

A PROOF OF THE MURNAGHAN–NAKAYAMA RULE USING SPECHT MODULES AND TABLEAU COMBINATORICS

JASDEEP KOCHHAR AND MARK WILDON

ABSTRACT. The Murnaghan–Nakayama rule is a combinatorial rule for the character values of symmetric groups. We give a new combinatorial proof by explicitly finding the trace of the representing matrices in the standard basis of Specht modules. This gives an essentially bijective proof of the rule. A key lemma is an extension of a straightening result proved by the second author to skew tableaux. Our module theoretic methods also give short proofs of Pieri’s rule and Young’s rule.

1. INTRODUCTION

In this article we give a new combinatorial proof of the Murnaghan–Nakayama rule for the values of the ordinary character χ^λ of S_n canonically labelled by the partition λ of $n \in \mathbf{N}$. To state the rule, we require the following definitions.

Let $\ell(\lambda)$ denote the number of parts of λ . Given partitions μ and λ of m and $m+n$ respectively, we say that μ is a *subpartition* of λ , and write $\mu \subseteq \lambda$, if $\ell(\mu) \leq \ell(\lambda)$ and $\mu_i \leq \lambda_i$ for $1 \leq i \leq \ell(\mu)$. We define the *skew diagram* $[\lambda/\mu]$ to be the set of *boxes*

$$\{(i, j) : 1 \leq i \leq t \text{ and } \mu_i < j \leq \lambda_i\},$$

and call λ/μ a *skew partition*. We define *row* k (resp. *column* k) of λ/μ to be the subset of $[\lambda/\mu]$ of boxes whose first (resp. second) coordinate equals k . Let $\text{ht}(\lambda/\mu)$ be one less than the number of non-empty rows of $[\lambda/\mu]$. We define a *border strip* to be a skew partition whose skew diagram is connected and which contains no four boxes forming the partition $(2, 2)$.

Theorem 1.1 (Murnaghan–Nakayama rule). *Given $m, n \in \mathbf{N}$, let $\rho \in S_{m+n}$ be an n -cycle and let π be a permutation of the remaining m numbers. Then*

$$\chi^\lambda(\pi\rho) = \sum (-1)^{\text{ht}(\lambda/\mu)} \chi^\mu(\pi),$$

where the sum is over all $\mu \subset \lambda$ such that $|\mu| = m$ and λ/μ is a border strip.

Before we continue we provide an example of the Murnaghan–Nakayama rule, showing how it can be applied recursively to calculate single character values.

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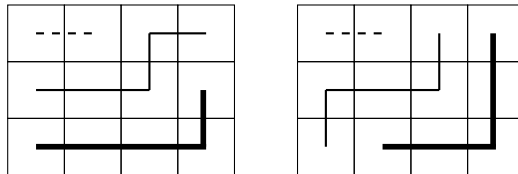


FIGURE 1. The border strips of size 5 (solid) and 2 (dashed) removed to compute $\chi^{(4,4,4)}(\sigma)$ in Example 1.2.

Example 1.2. Let $\sigma = (1, 2)(3, 4, 5, 6, 7)(8, 9, 10, 11, 12) \in S_{12}$. We evaluate $\chi^{(4,4,4)}(\sigma)$. Taking $\rho = (8, 9, 10, 11, 12)$, we begin by removing border strips of size 5 from $(4, 4, 4)$. As shown in Figure 1 there are two such strips, namely $(4, 4, 4)/(4, 3)$ and $(4, 4, 4)/(3, 3, 1)$, of heights 1 and 2, respectively. Therefore by the Murnaghan–Nakayama rule

$$\chi^{(4,4,4)}(\sigma) = (-\chi^{(4,3)} + \chi^{(3,3,1)})((1, 2)(3, 4, 5, 6, 7)).$$

Two further applications of the Murnaghan–Nakayama rule to each summand now show that $\chi^{(4,4,4)}(\sigma) = (\chi^{(2)} + \chi^{(2)})(1, 2) = 1 + 1 = 2$.

As Stanley notes in [13, page 401], the Murnaghan–Nakayama rule was first proved by Littlewood and Richardson in [7, §11]. Their proof derives it, essentially as stated in Theorem 1.1, as a corollary of the older Frobenius formula [3, page 519, (6)] for the characters of symmetric groups. (For a modern statement of the Frobenius formula see [13, (7.77)] or [4, (4.10)].) Later Murnaghan [10, page 462, (13)] gave a similar but independent derivation of the rule. Murnaghan’s paper was cited by Nakayama [11, page 183], who gave a more concise proof, still from the Frobenius formula. James gave a different proof in [5, Ch. 11] using the relatively deep Littlewood–Richardson rule. More recently, elegant involutive proofs have been given by Mendes and Remmel [9, Theorem 6.3] using Pieri’s rule and Young’s rule and by Loehr [8, §11] using his labelled abacus representation of antisymmetric functions.

The starting point for our proof is Corollary 2.9 of Theorem 2.2 below, which states that $\chi^\lambda(\pi\rho) = \sum_\mu \chi^\mu(\pi)\chi^{\lambda/\mu}(\rho)$, where $\chi^{\lambda/\mu}$ is the ordinary character of the skew Specht module $S^{\lambda/\mu}$ defined in §2.1. By this corollary, it suffices to show that if ρ is an n -cycle then

$$(1.1) \quad \chi^{\lambda/\mu}(\rho) = \begin{cases} (-1)^{\text{ht}(\lambda/\mu)} & \text{if } \lambda/\mu \text{ is a border strip of size } n \\ 0 & \text{otherwise.} \end{cases}$$

We do this by explicitly computing the trace of the matrix representing the n -cycle ρ in the standard basis (see Theorem 2.1) of $S^{\lambda/\mu}$. In the critical case where λ/μ is a border strip, we show that there is a unique basis element giving a non-zero contribution to the trace. This gives a new and essentially bijective proof of the Murnaghan–Nakayama rule.

Theorem 2.2 is the main result in [6]. The proof in [6] constructs skew Specht modules as ideals in the group algebra of S_n over a field. Our proof using polytabloids instead generalizes James' proof of the modular branching rule for Specht modules [5, Ch. 9]. In this way we obtain a stronger isomorphism for integral modules that replaces the lexicographic order used in [5] and [6] with the dominance order.

In §2.1 and §2.2 we define λ/μ -polytabloids and state Theorem 2.1, which says that the set of standard λ/μ -polytabloids is a \mathbf{Z} -basis of $S^{\lambda/\mu}$. In §2.3 we prove Theorem 2.2 and deduce Corollary 2.9. In §3 we use Theorem 2.2 to give short module-theoretic proofs of Pieri's rule and Young's rule. In §4 we prove Lemma 4.3, which gives a necessary condition for a standard polytabloid to appear with a non-zero coefficient when a given λ/μ -polytabloid is written as a linear combination of standard polytabloids. This generalises Proposition 4.1 in [14] to skew tableaux. In §5 we use Lemma 4.3 to give a bijective proof of (1.1) when λ/μ is a border strip. We then deal with the remaining case in §6 by a short argument using Pieri's rule and Young's rule.

2. BACKGROUND

2.1. Skew tableaux and skew Specht modules. Fix $m, n \in \mathbf{N}$. Let λ be a partition of $m+n$ and let μ be a subpartition of λ of size m . We define a λ/μ -tableau t to be a bijective function $t : [\lambda/\mu] \rightarrow \{1, 2, \dots, n\}$, and call t a *skew tableau* of *shape* λ/μ . We call $(i, j)t$ the *entry of t* in position (i, j) . Thus a λ/μ -tableau can be visualized as a filling of the boxes $[\lambda/\mu]$ with distinct entries from $\{1, \dots, n\}$. We draw skew diagrams using the 'English convention' in which the largest part appears at the top of the page: thus the *top row* is row 1, and so on. The *conjugate* partition of λ is the partition λ' whose diagram $[\lambda']$ is obtained by reflecting $[\lambda]$ in its leading diagonal. Equivalently, $\lambda'_j = |\{i : \lambda_i \geq j\}|$.

There is a natural action of S_n on the set of λ/μ -tableaux defined by $(i, j)(t\sigma) = ((i, j)t)\sigma$ for $\sigma \in S_n$. Given a λ/μ -tableau t , let $R(t)$ (resp. $C(t)$) be the subgroup of S_n consisting of all permutations that setwise fix the entries in each row (resp. column) of t . We define an equivalence relation \sim on the set of λ/μ -tableaux by $t \sim u$ if and only if there exists $\pi \in R(t)$ such that $u = t\pi$. The λ/μ -*tabloid* $\{t\}$ is the equivalence class of t . A short calculation shows that S_n acts on the set of λ/μ -tabloids by $\{t\}\sigma = \{t\sigma\}$.

