ in its bottom left position. We may therefore assume, by Lemma 5.5, that 1 is swapped with b^+. After column straightening, which introduces the sign $(-1)^\ell$, a sequence of horizontal swaps gives the standard tableau having $\{1, 1^+, y_1^+, \ldots, y_{\ell-1}^+\}$ in its first column. Thus if s is trace-contributing then $\{1^+, y_1^+, \ldots, y_{\ell-1}^+\} = \{y_1, \ldots, y_{\ell-1}, b\}$. By Lemma 5.6, $\{y_1, \ldots, y_{\ell-1}, b\} = \{2, \ldots, \ell+1\}$. Therefore $s = s_{(n-\ell,1^\ell)}$ and the coefficient of $e(s_{(n-\ell,1^\ell)})$ in $e(s^+_{(n-\ell,1^\ell)})$ is $(-1)^\ell$, as required.

5.3. **Inductive step.** Let $\delta(i) \in \mathbf{N}_0^{\ell(\lambda)}$ denote the composition defined by $\delta(i)_i = 1$ and $\delta(i)_k = 0$ if $k \neq i$.

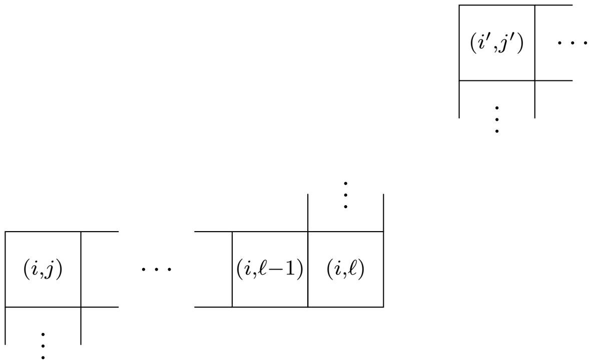
Proposition 5.7. *Let s be a standard λ/μ-tableau. Let $c \in \mathbf{N}$ and suppose that* either $c = 1$ or $c > 1$ *and the entries* $1, \ldots, c^-$ *and n lie in the same column of s. Let (i, j) be the box of s containing c, and let (i', j') be the box of $\widetilde{s^+}$ containing c. If s is a trace-contributing tableau, then $i = i'$.*

Proof. By hypothesis, the highest c^- entries in column j' of s and $\widetilde{s^+}$ are $1, \ldots, c^-$. Let $t = \widetilde{s^+}$. Setting $\beta = \mathrm{sh}_{<c}(s) = \mathrm{sh}_{<c}(\bar{t})$ we have $\mathrm{sh}_{\leq c}(s) = \beta + \delta(i)$ and $\mathrm{sh}_{\leq c}(\bar{t}) = \beta + \delta(i')$. By Lemma 4.3, the hypothesis that s is trace-contributing implies that $\mathrm{sh}_{\leq c}(\bar{t}) \trianglerighteq \mathrm{sh}_{\leq c}(s)$. Therefore $i \geq i'$.

If $j = j'$ then *either* $c = 1$ and 1 is at the top of the column of s which has n at its bottom, *or* $c > 1$ and c is immediately below c^- in both s and t. In either case $i = i'$.

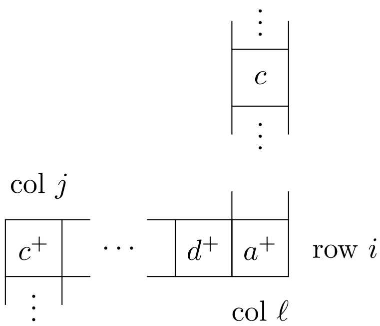
We may therefore suppose, for a contradiction, that $i > i'$ and $j < j'$. By hypothesis the box (i, j) of s containing c is the top corner box in row i. Let

(i, ℓ) be the bottom corner box in row i; note that $\ell \leq j'$, as shown in the diagram below.



By the hypothesis that s is trace-contributing and Lemma 5.5 there is a sequence of horizontal swaps, Garnir swaps, and column straightenings from $\widetilde{s^+}$ to s. Suppose that in such a sequence an entry $b < c$ is moved. If b is the first such entry moved in this sequence, and u is the tableau obtained after column straightening, then, by Corollary 4.6 applied with $x = b$, the coefficient of $e(s)$ in $e(u)$ is zero. Therefore the entries $\{1, \ldots, c^-\}$ are fixed and c is the smallest number moved. Take such a sequence and stop it immediately after the first swap in which c enters row i. Let v be the column standard tableau so obtained, and let u be its immediate predecessor.

