THE MULTISTEP HOMOLOGY OF THE SIMPLEX AND REPRESENTATIONS OF SYMMETRIC GROUPS

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Abstract. The symmetric group on a set acts transitively on its subsets of a given size. We define homomorphisms between the corresponding permutation modules, defined over a field of characteristic two, which generalize the boundary maps from simplicial homology. The main results determine when these chain complexes are exact and when they are split exact. As a corollary we obtain a new explicit construction of the basic spin modules for the symmetric group.

1. Introduction

Fix $n \in \mathbb{N}$ and let S_n denote the symmetric group of degree n. For each $k \in \mathbb{Z}$, let Ω_k denote the set of all k-subsets of $\{1, \ldots, n\}$, permuted by the action of S_n. Let $\mathbb{F}$ be a field and let $\mathbb{F}\Omega_k$ be the $\mathbb{F}$-vector space of all formal $\mathbb{F}$-linear combinations of the elements of Ω_k. Thus $\mathbb{F}\Omega_k$ is an $\mathbb{F}S_n$-module of dimension $\binom{n}{k}$ having Ω_k as a permutation basis. For instance if $n \geq 5$ then $\{1,2,3\} + \{3,4,5\} \in \mathbb{F}\Omega_3$ is sent to $\{1,2,3\} + \{1,4,5\}$ by the transposition swapping 1 and 3.

Given $t \in \mathbb{N}_0$ and $k \in \mathbb{Z}$, let $\varphi_k^{(t)} : \mathbb{F}\Omega_k \to \mathbb{F}\Omega_{k-t}$ be the $\mathbb{F}S_n$-module homomorphism defined on each $Y \in \Omega_k$ by

$$Y\varphi_k^{(t)} = \sum_{\substack{X \subseteq Y \\ |X| = |Y| - t}} X. \tag{1}$$

(Throughout we work with right-modules and write maps on the right.) Motivated by the connection with simplicial homology discussed below, we call $\varphi_k^{(t)}$ a *multistep boundary map*. This article concerns the remarkably intricate behaviour of the multistep boundary maps when $\mathbb{F}$ has characteristic two.

Given $Z \in \Omega_k$ and $t \in \mathbb{N}$, we may compute $Z\varphi_k^{(t)}\varphi_{k-t}^{(t)}$ by summing over all chains $Z \supseteq Y \supseteq X$ with $Y \in \Omega_{k-t}$ and $X \in \Omega_{k-2t}$. For each X there are $\binom{2t}{t}$ choices for Y; since $\binom{2t}{t} \equiv 0 \bmod 2$, and $\mathbb{F}$ has characteristic two, $Z\varphi_k^{(t)}\varphi_{k-t}^{(t)} = 0$. Hence if $a < t$ and $c \in \mathbb{N}_0$ is maximal such that $a + ct \leq n$ then

$$0 \to \mathbb{F}\Omega_{a+ct} \xrightarrow{\varphi_{a+ct}^{(t)}} \mathbb{F}\Omega_{a+(c-1)t} \xrightarrow{\varphi_{a+(c-1)t}^{(t)}} \cdots \xrightarrow{\varphi_{a+2t}^{(t)}} \mathbb{F}\Omega_{a+t} \xrightarrow{\varphi_{a+t}^{(t)}} \mathbb{F}\Omega_a \to 0 \tag{2}$$

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is a chain complex of $\mathbb{F}S_n$-modules, each non-zero except at the beginning and end. Its *homology* in *degree* k is, by definition, the $\mathbb{F}S_n$-module $\ker\varphi_k^{(t)}/\operatorname{im}\varphi_{k+t}^{(t)}$.

If $t = 1$ then the chain complex (2) is exact in every degree. Moreover (2) is *split exact*, in the sense that, for each k, there is an $\mathbb{F}S_n$-submodule C_k of $\mathbb{F}\Omega_k$ such that $\mathbb{F}\Omega_k = \ker\varphi_k^{(1)} \oplus C_k$, if and only if n is odd. We give short proofs of these results in §2 below.