Generalizing the usual definitions to skew partitions, we say that a λ/μ -tableau is *row standard* if the entries in its rows are increasing when read from left to right, and *column standard* if the entries in its columns are increasing when read from top to bottom. A tableau t that is both row standard and column standard is a *standard* tableau.

Let $M^{\lambda/\mu}$ be the $\mathbf{Z}S_n$ -permutation module spanned by the λ/μ -tabloids. We define the λ/μ -polytabloid $e(t) \in M^{\lambda/\mu}$ by

$$e(t) = \sum_{\sigma \in C(t)} \text{sgn}(\sigma) \{t\} \sigma.$$

If t is a standard tableau then we say that $e(t)$ is a *standard polytabloid*. The *skew Specht module* $S^{\lambda/\mu}$ is then the $\mathbf{Z}S_n$ -module spanned by all λ/μ -polytabloids. Taking $\mu = \emptyset$ this is the Specht module S^λ , defined over \mathbf{Z} . By definition, χ^λ is the character of $S^\lambda \otimes_{\mathbf{Z}} \mathbf{C}$, and more generally, $\chi^{\lambda/\mu}$ is the character of $S^{\lambda/\mu} \otimes_{\mathbf{Z}} \mathbf{C}$.

2.2. Garnir relations and the Standard Basis Theorem. If $\sigma \in S_n$ then an easy calculation shows that

$$(2.1) \quad e(t)\sigma = e(t\sigma).$$

Hence $S^{\lambda/\mu}$ is cyclic, generated by any λ/μ -polytabloid. Moreover given $\tau \in C(t)$ then

$$(2.2) \quad e(t)\tau = \text{sgn}(\tau)e(t)$$

so $S^{\lambda/\mu}$ is spanned by the λ/μ -polytabloids $e(t)$ for t a column standard λ/μ -tableau. Let \tilde{t} be the unique column standard λ/μ -tableau whose columns agree setwise with t and let $\varepsilon_t \in \{+1, -1\}$ be defined by $e(\tilde{t}) = \varepsilon_t e(t)$. We call \tilde{t} the *column straightening* of t .

Suppose that (i, j) and $(i, j + 1)$ are boxes in $[\lambda/\mu]$. Given a λ/μ -tableau t , let

$$X = \{(i, j)t, (i + 1, j)t, \dots\}$$

be the set of entries in column j of t weakly below box (i, j) , and let

$$Y = \{\dots, (i - 1, j + 1)t, (i, j + 1)t\}$$

be the set of entries in column $j + 1$ of t weakly above box $(i, j + 1)$. Let $C_{X,Y}$ be the set of all products of transpositions $(x_1, y_1) \dots (x_k, y_k)$ for $x_1 < \dots < x_k$ and $y_1 < \dots < y_k$ where $\{x_1, \dots, x_k\} \subseteq X$ and $\{y_1, \dots, y_k\} \subseteq Y$ are non-empty k -sets. We define the *Garnir element for X and Y* by

$$(2.3) \quad G_{X,Y} = 1 + \sum_{\sigma \in C_{X,Y}} \text{sgn}(\sigma) \sigma \in \mathbf{Z}S_{X \cup Y}.$$

Restated, replacing ideals in the group ring $\mathbf{Z}S_n$ with polytabloids, (3.8) in [2] implies that

$$(2.4) \quad e(t)G_{X,Y} = 0.$$

Similarly restated, Theorem 3.9 in [2] is as follows.

Theorem 2.1 (Standard Basis Theorem).

- (i) Any λ/μ -polytabloid can be expressed as a \mathbf{Z} -linear combination of standard λ/μ -polytabloids by applications of column relations (2.2) and Garnir relations (2.4).
- (ii) The $\mathbf{Z}S_n$ -module $S^{\lambda/\mu}$ has the set of standard λ/μ -polytabloids as a \mathbf{Z} -basis.

We remark that the proofs of Theorem 7.2 and 8.4 in [5], for Specht modules labelled by partitions, but defined using polytabloids, generalize easily to prove (2.4) and Theorem 2.1 exactly as stated above. We give a small example of Garnir relations in Example 2.8 below.

2.3. A filtration for Specht modules. We require the following notation. Given finite groups G and H , a $\mathbf{Z}G$ -module U and a $\mathbf{Z}H$ -module V , we denote by $U \boxtimes V$ the $\mathbf{Z}[G \times H]$ -module given by the outer tensor product (see [1, (43.1)]) of U and V . The induction and restriction of modules and characters, defined as in [1, §12D, §43], are denoted by \uparrow and \downarrow , respectively.

Fix throughout this section $m, n \in \mathbf{N}$ and a partition λ of $m + n$. Let $S_{(m,n)} = S_{\{1,2,\dots,m\}} \times S_{\{m+1,m+2,\dots,m+n\}}$. We shall prove the following theorem.

Theorem 2.2 ([6, Theorem 3.1]). *The restricted Specht module $S^\lambda \downarrow_{S_{(m,n)}}$ has a descending chain of $\mathbf{Z}S_{(m,n)}$ -submodules whose successive quotients are isomorphic to $S^\mu \boxtimes S^{\lambda/\mu}$, where each subpartition μ of λ of size m occurs exactly once.*

Suppose that λ has first part c . Given a λ -tableau t we define the m -shape of t to be the composition $(\gamma_1, \dots, \gamma_c)$ such that γ_j equals the number of entries in column j of t not exceeding m . Let \succeq denote the dominance order on compositions of the same size, defined by $\delta \succeq \gamma$ if and only if $\ell(\delta) \leq \ell(\gamma)$ and $\sum_{i=1}^k \delta_i \geq \sum_{i=1}^k \gamma_i$ whenever $1 \leq k \leq \ell(\delta)$. For each composition γ such that $\ell(\gamma) \leq c$ we define

$$V^{\succeq \gamma} = \langle e(t) : t \text{ a column standard } \lambda\text{-tableau of } m\text{-shape } \delta \text{ where } \delta \succeq \gamma \rangle_{\mathbf{Z}}.$$

Note that the definition of the m -shape agrees with the notation $b(y)$ in the proof of [6, Theorem 3.1]. We require the following total ordering on the set of column standard λ -tableaux, defined implicitly in [5, page 30].

Definition 2.3. Let u and t be column standard λ -tableaux. We write $u > t$ if and only if the greatest entry appearing in a different column in u to t appears further right in u than t .

For instance, the $>$ order on column standard $(2, 2)$ -tableaux is

$$\begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} > \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} > \begin{array}{|c|c|} \hline 2 & 1 \\ \hline 3 & 4 \\ \hline \end{array} > \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 4 & 3 \\ \hline \end{array} > \begin{array}{|c|c|} \hline 2 & 1 \\ \hline 4 & 3 \\ \hline \end{array} > \begin{array}{|c|c|} \hline 3 & 1 \\ \hline 4 & 2 \\ \hline \end{array}.$$

Note that here, as in general, the greatest tableau under $>$ is standard. Several times below we use that if $x > y$ and x is to the left of y in the column standard tableau u then $\widetilde{u(x, y)} > u$.

Proposition 2.4. *Let u be a column standard λ -tableau of m -shape γ . Then $e(u)$ is equal to a \mathbf{Z} -linear combination of standard λ -polytabloids $e(t)$ where each t has m -shape μ' for some partition μ' such that $\mu' \succeq \gamma$.*

Proof. If u is standard then γ is a partition, and there is nothing to prove. If u is not standard then there exists $(i, j) \in [\lambda]$ such that $(i, j)u > (i, j+1)u$. Let X and

Y be as defined in (2.3). By (2.4) we have

$$0 = e(u) + \sum_{\sigma \in C_{X,Y}} \varepsilon_{u\sigma} \operatorname{sgn}(\sigma) e(\widetilde{u\sigma})$$

where $\widetilde{u\sigma}$ and $\varepsilon_{u\sigma} \in \{+1, -1\}$ are as defined at the start of §2.2. Let $\sigma \in C_{X,Y}$. Since the minimum of X exceeds the maximum of Y , we have $x > y$ for each transposition (x, y) in σ . Hence $\widetilde{u\sigma} > u$. Write δ for the m -shape of $\widetilde{u\sigma}$. If there are exactly k transpositions (x, y) in σ such that $x > m \geq y$, then $\delta_j = \gamma_j + k$, $\delta_{j+1} = \gamma_{j+1} - k$ and $\delta_{j'} = \gamma_{j'}$ for $j' \neq j, j+1$. Hence $\delta \triangleright \gamma$. The lemma now follows by induction on the \geq and \triangleright orders. \square