When c enters row i of v, it is swapped with the entry, d^+ say, in box $(i, \ell - 1)$ of u. Let a^+ be the entry in box (i, ℓ) of u. Thus the column standard tableau u is as shown below and $v = u\widetilde{(c, d^+)}$.



Note that $d^+ > a^+$ since otherwise u is standard with respect to all boxes weakly to the left of column ℓ, and so d^+ cannot be moved in a Garnir swap.

To complete the proof we require the following critical quantity. *Let r be maximal such that entries $c, \ldots, r$ are strictly to the left of column ℓ in the original tableau s.* If $r = d$ then, since $d > a$, a is strictly to the left of column ℓ in s; this is impossible since a^+ appears in column ℓ in u. Therefore $r < d$. Since d is in position $(i, \ell - 1)$ of s and $r \geq c$, it follows that $c \neq d$.

Claim. We have $v \ntrianglerighteq s$. *Proof of claim.* Let $\mathrm{sh}_{\{c^+,\dots,r^+\}}(u) = \delta$. By hypothesis and our stopping condition on swaps, if $q \leq r$ then the box of q^+ in u is the box of q in s. Hence $\mathrm{sh}_{\{c,\dots,r\}}(s) = \delta$. Since $d > r$ and d is in position $(i, \ell - 1)$ of s, we see that r^+ is not in row i of s. By maximality of r, the row of s containing r^+ is row h for some $h < i$. Clearly the row of c in v is i. Therefore $\mathrm{sh}_{\{c,\dots,r^+\}}(\overline{v}) = \delta + \delta(i)$ and $\mathrm{sh}_{\{c,\dots,r^+\}}(s) = \delta + \delta(h)$. Since $1, \dots, c^-$ are in the same positions in both v and s, it follows that

$$\mathrm{sh}_{\leq r^+}(s) \rhd \mathrm{sh}_{\leq r^+}(\overline{v})$$

which implies the claim.

It now follows from Lemma 4.3, as before, that $e(s)$ does not appear in $e(v)$, a final contradiction. □

Corollary 5.8. *If s is a trace-contributing tableau then either* 1 *and n are in the same column of s, or* 1 *and n are in the top row of s.*

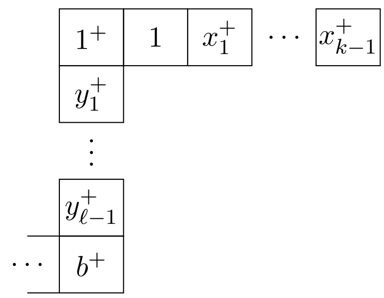
Proof. Let 1 and n be in positions (i, j) of s and (i', j') of s, respectively. If column j' is singleton then n is the top right entry of s and, taking $c = 1$ in Proposition 5.7, we get $i = i'$; thus 1 and n are in the top row of s. Otherwise, when we column straighten s^+ to obtain $\widetilde{s^+}$, the entry 1 in position (i', j') moves up to position (i'', j') where $i'' < i'$. Again taking $c = 1$ in Proposition 5.7, we get $i = i''$. Since (i'', j') is the top corner box in its row, and so is (i, j), we see that $j = j'$. Hence 1 and n are in the same column of s. □

Proof of Proposition 5.2. We now complete the inductive step of the proof.

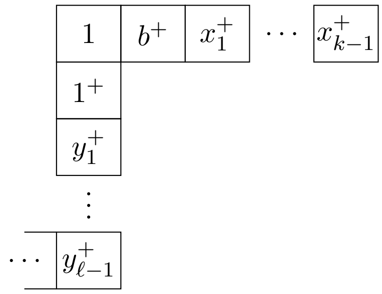
Suppose that λ/μ has more than one top corner box, and that s is a trace-contributing λ/μ-tableau. Let 1 be in position (i, j) of s and in position (i', j') of $\widetilde{s^+}$. By Proposition 5.7, we have $i = i'$.

Case (1). Suppose that 1 and n lie in the same row of s. By Corollary 5.8, this is the top row. Let the entries in the top row be $\{1, x_1, \dots, x_{k-1}, n\}$, and let the entries in the column of 1 be $\{1, y_1, \dots, y_{\ell-1}, b\}$.