Our first main theorem gives a complete description of the homology modules when $t = 2$. The following notation is required: for k such that $2k \le n$, define $G_{k-1} = \langle(1,2)\rangle \times \cdots \times \langle\big(2(k-1)-1, 2(k-1)\big)\rangle$ and

$$v_k = \{2, 4, \ldots, 2k\} \sum_{\sigma \in G_{k-1}} \sigma.$$

(These elements are illustrated in Example 1.4.) Let $D^{(n-k,k)}$ denote the simple $\mathbb{F}S_n$-module defined, with its usual definition, in §3 below.

Theorem 1.1. *Let $\varepsilon_k : \mathbb{F}\Omega_k \to \mathbb{F}\Omega_{k-2}$ denote the two-step boundary map $\varphi_k^{(2)}$, as defined in* (1)*, and let $H_k = \ker\varepsilon_k/\operatorname{im}\varepsilon_{k+2}$. Then*

$$H_k \cong \begin{cases} E^{(m+1,m-1)} & \text{if } n = 2m \text{ is even and } k = m \\ D^{(m+1,m)} & \text{if } n = 2m+1 \text{ is odd and } k = m \text{ or } k = m+1 \\ 0 & \text{otherwise,} \end{cases}$$

where $E^{(m+1,m-1)}$ is a non-split extension of $D^{(m+1,m-1)}$ by itself. Moreover, if $n = 2m$ or $n = 2m+1$ then H_m is the submodule of $\mathbb{F}\Omega_m/\operatorname{im}\varepsilon_{m+2}$ generated by $v_m + \operatorname{im}\varepsilon_{m+2}$ and, for each $m \in \mathbb{N}$, there are isomorphisms $D^{(m+1,m)}{\downarrow}_{S_{2m}} \cong E^{(m+1,m-1)}$, $D^{(m+1,m-1)}{\downarrow}_{S_{2m-1}} \cong D^{(m,m-1)}$.

The results on the restrictions of $D^{(m+1,m-1)}$ and $D^{(m+1,m)}$ in Theorem 1.1 are originally due to Danz and Külshammer [6, Proposition 3.3]; they are included so that the theorem can be proved by induction as it is stated. In Corollary 4.10 we take $n = 2m$ and construct an $\mathbb{F}S_{2m}$-endomorphism ϑ of H_m such that ϑ is non-zero and $\vartheta^2 = 0$, making explicit the structure of the non-split extension $E^{(m+1,m-1)}$.

In particular, Theorem 1.1 implies that the chain complex of $\mathbb{F}S_{2m}$-modules

$$0 \to \mathbb{F}\Omega_{2m} \xrightarrow{\varepsilon_{2m}} \mathbb{F}\Omega_{2m-2} \xrightarrow{\varepsilon_{2m-2}} \cdots \xrightarrow{\varepsilon_4} \mathbb{F}\Omega_2 \xrightarrow{\varepsilon_2} \mathbb{F}\Omega_0 \to 0$$

is exact whenever m is odd; if m is even then it has non-zero homology of $E^{(m+1,m-1)}$ uniquely in degree m. This categorifies the binomial identity

$$\sum_{j=0}^{m} (-1)^j \binom{2m}{2j} = \begin{cases} (-1)^{m/2} 2^m & \text{if } m \text{ is even} \\ 0 & \text{if } m \text{ is odd.} \end{cases} \tag{3}$$

Our second main theorem determines the degrees in which the chain complex (2) is exact. In particular, case (ii) determines when one of the maps is surjective or injective.