Corollary 2.5. *Let μ be a subpartition of λ of size m . Then $V^{\triangleright\mu'}$ is a $\mathbf{Z}S_{(m,n)}$ -submodule of S^λ with \mathbf{Z} -basis given by the standard λ -tableaux of m -shape ν' such that $\nu' \triangleright \mu'$.*

Proof. Since the standard λ -polytabloids are linearly independent by Theorem 2.1(ii), it follows immediately from Proposition 2.4 that $V^{\triangleright\mu'}$ has a \mathbf{Z} -basis as claimed. If $\pi \in S_{(m,n)}$ and s is a standard λ -tableau of m -shape ν' then $s\pi$ also has m -shape ν' , as does $\widetilde{s\pi}$. By (2.2) and Proposition 2.4, $e(s\pi) = \pm e(\widetilde{s\pi}) \in V^{\triangleright\nu'} \subseteq V^{\triangleright\mu'}$. Hence $V^{\triangleright\mu'}$ is a $\mathbf{Z}S_{(m,n)}$ -module. \square

Given a μ -tableau u with (as usual) entries $\{1, \dots, m\}$ and a λ/μ -tableau v with entries $\{m+1, \dots, m+n\}$, let $u \cup v$ denote the λ -tableau defined by

$$(i, j)(u \cup v) = \begin{cases} (i, j)u & \text{if } (i, j) \in [\mu] \\ (i, j)v & \text{if } (i, j) \in [\lambda/\mu]. \end{cases}$$

Clearly every λ -tableau of m -shape μ' is of this form. We shall show that the action of $S_{(m,n)}$ on standard λ -polytabloids is compatible with this factorization. We require the following lemma and proposition, which are illustrated in Example 2.8 below.

Lemma 2.6. *Let μ be a subpartition of λ of size m . Let u be a column standard μ -tableau and let v be a λ/μ -tableau. Let $(i, j) \in [\mu]$ be a box such that*

$$m \geq (i, j)u > (i, j+1)u.$$

Let $r = \mu'_j$ so (r, j) is the lowest box in column j of u , and define

$$X = \{(i, j)u, (i+1, j)u, \dots, (r, j)u, (r+1, j)v, \dots\},$$

$$Y = \{\dots, (i-1, j+1)u, (i, j+1)u\},$$

$$X^* = \{(i, j)u, (i+1, j)u, \dots, (r, j)u\}.$$

Let $C_{X^, Y} = \{\sigma \in C_{X, Y} : x\sigma = x \text{ for all } x \in X \setminus X^*\}$. Then*

$$0 = e(u \cup v) + \sum_{\sigma^* \in C_{X^*, Y}} \operatorname{sgn}(\sigma^*) e(u \cup v)\sigma^* + \sum_{\sigma \in C_{X, Y} \setminus C_{X^*, Y}} \operatorname{sgn}(\sigma) e(u \cup v)\sigma$$

where

- (i) for each σ^* , we have $e(u \cup v)\sigma^* = e(u\sigma^* \cup v)$ and $\widetilde{u\sigma^*} > u$;
- (ii) for each σ , $e(u \cup v)\sigma$ is a \mathbf{Z} -linear combination of polytabloids $e(s)$ for standard tableaux s of m -shape ν' where $\nu' \triangleright \mu'$.

Proof. Since $G_{X,Y} = 1 + \sum_{\sigma^* \in C_{X^*,Y}} \text{sgn}(\sigma^*)\sigma^* + \sum_{\sigma \in C_{X,Y} \setminus C_{X^*,Y}} \text{sgn}(\sigma)\sigma$, the displayed equation follows from (2.4). Since $C_{X^*,Y} \subseteq S_{\{1,\dots,m\}}$, (i) follows from the observation after Definition 2.3. Take $\sigma \in C_{X,Y} \setminus C_{X^*,Y}$ and let $w = (u \cup v)\sigma$. Since σ involves a transposition (x,y) with $x > m \geq y$, the statistic k in the proof of Proposition 2.4 is non-zero. Hence the m -shape of $e(\widetilde{w})$ is δ for some composition δ with $\delta \triangleright \mu'$. The statement of Proposition 2.4 now implies that $e(\widetilde{w})$ is a \mathbf{Z} -linear combination of standard polytabloids $e(s)$ for s of m -shape ν' where $\nu' \triangleright \delta$. Hence $\nu' \triangleright \mu'$, as required for (ii). \square

Proposition 2.7. *Let μ be a subpartition of λ of size m . Let u be a column standard μ -tableau and let t be a standard λ/μ -tableau. If $e(u) = \sum_S \alpha_S e(S)$ where the sum is over all standard μ -tableaux S and $\alpha_S \in \mathbf{Z}$ for each S then*

$$e(u \cup t) \in \sum_S \alpha_S e(S \cup t) + \sum_{\nu' \triangleright \mu'} V^{\triangleright \nu'}.$$

Proof. If u is standard the result is obvious. If not, there exists a box $(i,j) \in [\mu]$ such that $m \geq (i,j)u > (i+1,j)u$. Let X^* and Y be as in Lemma 2.6. By Lemma 2.6(ii) we have

$$e(u \cup t) \in - \sum_{\sigma^* \in C_{X^*,Y}} \text{sgn}(\sigma^*) e(u \cup t)\sigma^* + \sum_{\nu' \triangleright \mu'} V^{\triangleright \nu'}.$$

Using Lemma 2.6(i), the result now follows by induction on the \geq order. \square

We also need the analogous lemma in which u is a λ/μ -tableau, $(i,j) \in [\lambda/\mu]$ and $(i,j)u > (i,j+1)u > m$, and $Y^* = \{(r,j+1)u, \dots, (i,j+1)u\}$ where now $r = \mu'_{j+1} + 1$, and the relevant sets of coset representatives are C_{X,Y^*} and $C_{X,Y} \setminus C_{X,Y^*}$. It implies the analogous proposition in which $e(t \cup v)$ is straightened, where t is a standard μ -tableau and v is a column standard λ/μ -tableau. The proofs are entirely analogous.

The following example makes explicit the statements of Lemma 2.6 and Proposition 2.7.

Example 2.8. Let u , t and $u \cup t$ be the skew tableaux shown below.

$$u = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 4 & 3 \\ \hline \end{array}, \quad t = \begin{array}{|c|} \hline 5 \\ \hline 7 \\ \hline 6 & 8 \\ \hline \end{array}, \quad u \cup t = \begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 4 & 3 & 7 \\ \hline 6 & 8 & \\ \hline \end{array}.$$

As $4 = (2, 1)(u \cup t) > (2, 2)(u \cup t) = 3$, to straighten $u \cup t$ we define $X = \{4, 6\}$ and $Y = \{2, 3\}$. The relation $e(u \cup t)G_{X,Y} = 0$ gives

$$\begin{aligned} e(u \cup t) &= -e\left(\begin{array}{|c|c|c|} \hline 1 & 3 & 5 \\ \hline 2 & 4 & 7 \\ \hline 6 & 8 & \\ \hline \end{array}\right) + e\left(\begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 & 4 & 7 \\ \hline 6 & 8 & \\ \hline \end{array}\right) \\ &+ e\left(\begin{array}{|c|c|c|} \hline 1 & 3 & 5 \\ \hline 2 & 6 & 7 \\ \hline 4 & 8 & \\ \hline \end{array}\right) - e\left(\begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 & 6 & 7 \\ \hline 4 & 8 & \\ \hline \end{array}\right) - e\left(\begin{array}{|c|c|c|} \hline 1 & 4 & 5 \\ \hline 2 & 6 & 7 \\ \hline 3 & 8 & \\ \hline \end{array}\right). \end{aligned}$$

In the notation of Lemma 2.6, we have $X^* = \{4\}$. The standard polytabloids in the top and bottom lines come from the permutations in $C_{X^*,Y}$ and $C_{X,Y} \setminus C_{X^*,Y}$, respectively. Furthermore, the 4-shape of each polytabloid in the top line is $(2, 2)$ and in the bottom line is $(3, 1)$. Therefore

$$e(u \cup t) \in -e\left(\begin{array}{|c|c|c|} \hline 1 & 3 & 5 \\ \hline 2 & 4 & 7 \\ \hline 6 & 8 & \\ \hline \end{array}\right) + e\left(\begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 & 4 & 7 \\ \hline 6 & 8 & \\ \hline \end{array}\right) + V^{\triangleright(3,1)},$$

as expected from Proposition 2.7.