Straightening the top row of s^+ by a sequence of $k - 1$ horizontal swaps moves 1^+ and 1 into adjacent positions, giving the tableau u shown below.



As in the base case, the only Garnir swap that can lead to s is $(1, b^+)$, which introduces the sign $(-1)^\ell$. Let $v = \widetilde{u(1, b^+)}$, as shown below.



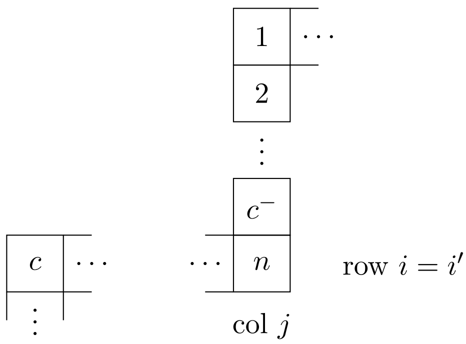
By Lemma 5.5 and Corollary 4.6, v can be straightened by a sequence of horizontal swaps, Garnir swaps and column straightenings which either fix 1, and so leave invariant the content of its top row, or move 1 into a lower row, giving a tableau, w say, such that, $e(s)$ does not appear in $e(w)$. Since $e(s)$ has a non-zero coefficient in $e(v)$, we have

$$\{b^+, x_1^+, \ldots, x_{k-1}^+\} = \{x_1, \ldots, x_{k-1}, n\}.$$

Lemma 5.6 implies that $b^+ = x_1 = n-k+1$, $x_{k-1}^+ = n$ and $\{x_1, \ldots, x_{k-1}\} = \{n-k+1, \ldots, n-1\}$. Thus s and v have top row entries $\{1, n-k+1, \ldots, n\}$.

Let S and V be the tableaux obtained from s and v by deleting all but the top corner box in their top rows. This removes entries $\{n-k+1, \ldots, n\}$. Let $\lambda^\star/\mu$ be the common shape of S and V. Observe that S has greatest entry $n-k = b$ in the bottom corner box of its rightmost column and that V is the column straightening of $S^\dagger$, where $\dagger$ is defined as $+$ on tableaux, but replacing n with $n-k$. By induction, $S = s_{\lambda^\star/\mu}$, and since s has $n-k+1, \ldots, n$ in its top row, we have $s = s_{\lambda/\mu}$. Moreover, the coefficient of $e(S)$ in $e(S^\dagger)$ is $(-1)^{\mathrm{ht}(\lambda^\star/\mu)}$, Since $\mathrm{ht}(\lambda^\star/\mu) = \mathrm{ht}(\lambda/\mu)$, the coefficient of $e(s)$ in $e(s^+)$ is $(-1)^{\mathrm{ht}(\lambda/\mu)}$, as required.

Case (2). If Case (1) does not apply then, since $i = i'$, 1 and n are in the same column of s and so $j = j'$. Take c maximal such that $1, \ldots, c^-$ are in column j of s. By Proposition 5.7, the row of c in s is the same as the row of c in $\widetilde{s^+}$. By the maximality of c it follows that column j of s has entries $1, \ldots, c^-, n$, as shown below.



By Lemma 5.5 there is a sequence of horizontal swaps, Garnir swaps and column straightenings from $\widetilde{s^+}$ to s. As seen in the proof of Proposition 5.7, it follows easily from Lemma 4.3 that $1, \ldots, c^-$ do not move. Let X be the set of entries of s lying strictly to the right of column j. These entries become X^+ in $\widetilde{s^+}$, which is standard with respect to these columns. No permutation in our chosen sequence can involve a entry in one of these columns. Hence $X^+ = X$, and so $X = \varnothing$.

We have shown that j is the rightmost column of s, and that s agrees with $s_{\lambda/\mu}$ in this column. Let S be the tableau obtained from s by deleting all but the bottom corner box in column j and subtracting c^- from each remaining entry. Thus the top row of S has entries $1, \ldots, n-c^-$ and $n-c^-$ is its greatest entry. Let S have shape $\lambda^\star/\mu^\star$. By induction, $S = s_{\lambda^\star/\mu^\star}$, and hence $s = s_{\lambda/\mu}$. Let $S^\dagger$ be defined as S^+, but replacing n with $n-c^-$. By induction, the coefficient of $e(S)$ in $e(S^\dagger)$, is $(-1)^{\mathrm{ht}(\lambda^\star/\mu^\star)}$. Since $\mathrm{ht}(\lambda^\star/\mu^\star)+c^- = \mathrm{ht}(\lambda/\mu)$, and the sign introduced by column straightening s^+ is $(-1)^{c^-}$, the coefficient of $e(s)$ in $e(s^+)$ is $(-1)^{\mathrm{ht}(\lambda/\mu)}$, as required. □