Theorem 1.2. *Let $t \in \mathbb{N}$, let $n \in \mathbb{N}$ and let $0 \le k \le n$. Let 2^τ be the least two-power appearing in the binary form of t. The sequence*

$$\mathbb{F}\Omega_{k+t} \xrightarrow{\varphi^{(t)}_{k+t}} \mathbb{F}\Omega_k \xrightarrow{\varphi^{(t)}_{k}} \mathbb{F}\Omega_{k-t} \tag{4}$$

is exact if and only if one of

(i) $t = 1$;

(ii) $k < 2^\tau$ *and* $k + t \le n - k$ or $n - k < 2^\tau$ *and* $n - k + t \le k$;

(iii) t *is a two-power* and $n \ge 2k + t$ *or* $n \le 2k - t$.

We also characterize when (2) is exact in every degree. It seems remarkable that this is the case if and only if it is split exact in every degree.

Theorem 1.3. *Let 2^τ be the least two-power appearing in the binary form of t. The chain complex* (2) *is exact in every degree if and only if one of*

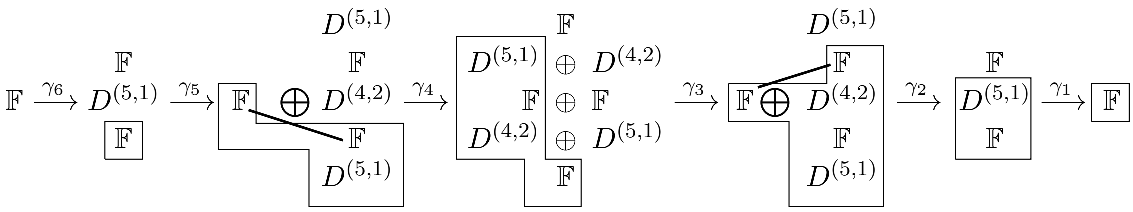
(a) $n = 2a + t$ *and* $a < 2^\tau$;

(b) t *is a two-power and* $n \equiv 2a + t$ *mod* $2t$.

Moreover, if either (a) *or* (b) *holds then* (2) *is split exact in every degree.*

We end this introduction with two examples showing some of the rich behaviour of the kernels and images of the multistep boundary maps. For readability we write γ_k for $\varphi^{(1)}_k$.

Example 1.4. When $n = 6$ the Loewy layers of the modules in the exact chain complex $\mathbb{F}\Omega_6 \xrightarrow{\gamma_6} \mathbb{F}\Omega_5 \xrightarrow{\gamma_5} \cdots \xrightarrow{\gamma_2} \mathbb{F}\Omega_1 \xrightarrow{\gamma_1} \mathbb{F}\Omega_0$ are shown below.



As predicted by Theorem 1.1, $\ker \varepsilon_4 \cong \mathbb{F}$ is a direct summand of $\mathbb{F}\Omega_4$ and $\ker \varepsilon_2$ is the (unique) co-dimension 1 direct summand of $\mathbb{F}\Omega_2$. Thus the chain complex $0 \to \mathbb{F}\Omega_6 \xrightarrow{\varepsilon_6} \mathbb{F}\Omega_4 \xrightarrow{\varepsilon_4} \mathbb{F}\Omega_2 \xrightarrow{\varepsilon_2} \mathbb{F}\Omega_0 \to 0$ is split exact. Moreover $0 \to \mathbb{F}\Omega_5 \xrightarrow{\varepsilon_5} \mathbb{F}\Omega_3 \xrightarrow{\varepsilon_3} \mathbb{F}\Omega_1 \to 0$ is exact except in degree 3, where it has homology $E^{(4,2)}$. By Theorem 1.1 the homology is generated by $v_3 + \operatorname{im} \varepsilon_5$, where $v_3 = \{2,4,6\} + \{1,4,6\} + \{2,3,6\} + \{1,3,6\}$.