Proof of Theorem 2.2. We start by proving that there is an isomorphism ϕ of $\mathbf{Z}S_{(m,n)}$ -modules

$$\frac{V^{\triangleright\mu'}}{\sum_{\nu' \triangleright \mu'} V^{\triangleright\nu'}} \xrightarrow{\phi} S^\mu \boxtimes S^{\lambda/\mu}.$$

By Corollary 2.5, the module on the left-hand side has a \mathbf{Z} -basis given by the set of standard λ -tableaux of m -shape μ' . Therefore the linear extension ϕ of the map $e(s \cup t)\phi = e(s) \otimes e(t)$, where $s \cup t$ is a standard λ -tableau of m -shape μ' , is a well-defined \mathbf{Z} -linear morphism. Since the tensors $e(s) \otimes e(t)$ for s a standard μ -tableau and t a standard λ/μ -tableau form a basis for $S^\mu \boxtimes S^{\lambda/\mu}$, ϕ is a \mathbf{Z} -linear isomorphism.

To show that ϕ is a $\mathbf{Z}S_{(m,n)}$ -module homomorphism, it suffices to consider the actions of $S_{\{1, \dots, m\}}$ and $S_{\{m+1, \dots, m+n\}}$ separately. Let $\pi \in S_{\{1, \dots, m\}}$ and let $s \cup t$ be a standard λ -tableau. Observe that $(s \cup t)\pi = \widetilde{s\pi} \cup t$ and $\varepsilon_{(s \cup t)\pi} = \varepsilon_{s\pi}$. Suppose that $e(\widetilde{s\pi}) = \sum_S \alpha_S e(S)$ where the sum is over all standard μ -tableaux S . On the one hand

$$(e(s) \otimes e(t))\pi = -\varepsilon_{s\pi} \sum_S \alpha_S e(S) \otimes e(t).$$

On the other hand, by Proposition 2.7 we have

$$e(s \cup t)\pi \in -\varepsilon_{s\pi} \sum_S \alpha_S e(S \cup t) + \sum_{\nu' \triangleright \mu'} V^{\triangleright\nu'}.$$

The argument is entirely analogous for the action of $S_{\{m+1, \dots, m+n\}}$.

We now write \geq for the lexicographic order of compositions. We define $V^{\geq\mu'}$ in a similar way to $V^{\triangleright\mu'}$, replacing the condition $\delta \triangleright \mu'$ with $\delta \geq \mu'$. Since $\nu' \triangleright \mu'$ implies that $\nu' \geq \mu'$, replacing every instance of \triangleright with \geq in Proposition 2.4 and Corollary 2.5 implies that $V^{\geq\mu'}$ is also a $\mathbf{Z}S_{(m,n)}$ -module. Moreover, $V^{\geq\mu'}$ has a

\mathbf{Z} -basis given by the standard λ -tableaux of m -shape ν' such that $\nu' \geq \mu'$, and so there is an isomorphism

$$\frac{V^{\geq \mu'}}{\sum_{\nu' > \mu'} V^{\geq \mu'}} \cong \frac{V^{\triangleright \mu'}}{\sum_{\nu' \triangleright \mu'} V^{\triangleright \nu'}} \cong S^\mu \boxtimes S^{\lambda/\mu}.$$

Therefore the modules $V^{\geq \mu'}$, where μ ranges over all subpartitions of λ of size m , give the required chain of submodules. \square

Corollary 2.9. *Let $\rho \in S_{m+n}$ be an n -cycle and let π be a permutation of the remaining m numbers. Then*

$$\chi^\lambda(\pi\rho) = \sum_{\mu} \chi^\mu(\pi) \chi^{\lambda/\mu}(\rho)$$

where the sum is over all subpartitions μ of λ of size m .

Proof. By taking a suitable conjugate of $\pi\rho$ we may assume that $\pi \in S_{\{1, \dots, m\}}$ and $\rho \in S_{\{m+1, \dots, m+n\}}$. Taking characters in Theorem 2.2 gives

$$(2.5) \quad \chi^\lambda \downarrow_{S_{(m,n)}} = \sum_{\mu} \chi^\mu \times \chi^{\lambda/\mu}$$

where the sum is over all subpartitions μ of λ of size m . Now evaluate both sides at $\pi\rho$. \square

3. PIERI'S RULE AND YOUNG'S RULE

A skew partition λ/μ is a *vertical* (resp. *horizontal*) *strip* if $[\lambda/\mu]$ has at most one box in each row (resp. column). Given $n \in \mathbf{N}$, we write sgn_{S_n} for the character and the $\mathbf{C}S_n$ -module afforded by the sign representation of S_n .

Theorem 3.1 (Pieri's rule). *Let λ be a partition of $m+n$. If μ is a subpartition of λ of size m then*

$$\langle \chi^\lambda \downarrow_{S_m \times S_n}, \chi^\mu \times \text{sgn}_{S_n} \rangle = \begin{cases} 1 & \text{if } \lambda/\mu \text{ is a vertical strip} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By Maschke's Theorem (see [1, (10.8)]) and (2.5), applied to a suitable conjugate of $S_m \times S_n$, it suffices to prove that the multiplicity of sgn_{S_n} as a direct summand of $S^{\lambda/\mu} \otimes_{\mathbf{Z}} \mathbf{C}$ is 1 if λ/μ is a vertical strip and otherwise 0. For this we use the corresponding idempotent $E = \frac{1}{n!} \sum_{\tau \in S_n} \tau \text{sgn}(\tau) \in \mathbf{C}S_n$.

If λ/μ is not a vertical strip then it contains boxes (i, j) , $(i, j+1)$ in the same row. If t is a λ/μ -tableau then $\{t\}(1 - (x, y)) = 0$ where $x = (i, j)t$ and $y = (i, j+1)t$. Since $E = \frac{1}{n!} (1 - (x, y)) \sum_{\pi} \pi \text{sgn}(\pi)$, where the sum is over a set of right coset representatives for the cosets of $\langle (x, y) \rangle$ in S_n , it follows that $M^{\lambda/\mu} E = 0$. Hence $S^{\lambda/\mu} E = 0$ as required.

Suppose that λ/μ is a vertical strip, and let t be a λ/μ -tableau. Let Y_1, \dots, Y_c be the sets of entries in each column of t . Let $G = S_{Y_1} \times \dots \times S_{Y_c}$ and let π_1, \dots, π_d

be a set of right coset representatives for the cosets of G in S_n . Observe that

$$\{(Y_1\pi_j, \dots, Y_c\pi_j) : 1 \leq j \leq d\}$$

is the complete set of set compositions of $\{1, \dots, n\}$ into c non-empty parts of sizes $|Y_1|, \dots, |Y_c|$. Let $M = |Y_1|! \dots |Y_c|!$. By (2.2), $e(t)\tau = \text{sgn}(\tau)e(t)$ for each $\tau \in G$. The observation now implies that

$$e(t)E = \frac{M}{n!} \sum_{i=1}^d \text{sgn}(\pi_i)e(t\pi_i)$$

is non-zero and depends on t only up to a sign. Hence the multiplicity of sgn_{S_n} in $S^{\lambda/\mu}$ is 1. This completes the proof. \square

For example, the unique submodule of $S^{(2,1,1)/(1)} \otimes_{\mathbf{Z}} \mathbf{C}$ affording sgn_{S_3} is spanned by $e(t)E = \frac{1}{3}e(t) - \frac{1}{3}e(t(1, 2)) + \frac{1}{3}e(t(1, 3, 2))$ where

$$t = \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}, \quad t(1, 2) = \begin{array}{|c|} \hline 2 \\ \hline 1 \\ \hline 3 \\ \hline \end{array}, \quad t(1, 3, 2) = \begin{array}{|c|} \hline 3 \\ \hline 1 \\ \hline 2 \\ \hline \end{array}.$$

The following lemma is also used in §6.

Lemma 3.2. *Let λ be a partition of $m+n$ and let μ be a subpartition of λ of size m . If ψ is a character of S_n then*

$$\langle \chi^{\lambda/\mu}, \psi \rangle_{S_n} = \langle \chi^\lambda, \chi^\mu \times \psi \uparrow_{S_m \times S_n}^{S_{m+n}} \rangle_{S_{m+n}}.$$

Proof. By Frobenius reciprocity (see [1, Theorem 38.8]) and Corollary 2.9,

$$\begin{aligned} \langle \chi^\lambda, \chi^\mu \times \psi \uparrow_{S_m \times S_n}^{S_{m+n}} \rangle &= \langle \chi^\lambda \downarrow_{S_m \times S_n}^{S_{m+n}}, \chi^\mu \times \psi \rangle \\ &= \left\langle \sum_{\nu} \chi^\nu \times \chi^{\lambda/\nu}, \chi^\mu \times \psi \right\rangle \end{aligned}$$

where the sum runs over all partitions ν of m such that $\nu \subset \lambda$. The only non-zero summand is $\langle \chi^\mu \times \chi^{\lambda/\mu}, \chi^\mu \times \psi \rangle = \langle \chi^{\lambda/\mu}, \psi \rangle$. \square

Using Lemma 3.2 we immediately obtain the more usual statement of Pieri's rule that if ν is a partition of n then $(\chi^\nu \times \text{sgn}_{S_\ell}) \uparrow_{S_n \times S_\ell}^{S_{n+\ell}} = \sum_{\kappa} \chi^\kappa$ where the sum is over all partitions κ of $n+\ell$ such that κ/ν is a vertical strip. Multiplying by the sign character using the basic result that $\chi^\nu \times \text{sgn}_{S_n} = \chi^{\nu'}$ (see for instance [5, (6.6)]) then gives Young's rule: $(\chi^\nu \times 1_{S_\ell}) \uparrow_{S_n \times S_\ell}^{S_{n+\ell}} = \sum_{\kappa} \chi^\kappa$ where the sum is over all partitions κ of $n+\ell$ such that κ/ν is a horizontal strip.