6. Proof of Theorem 1.1

Let λ/μ be a skew-partition of size n and let $\rho \in S_n$ be an n-cycle. Following the outline in §1.1, to complete the proof of Theorem 1.1, we must show that $\chi^{\lambda/\mu}(\rho) = 0$ if λ/μ is not a border strip. We require the following two lemmas.

Lemma 6.1. *Let $0 \le \ell \le n$. If*

$$\langle \chi^\lambda, \chi^\mu \times 1_{S_\ell} \times \mathrm{sgn}_{S_{n-\ell}} \uparrow_{S_m \times S_\ell \times S_{n-\ell}}^{S_{m+n}} \rangle > 0$$

then $[\lambda/\mu]$ has no four boxes making the shape $(2,2)$.

Proof. By the versions of Pieri's rule and Young's rule proved at the end of §3, the hypothesis implies that λ is obtained from μ by adding a horizontal strip of size ℓ and then a vertical strip of size $n-\ell$. If two boxes from a horizontal strip are added to row i then at most one box can be added below them in row $i+1$ by a vertical strip. The result follows. □

Lemma 6.2. *If λ is a partition of n and ρ is an n-cycle then $\chi^\lambda(\rho) \neq 0$ if and only if $\lambda = (n-\ell, 1^\ell)$ where $0 \le \ell < n$.*

Proof. By a column orthogonality relation

$$\sum_\lambda \chi^\lambda(\rho)^2 = |\mathrm{Cent}_{S_n}(\rho)| = n,$$

where the sum is over all partitions λ of n. By (1.1) in the case proved in §5, we have $\chi^{(n-\ell,1^\ell)}(\rho) = (-1)^{\ell-1}$ for $0 \le \ell < n$. Therefore the partitions $(n-\ell, 1^\ell)$ give all the non-zero summands. □

Proposition 6.3. *Let λ/μ be a skew-partition of size n and let $\rho \in S_n$ be an n-cycle. If λ/μ is not a border strip then $\chi^{\lambda/\mu}(\rho) = 0$.*

Proof. If $[\lambda/\mu]$ is disconnected then it is clear from the Standard Basis Theorem (Theorem 2.1(ii)) that $S^{\lambda/\mu}$ is isomorphic to a module induced from a proper Young subgroup $S_{n-\ell} \times S_\ell$ of S_n. Since no conjugate of ρ lies in this subgroup, we have $\chi^{\lambda/\mu}(\rho) = 0$.

In the remaining case λ/μ has four boxes making the shape $(2,2)$. By either Pieri's rule or Young's rule, we have $\langle 1_{S_\ell} \times \mathrm{sgn}_{S_{n-\ell}} \uparrow_{S_\ell \times S_{n-\ell}}^{S_n}, \chi^{(n-\ell,1^\ell)} \rangle = 1$. Hence

$$\begin{aligned}\langle \chi^\lambda, \chi^\mu \times 1_{S_\ell} \times \mathrm{sgn}_{S_{n-\ell}} \uparrow_{S_m \times S_\ell \times S_{n-\ell}}^{S_{m+n}} \rangle &\geq \langle \chi^\lambda, \chi^\mu \times \chi^{(n-\ell,1^\ell)} \uparrow_{S_m \times S_n}^{S_{m+n}} \rangle \\ &= \langle \chi^{\lambda/\mu}, \chi^{(n-\ell,1^\ell)} \rangle\end{aligned}$$

where the equality follows from Lemma 3.2. By Lemma 6.1 the left-hand size is 0. Hence $\langle \chi^{\lambda/\mu}, \chi^{(n-\ell,1^\ell)} \rangle = 0$ for $0 \leq \ell < n$. By Lemma 6.2, this implies the result. □

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Department of Mathematics, Royal Holloway University of London, United Kingdom

E-mail address: `Jasdeep.Kochhar.2015@rhul.ac.uk`

E-mail address: `mark.wildon@rhul.ac.uk`