Remark 3.3. A similarly explicit proof of Young's rule can be given, using a similar argument to the proof of Theorem 3.1. To reduce to horizontal strips, observe that if t is a standard λ/μ -tableau with boxes (i, j) and $(i+1, j)$ then $e(t)(1+(x, y)) = 0$ where $x = (i, j)t$ and $y = (i+1, j)t$.

4. THE DOMINANCE LEMMA FOR SKEW TABLEAUX

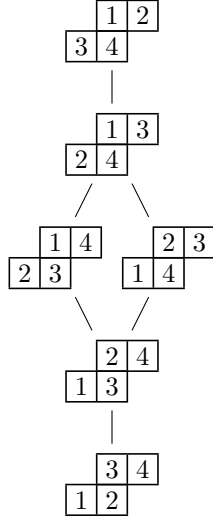
The dominance order for tabloids is defined in [5, Definition 3.11], or, in a way more convenient for us, in [12, Definition 2.5.4]. We extend it to compare row standard skew tableaux of shape a fixed skew partition.

Definition 4.1. Let t be a row standard λ/μ -tableau where $|\lambda/\mu| = n$. Given $1 \leq y \leq n$, we define $\text{sh}_{\leq y}(t)$ to be the composition β such that

$$\beta_i = |\{x : x \in \text{row } i \text{ of } t, x \leq y\}|$$

for $1 \leq i \leq \ell(\lambda)$. If s is another row standard λ/μ -tableau, then we say that s *dominates* t , and write $s \triangleright t$, if $\text{sh}_{\leq y}(s) \triangleright \text{sh}_{\leq y}(t)$ for all $y \in \{1, \dots, n\}$, where on the right-hand side \triangleright denotes the dominance order of compositions defined in §2.3.

Example 4.2. The \triangleright order on the row standard $(3, 2)/(1)$ -tableaux is shown below, with the largest tableau at the top.



Given a λ/μ -tableau t , we define its *row straightening* \bar{t} to be the unique row standard λ/μ -tableau whose rows agree setwise with t . We extend the dominance order to λ/μ -tabloids by setting $\{s\} \triangleright \{t\}$ if and only if $\bar{s} \triangleright \bar{t}$.

Lemma 4.3 (Dominance Lemma). *If t is a column standard λ/μ -tableau then \bar{t} is standard and*

$$e(t) = e(\bar{t}) + w,$$

where w is a \mathbf{Z} -linear combination of standard polytabloids $e(s)$ such that $s \triangleleft \bar{t}$.

We first show that \bar{t} is standard. Suppose, for a contradiction, that there exist boxes (i, j) and $(i + 1, j) \in [\lambda/\mu]$ such that $(i, j)\bar{t} > (i + 1, j)\bar{t}$. Define

$$R = \{(i, k)\bar{t} : j \leq k \leq \lambda_i\}$$

$$S = \{(i + 1, k)\bar{t} : \mu_{i+1} < k \leq j\}.$$

Since

$$(i+1, \mu_{i+1}+1)\bar{t} < \dots < (i+1, j)\bar{t} < (i, j)\bar{t} < \dots < (i, \lambda_i)\bar{t}$$

we have $x > y$ for each $x \in R$ and $y \in S$. But since $|R| + |S| = \lambda_i - \mu_{i+1} + 1$, the pigeonhole principle implies that there exist $x \in R$ and $y \in S$ lying in the same column of the column standard skew tableau t , a contradiction.

The next two lemmas generalise Lemmas 3.15 and 8.3 in [5] to skew tableaux.

Lemma 4.4. *Let t be a λ/μ -tableau. Let $x, y \in \{1, \dots, n\}$ be such that $x < y$. If x is strictly higher than y in t then $\overline{t(x, y)} \triangleleft \bar{t}$.*

Proof. Let x be in row k of t and let y be in row ℓ of t . By hypothesis, $k < \ell$. Let $z \in \{1, \dots, n\}$. If $x \leq z < y$ then

$$\begin{aligned} \text{sh}_{\leq z}(\overline{t(x, y)})_k &= \text{sh}_{\leq z}(\bar{t})_k - 1 \\ \text{sh}_{\leq z}(\overline{t(x, y)})_\ell &= \text{sh}_{\leq z}(\bar{t})_\ell + 1. \end{aligned}$$

Whenever $i \notin \{k, \ell\}$ or $z < x$ or $y \leq z$ we have $\text{sh}_{\leq z}(\overline{t(x, y)})_i = \text{sh}_{\leq z}(\bar{t})_i$. It easily follows from these equations and the definition of the dominance order for compositions that $\overline{t(x, y)} \triangleleft \bar{t}$. \square

Lemma 4.5. *Let t be a column standard λ/μ -tableau. Then $e(t) = \{t\} + w$, where w is a \mathbf{Z} -linear combination of λ/μ -tabloids $\{s\}$ such that $\{s\} \triangleleft \{t\}$.*

Proof. The proof of Lemma 8.3 in [5] still holds, replacing Lemma 3.15 in [5] with our Lemma 4.4. \square

Proof of Lemma 4.3. Let $e(t) = \sum_s \alpha_s e(s)$ where the sum is over all standard λ/μ -tableaux and $\alpha_s \in \mathbf{Z}$ for each s . Let u be a standard tableau maximal in the dominance order such that $\alpha_u \neq 0$. Applying Lemma 4.5 to $e(u)$ gives

$$e(u) = \{u\} + w^{\triangleleft \{u\}},$$

where $w^{\triangleleft \{u\}}$ is a \mathbf{Z} -linear combination of λ/μ -tabloids each dominated by $\{u\}$. By Lemma 4.5 and the maximality of u , there is no other standard λ/μ -tableau s with $\alpha_s \neq 0$ such that $e(s)$ has $\{u\}$ as a summand. Therefore the coefficient of $\{u\}$ in $e(t)$ is α_u . Applying Lemma 4.5, now to $e(t)$, gives

$$e(t) = \{t\} + w^{\triangleleft \{t\}},$$

where $w^{\triangleleft \{t\}}$ is a \mathbf{Z} -linear combination of λ/μ -tabloids each dominated by $\{t\}$. In particular $\{t\} \trianglerighteq \{u\}$, and so we have that $\bar{t} = u$ by the maximality of u . Hence

$$e(t) = \alpha_{\bar{t}} e(\bar{t}) + w,$$

where w is a \mathbf{Z} -linear combination of standard polytabloids $e(v)$ for standard tableaux v such that $v \triangleleft \bar{t}$. It follows that $\{t\}$ cannot be a summand of w in the equation immediately above. Since the coefficient of $\{t\}$ in $e(t)$ is 1, we have $\alpha_{\bar{t}} = 1$. \square

We isolate the following corollary of Lemma 4.3.

Corollary 4.6. *Let s be a standard λ/μ -tableau, and let u be a column standard λ/μ -tableau. Suppose that there exists $x \in \{1, 2, \dots, n\}$ such that the boxes containing $1, 2, \dots, x-1$ are the same in s and u , and x is lower in u than s . If*

$$e(u) = \sum \alpha_v e(v),$$

where the sum is over all standard λ -tableaux v , then $\alpha_s = 0$.

Proof. By assumption, $\text{sh}_{\leq z}(s) = \text{sh}_{\leq z}(\bar{u})$ if $1 \leq z < x$. As x is in a lower row in u than in s , we have $\text{sh}_{\leq x}(\bar{u}) \not\leq \text{sh}_{\leq x}(s)$. Now apply Lemma 4.3. \square

5. THE MURNAGHAN–NAKAYAMA RULE FOR BORDER STRIPS

In this section we give a bijective proof that $\chi^{\lambda/\mu}(\rho) = (-1)^{\text{ht}(\lambda/\mu)}$ when λ/μ is a border strip of size n and ρ is the n -cycle $(1, 2, \dots, n)$. This deals with one of the two cases in (1.1). Our proof shows that the matrix representing ρ in the standard basis of $S^{\lambda/\mu}$ has a unique non-zero entry on its diagonal. The relevant standard tableau is defined as follows.

Definition 5.1. Let λ/μ be a border strip of size n . Say that a box $(i, j) \in [\lambda/\mu]$ is *columnar* if $(i+1, j) \in [\lambda/\mu]$. We define the standard λ/μ -tableau $t_{\lambda/\mu}$ as follows:

- (i) assign the numbers $\{1, \dots, z\}$ in ascending order to the z columnar boxes of λ/μ , starting with 1 in row 1 and finishing with z in the row above the bottom row;
- (ii) then assign the numbers $\{z+1, \dots, n\}$ in ascending order to the $n-z$ non-columnar boxes, starting with $z+1$ in column 1 and finishing with n in the rightmost column.

For example, $t_{(5,3,3)/(2,2)}$, $t_{(5,3,2)/(2,1)}$ and $t_{(5,1,1)/\emptyset}$ are respectively

$$\begin{array}{|c|c|c|} \hline 1 & 6 & 7 \\ \hline 2 & & \\ \hline 3 & 4 & 5 \\ \hline \end{array}, \quad \begin{array}{|c|c|c|} \hline 1 & 6 & 7 \\ \hline 2 & 5 & \\ \hline 3 & 4 & \\ \hline \end{array} \quad \text{and} \quad \begin{array}{|c|c|c|c|c|} \hline 1 & 4 & 5 & 6 & 7 \\ \hline 2 & & & & \\ \hline 3 & & & & \\ \hline \end{array}$$

where 1 and 2 are the entries in columnar boxes in each case. We remark that there are no columnar boxes if and only if λ/μ is a horizontal strip, as defined in §3.

As useful pieces of notation, we define x^- and x^+ for $x \in \{1, \dots, n\}$ by $x^- = x-1$ and

$$x^+ = \begin{cases} x+1 & \text{if } 1 \leq x < n \\ 1 & \text{if } x = n. \end{cases}$$

Thus $x\rho = x^+$ for all $x \in \{1, \dots, n\}$ and $1^- = 0$. Given a λ/μ -tableau t , we define t^+ by $(i, j)t^+ = ((i, j)t)^+$. By (2.1), $e(t\rho) = e(t^+)$.

We say that a standard λ/μ -tableau t such that $e(t)$ has a non-zero coefficient in the unique expression of $e(t^+)$ as a \mathbf{Z} -linear combination of standard polytabloids is *trace-contributing*. Since $\chi^{\lambda/\mu}(\rho)$ is the trace of the matrix representing ρ in the standard basis, it suffices to prove the following proposition.

Proposition 5.2. *Let λ/μ be a border strip. The unique trace-contributing λ/μ -tableau is $t_{\lambda/\mu}$. The coefficient of $e(t_{\lambda/\mu})$ in $e(t_{\lambda/\mu}^+)$ is $(-1)^{\text{ht}(\lambda/\mu)}$.*

The proof of Proposition 5.2 is by induction on the number of top corner boxes of λ/μ , as defined in Definition 5.3 below. The necessary preliminaries are collected below. We then prove the base case, when $\lambda/\mu = (n - \ell, 1^\ell)$ for some $\ell \in \mathbf{N}_0$; this gives a good flavour of the general argument. In the remainder of this section we give the inductive step.

We assume, without loss of generality, that $\mu_1 < \lambda_1$ and $\mu_{\ell(\lambda)} = 0$, so the non-empty rows of λ/μ are $1, \dots, \ell(\lambda)$ and column 1 of λ/μ is non-empty. We can do this since the character indexed by a skew diagram is equal to the character indexed by the same skew diagram with its empty rows and columns removed.

5.1. Preliminaries for the proof of Proposition 5.2. For $Z \subseteq \{1, \dots, n\}$ and t a row standard λ/μ -tableau we define $\text{sh}_Z(t)$ to be the composition β such that

$$\beta_i = |\{x : x \in \text{row } i \text{ of } t, x \in Z\}|$$

for $1 \leq i \leq \ell(\lambda)$. Set $\text{sh}_{<y}(t) = \text{sh}_{\{1, \dots, y^-\}}(t)$. We also use $\text{sh}_{\leq y}(t)$, as already defined in Definition 4.1.

Definition 5.3. Let λ/μ be a border strip. We say that column j of λ/μ is *singleton* if it contains a unique box. We define a *top corner box* to be a box $(i, j) \in [\lambda/\mu]$ such that $(i, j - 1), (i - 1, j) \notin [\lambda/\mu]$ and a *bottom corner box* to be a box $(i, j) \in [\lambda/\mu]$ such that $(i + 1, j), (i, j + 1) \notin [\lambda/\mu]$.

Lemma 5.4. *Let λ/μ be a border strip and let t be a λ/μ -tableau. If columns j and $j + 1$ of λ/μ are singleton, with their unique box in row i , then $e(t) = e(t)(x, y)$ where $x = (i, j)t$ and $y = (i, j + 1)t$.*

Proof. This follows immediately from the Garnir relation (2.4), taking $X = \{x\}$ and $Y = \{y\}$. \square

In fact, all the Garnir relations that we use can be reduced to single transpositions. Let x and y be entries in adjacent columns of a column standard tableau, with x left of y and $x > y$. We say that (x, y) is a *Garnir swap* if at least one of these column is not singleton, and otherwise that (x, y) is a *horizontal swap*.

Lemma 5.5. *Let t be a trace-contributing border strip tableau. Then t can be obtained from \widetilde{t}^+ by iterated horizontal swaps, Garnir swaps and column straightenings. If in such a sequence 1 moves, then 1 moves either left or down.*

Proof. The first claim is immediate from Theorem 2.1(i). The second follows from Corollary 4.6 taking $x = 1$. \square

Given $X \subseteq \{1, 2, \dots, n\}$, we define $X^+ = \{x^+ : x \in X\}$. We also define $\min X$ to be the minimum of X , and $\max X$ to be the maximum of X . The following combinatorial result on the map $x \mapsto x^+$ is used several times to restrict the possible entries of trace-contributing tableaux.

Lemma 5.6. *Let X be a set of natural numbers such that $1, n \notin X$. Also suppose that b, c are not contained in X . We have $\{b^+\} \cup X^+ = X \cup \{c\}$ if and only if $b^+ = \min X$, $c = \max X^+$ and $X = \{b^+, \dots, c^-\}$.*

Proof. Since $\min X \notin X^+$ we have $\min X = b^+$. Similarly, since $\max X^+ \notin X$ we have $\max X^+ = c$. Suppose for a contradiction that X is a proper subset of $\{b^+, \dots, c^-\}$. Setting

$$d = \min(\{b^+, \dots, c^-\} \setminus X)$$

we see that since $b^+ = \min X \in X$, we have $d > b^+$. The minimality of d implies that $d^- \in X$ and so $d \in X^+$; since $d < c$ and $\{b^+\} \cup X^+ = X \cup \{c\}$, we have $d \in X$, a contradiction. The converse is obvious. \square

Finally, as a notational convention, when we specify a set, we always list the elements in increasing order. In diagrams the symbol \star marks an entry we have no need to specify more explicitly.

5.2. Base case: one top corner box. In this case $\mu = \emptyset$ and $\lambda = (n - \ell, 1^\ell)$ for some $\ell \in \mathbf{N}_0$. If $\ell = 0$ then there is a unique standard (n) -tableau and the result is clear. Suppose that $\ell > 0$ and let t be a standard $(n - \ell, 1^\ell)$ -tableau with entries $\{1, y_1, \dots, y_{\ell-1}, c\}$ in column 1. (By our notational convention, $1 < y_1 < \dots < y_{\ell-1} < c$.) If $c = n$ then \widetilde{t}^+ is standard with first column entries $\{1, 1^+, y_1^+, \dots, y_{\ell-1}^+\}$. Hence, assuming that t is trace-contributing, we have $c < n$. After a sequence of horizontal swaps applied to \widetilde{t}^+ we obtain the tableau shown below.

1^+	1	\star	\dots	\star
y_1^+				
\vdots				
$y_{\ell-1}^+$				
c^+				

A Garnir swap of 1 with 1^+ or any y_i^+ gives, after column straightening and a sequence of horizontal swaps, a standard tableau having c^+ in its bottom left position. We may therefore assume, by Lemma 5.5, that 1 is swapped with c^+ . After column straightening, which introduces the sign $(-1)^\ell$, a sequence of horizontal swaps gives the standard tableau having $\{1, 1^+, y_1^+, \dots, y_{\ell-1}^+\}$ in its first column. Thus if t is trace-contributing then $\{1^+, y_1^+, \dots, y_{\ell-1}^+\} = \{y_1, \dots, y_{\ell-1}, c\}$. By Lemma 5.6, $\{y_1, \dots, y_{\ell-1}, c\} = \{2, \dots, \ell + 1\}$. Therefore $t = t_{(n-\ell, 1^\ell)}$ and the coefficient of $e(t_{(n-\ell, 1^\ell)})$ in $e(t_{(n-\ell, 1^\ell)}^+)$ is $(-1)^\ell$, as required.

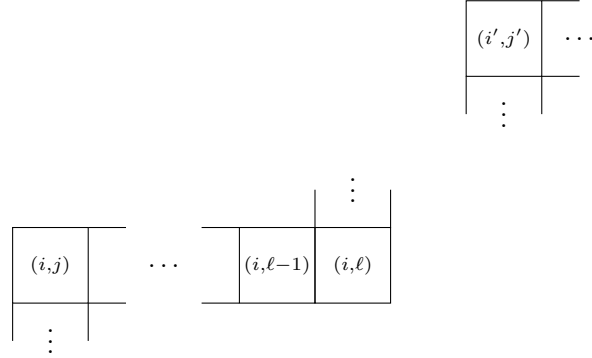
5.3. Inductive step. Let $\delta(i) \in \mathbf{N}_0^{\ell(\lambda)}$ denote the composition defined by $\delta(i)_i = 1$ and $\delta(i)_k = 0$ if $k \neq i$.

Proposition 5.7. *Let λ/μ be a border strip, and let t be a standard λ/μ -tableau. Let $c \in \mathbf{N}$ and suppose that either $c = 1$ or $c > 1$ and the entries $1, \dots, c^-$ and n lie in the same column of t . Let (i, j) be the box of t containing c , and let (i', j') be the box of \widetilde{t}^+ containing c . If t is a trace-contributing tableau, then $i = i'$.*

Proof. By hypothesis, the highest c^- entries in column j' of t and \widetilde{t}^+ are $1, \dots, c^-$. Let $s = \widetilde{t}^+$. Setting $\beta = \text{sh}_{<c}(t) = \text{sh}_{<c}(\overline{s})$ we have $\text{sh}_{\leq c}(t) = \beta + \delta(i)$ and $\text{sh}_{\leq c}(\overline{s}) = \beta + \delta(i')$. By Lemma 4.3, the hypothesis that t is trace-contributing implies that $\text{sh}_{\leq c}(\overline{s}) \supseteq \text{sh}_{\leq c}(t)$. Therefore $i \geq i'$.

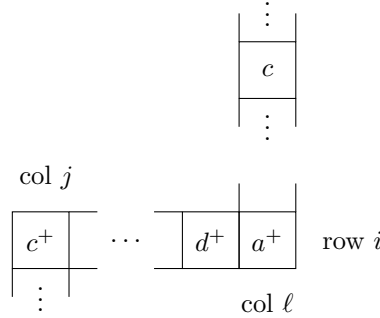
If $j = j'$ then either $c = 1$ and 1 is at the top of the column of t which has n at its bottom, or $c > 1$ and c is immediately below c^- in both s and t . In either case $i = i'$.

We may therefore suppose, for a contradiction, that $i > i'$ and $j < j'$. By hypothesis the box (i, j) of t containing c is the top corner box in row i . Let (i, ℓ) be the bottom corner box in row i ; note that $\ell \leq j'$, as shown in the diagram below.



By the hypothesis that t is trace-contributing and Lemma 5.5 there is a sequence of horizontal swaps, Garnir swaps, and column straightenings from \widetilde{t}^+ to t . Suppose that in such a sequence an entry $b < c$ is moved. If b is the first such entry moved in this sequence, and u is the tableau obtained after column straightening, then, by Corollary 4.6 applied with $x = b$, the coefficient of $e(t)$ in $e(u)$ is zero. Therefore the entries $\{1, \dots, c^-\}$ are fixed and c is the smallest number moved. Take such a sequence and stop it immediately after the first swap in which c enters row i . Let v be the column standard tableau so obtained, and let u be its immediate predecessor.

When c enters row i of v , it is swapped with the entry, d^+ say, in box $(i, \ell - 1)$ of u . Observe that the entries in boxes strictly to the left of column ℓ are the same in \widetilde{t}^+ and u , since no swap in the sequence from \widetilde{t}^+ to u involves an entry in these columns. Let a^+ be the entry in box (i, ℓ) of u . Thus the column standard tableau u is as shown overleaf and $v = u(\widetilde{c}, d^+)$.



Note that $d^+ > a^+$ since otherwise u is standard with respect to all boxes weakly to the left of column ℓ , and so d^+ cannot be moved in a Garnir swap.

To complete the proof we require the following critical quantity. *Let r be maximal such that entries c, \dots, r are strictly to the left of column ℓ in the original tableau t .* If $r = d$ then, since $d > a$, a is strictly to the left of column ℓ in t ; this is impossible since a^+ appears in column ℓ in u . Therefore $r < d$. Since d is in position $(i, \ell - 1)$ of t and $r \geq c$, it follows that $c \neq d$. Moreover, the entries c^+, \dots, r^+ are in the same boxes in t^+ and v .

Claim. We have $v \not\leq t$. *Proof of claim.* Let $\text{sh}_{\{c^+, \dots, r^+\}}(u) = \delta$. By hypothesis and our stopping condition on swaps, if $q \leq r$ then the box of q^+ in u is the box of q in t . Hence $\text{sh}_{\{c, \dots, r\}}(t) = \delta$. Since $d > r$ and d is in position $(i, \ell - 1)$ of t , we see that r^+ is not in row i of t . By maximality of r , the row of t containing r^+ is row h for some $h < i$. Clearly the row of c in v is i . Therefore $\text{sh}_{\{c, \dots, r^+\}}(\bar{v}) = \delta + \delta(i)$ and $\text{sh}_{\{c, \dots, r^+\}}(t) = \delta + \delta(h)$. Since $1, \dots, c^-$ are in the same positions in both v and t , it follows that

$$\text{sh}_{\leq r^+}(t) \triangleright \text{sh}_{\leq r^+}(\bar{v})$$

which implies the claim.

It now follows from Lemma 4.3, as before, that $e(t)$ does not appear in $e(v)$, a final contradiction. This completes the proof. \square

Corollary 5.8. *If t is a trace-contributing tableau then either 1 and n are in the same column of t , or 1 and n are in the top row of t .*

Proof. Let 1 and n be in positions (i, j) of t and (i', j') of t , respectively. If column j' is singleton then n is the top right entry of t and, taking $c = 1$ in Proposition 5.7, we get $i = i'$; thus 1 and n are in the top row of t . Otherwise, when we column straighten t^+ to obtain $\widetilde{t^+}$, the entry 1 in position (i', j') moves up to position (i'', j') where $i'' < i'$. Again taking $c = 1$ in Proposition 5.7, we get $i = i''$. Since (i'', j') is the top corner box in its row, and so is (i, j) , we see that $j = j'$. Hence 1 and n are in the same column of t . \square

Proof of Proposition 5.2. We now complete the inductive step of the proof.

Suppose that λ/μ has more than one top corner box, and that t is a trace-contributing λ/μ -tableau. Let 1 be in position (i, j) of t and in position (i', j') of \widetilde{t}^+ . By Proposition 5.7, we have $i = i'$.

Case (1). Suppose that 1 and n lie in the same row of t . By Corollary 5.8, this is the top row. Let the entries in the top row be $\{1, x_1, \dots, x_{k-1}, n\}$, and let the entries in the column of 1 be $\{1, y_1, \dots, y_{\ell-1}, c\}$.

Straightening the top row of t^+ by a sequence of $k-1$ horizontal swaps moves 1^+ and 1 into adjacent positions, giving the tableau u shown below.

$$\begin{array}{|c|c|c|c|} \hline 1^+ & 1 & x_1^+ & \cdots & x_{k-1}^+ \\ \hline y_1^+ & & & & \\ \hline \vdots & & & & \\ \hline y_{\ell-1}^+ & & & & \\ \hline \cdots & c^+ & & & \\ \hline \end{array}$$

As in the base case, the only Garnir swap that can lead to t is $(1, c^+)$, which introduces the sign $(-1)^\ell$. Let $v = u(\widetilde{1, c^+})$, as shown below.

$$\begin{array}{|c|c|c|c|} \hline 1 & c^+ & x_1^+ & \cdots & x_{k-1}^+ \\ \hline 1^+ & & & & \\ \hline y_1^+ & & & & \\ \hline \vdots & & & & \\ \hline \cdots & y_{\ell-1}^+ & & & \\ \hline \end{array}$$

By Lemma 5.5 and Corollary 4.6, v can be straightened by a sequence of horizontal swaps, Garnir swaps and column straightenings which either fix 1, and so leave invariant the content of its top row, or move 1 into a lower row, giving a tableau, w say, such that, $e(t)$ does not appear in $e(w)$. Since $e(t)$ has a non-zero coefficient in $e(v)$, we have

$$\{c^+, x_1^+, \dots, x_{k-1}^+\} = \{x_1, \dots, x_{k-1}, n\}.$$

Lemma 5.6 implies that $c^+ = x_1 = n - k + 1$, $x_{k-1}^+ = n$ and $\{x_1, \dots, x_{k-1}\} = \{n - k + 1, \dots, n - 1\}$. Thus t and v have top row entries $\{1, n - k + 1, \dots, n\}$.

Let T and V be the tableaux obtained from t and v by deleting all but the top corner box in their top rows. This removes entries $\{n - k + 1, \dots, n\}$. Let λ^*/μ be the common shape of T and V . Observe that T has greatest entry $n - k = c$ in the bottom corner box of its rightmost column and that V is the column straightening of T^\dagger , where \dagger is defined as $+$ on tableaux, but replacing n with $n - k$. By induction,

$T = t_{\lambda^*/\mu}$, and since t has $n - k + 1, \dots, n$ in its top row, we have $t = t_{\lambda/\mu}$. Moreover, the coefficient of $e(T)$ in $e(T^\dagger)$ is $(-1)^{\text{ht}(\lambda^*/\mu)}$. Since $\text{ht}(\lambda^*/\mu) = \text{ht}(\lambda/\mu)$, the coefficient of $e(t)$ in $e(t^+)$ is $(-1)^{\text{ht}(\lambda/\mu)}$, as required.

Case (2). If Case (1) does not apply then, since $i = i'$, 1 and n are in the same column of s and so $j = j'$. Take c maximal such that $1, 2, \dots, c^-$ are in column j of t . Suppose that in column j of t , the entry immediately below c^- equals d for some $d < n$. By Proposition 5.7, the row of c in t is the same as the row of c in $\widetilde{t^+}$. It follows that $c = d$, which contradicts the maximality of c unless column j of t has entries $1, 2, \dots, c^-, n$, as shown below.

$$\begin{array}{c}
 \begin{array}{|c|c|} \hline 1 & \cdots \\ \hline 2 & \\ \hline \vdots & \\ \hline c^- & \\ \hline \end{array} \\
 \vdots \\
 \begin{array}{|c|c|} \hline c & \cdots \\ \hline \vdots & \\ \hline \end{array}
 \end{array}
 \quad
 \begin{array}{c}
 \begin{array}{|c|c|} \hline \cdots & n \\ \hline \end{array} \\
 \text{col } j
 \end{array}
 \quad
 \text{row } i = i'$$

By Lemma 5.5 there is a sequence of horizontal swaps, Garnir swaps and column straightenings from $\widetilde{t^+}$ to t . As seen in the proof of Proposition 5.7, it follows easily from Lemma 4.3 that $1, \dots, c^-$ do not move. Let X be the set of entries of t lying strictly to the right of column j . These entries become X^+ in $\widetilde{t^+}$, which is standard with respect to these columns. No permutation in our chosen sequence can involve a entry in one of these columns. Hence $X^+ = X$, and so $X = \emptyset$.

We have shown that j is the rightmost column of t , and that t agrees with $t_{\lambda/\mu}$ in this column. Let T be the tableau obtained from t by deleting all but the bottom corner box in column j and subtracting c^- from each remaining entry. Thus the top row of T has entries $1, \dots, n - c^-$ and $n - c^-$ is its greatest entry. Let T have shape λ^*/μ^* . By induction, $T = t_{\lambda^*/\mu^*}$, and hence $t = t_{\lambda/\mu}$. Let T^\dagger be defined as T^+ , but replacing n with $n - c^-$. By induction, the coefficient of $e(T)$ in $e(T^\dagger)$, is $(-1)^{\text{ht}(\lambda^*/\mu^*)}$. Since $\text{ht}(\lambda^*/\mu^*) + c^- = \text{ht}(\lambda/\mu)$, and the sign introduced by column straightening t^+ is $(-1)^{c^-}$, the coefficient of $e(t)$ in $e(t^+)$ is $(-1)^{\text{ht}(\lambda/\mu)}$, as required. \square

6. PROOF OF THEOREM 1.1

Let λ/μ be a skew partition of size n and let $\rho \in S_n$ be an n -cycle. In order to complete the proof of Theorem 1.1, we must show that $\chi^{\lambda/\mu}(\rho) = 0$ if λ/μ is not a border strip. We require the following two lemmas.

Lemma 6.1. *Let $0 \leq \ell \leq n$. If*

$$\langle \chi^\lambda, \chi^\mu \times 1_{S_\ell} \times \text{sgn}_{S_{n-\ell}} \uparrow_{S_m \times S_\ell \times S_{n-\ell}}^{S_{m+n}} \rangle > 0$$

then $[\lambda/\mu]$ has no four boxes making the shape $(2, 2)$.

Proof. By the versions of Pieri's rule and Young's rule proved at the end of §3, the hypothesis implies that λ is obtained from μ by adding a horizontal strip of size ℓ and then a vertical strip of size $n - \ell$. If two boxes from a horizontal strip are added to row i then at most one box can be added below them in row $i + 1$ by a vertical strip. The result follows. \square

Lemma 6.2. *If λ is a partition of n and ρ is an n -cycle then $\chi^\lambda(\rho) \neq 0$ if and only if $\lambda = (n - \ell, 1^\ell)$ where $0 \leq \ell < n$.*

Proof. Write $\text{Cent}_{S_n}(\rho)$ for the centraliser subgroup of ρ in S_n . By a column orthogonality relation (see [1, (31.13)])

$$\sum_{\lambda} \chi^\lambda(\rho)^2 = |\text{Cent}_{S_n}(\rho)| = n,$$

and the sum is over all partitions λ of n . By (1.1) in the case proved in §5, we have $\chi^{(n-\ell, 1^\ell)}(\rho) = (-1)^{\ell-1}$ for $0 \leq \ell < n$. Therefore the partitions $(n - \ell, 1^\ell)$ give all the non-zero summands. \square

Proposition 6.3. *Let λ/μ be a skew partition of size n and let $\rho \in S_n$ be an n -cycle. If λ/μ is not a border strip then $\chi^{\lambda/\mu}(\rho) = 0$.*

Proof. If $[\lambda/\mu]$ is disconnected then it is clear from the Standard Basis Theorem (Theorem 2.1(ii)) that $S^{\lambda/\mu}$ is isomorphic to a module induced from a proper Young subgroup $S_{n-\ell} \times S_\ell$ of S_n . Since no conjugate of ρ lies in this subgroup, we have $\chi^{\lambda/\mu}(\rho) = 0$.

In the remaining case $[\lambda/\mu]$ has four boxes making the shape $(2, 2)$. By either Pieri's rule or Young's rule, we have

$$\langle 1_{S_\ell} \times \text{sgn}_{S_{n-\ell}} \uparrow_{S_\ell \times S_{n-\ell}}^{S_n}, \chi^{(n-\ell, 1^\ell)} \rangle = 1.$$

Hence

$$\begin{aligned} \langle \chi^\lambda, \chi^\mu \times 1_{S_\ell} \times \text{sgn}_{S_{n-\ell}} \uparrow_{S_m \times S_\ell \times S_{n-\ell}}^{S_{m+n}} \rangle &\geq \langle \chi^\lambda, \chi^\mu \times \chi^{(n-\ell, 1^\ell)} \uparrow_{S_m \times S_n}^{S_{m+n}} \rangle \\ &= \langle \chi^{\lambda/\mu}, \chi^{(n-\ell, 1^\ell)} \rangle \end{aligned}$$

where the equality follows from Lemma 3.2. By Lemma 6.1 the left-hand side is 0. It follows that $\langle \chi^{\lambda/\mu}, \chi^{(n-\ell, 1^\ell)} \rangle = 0$ for $0 \leq \ell < n$. By Lemma 6.2, this implies the result. \square

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DEPARTMENT OF MATHEMATICS, ROYAL HOLLOWAY UNIVERSITY OF LONDON, UNITED KINGDOM
E-mail address: `Jasdeep.Kochhar.2015@rhul.ac.uk`

DEPARTMENT OF MATHEMATICS, ROYAL HOLLOWAY UNIVERSITY OF LONDON, UNITED KINGDOM
E-mail address: `mark.wildon@rhul.ac.uk`