The boxes show the kernels of the maps γ_k. For example, by Theorem 1.2(i), $\ker \gamma_2$ is generated by $\{1,2,3\}\gamma_3 = \{1,2\} + \{2,3\} + \{3,1\}$. Since $\ker \varepsilon_2 = \langle X + Y : X, Y \in \Omega_2 \rangle$, the intersection $\ker \gamma_2 \cap \ker \varepsilon_2$ is generated by $\{1,2,3\}\gamma_3 + \{1,2,4\}\gamma_3 = \{1,3\} + \{2,3\} + \{1,4\} + \{2,4\}$; it is isomorphic to the Specht module $S^{(4,2)}$ and has composition factors $D^{(4,2)}$, $\mathbb{F}$, $D^{(5,1)}$. It follows that $\ker \gamma_2$ is not contained in either direct summand of $\mathbb{F}\Omega_2$. The line on the diagram above indicates a 'diagonally embedded' submodule; this submodule is unique if and only if $|\mathbb{F}| = 2$. The dual situation arises for $\ker \gamma_4$ and $\mathbb{F}\Omega_4$.

It is an amusing exercise to show that the outer automorphism of S_6 swaps the simple modules $D^{(4,2)}$ and $D^{(5,1)}$ and leaves $\mathbb{F}\Omega_3$ invariant. In particular, applying it to the homology module $\ker \varepsilon_3 / \operatorname{im} \varepsilon_5 \cong E^{(4,2)}$ gives a non-split extension of $D^{(5,1)}$ by itself.

Remark 1.5. In §2 we show that $\ker \varphi_k^{(1)}$ is isomorphic to the Specht module $S^{(n-k,1^k)}$, by an explicit isomorphism defined on a generator for $\operatorname{im} \varphi_{k+t}^{(t)}$. For small k, there are some interesting isomorphisms between the kernels of the multistep boundary maps and Young modules. For example, it follows from Proposition 5.8 that $\ker \varepsilon_2 \cong Y^{(n-2,2)}$ whenever $n \equiv 2 \bmod 4$; Example 1.4 shows the case $n = 6$. In general, however, $\ker \varphi_k^{(t)}$ appears to have no more explicit description than that given in the main theorems.

The second example shows that (4) may be split exact in cases when the full chain complex (2) containing it fails even to be exact.

Example 1.6. Take $n = 13$. When $t = 4$ and $a = 0$, the chain complex (2) is

$$0 \to \mathbb{F}\Omega_{12} \xrightarrow{\varphi_{12}^{(4)}} \mathbb{F}\Omega_8 \xrightarrow{\varphi_8^{(4)}} \mathbb{F}\Omega_4 \xrightarrow{\varphi_4^{(4)}} \mathbb{F}\Omega_0 \to 0.$$

Since $\binom{13}{4}$ is odd, the trivial module is a direct summand of $\mathbb{F}\Omega_4$; since $\ker \varphi_4^{(4)} = \langle X + Y : X, Y \in \Omega_4 \rangle$, we have $\mathbb{F}\Omega_4 = \ker \varphi_4^{(4)} \oplus \langle \sum_{X \in \Omega_4} X \rangle$. By Theorem 1.2, $\ker \varphi_4^{(4)} = \operatorname{im} \varphi_8^{(4)}$. Therefore $\mathbb{F}\Omega_8 \to \mathbb{F}\Omega_4 \twoheadrightarrow \mathbb{F}\Omega_0$ is split exact. But, again by Theorem 1.2, $\mathbb{F}\Omega_{12} \hookrightarrow \mathbb{F}\Omega_8 \to \mathbb{F}\Omega_4$ is not exact; the proof of Lemma 5.1 shows that the homology module $\ker \varphi_8^{(4)} / \operatorname{im} \varphi_{12}^{(4)}$ has $D^{(8,5)}$ as a composition factor. Calculation shows that in fact it is isomorphic to $D^{(8,5)}$.

Outline. In §2 below we give some further motivation from simplicial homology. This section also collects several results on hook-Specht modules and discusses earlier related work. In §3 we give the logical preliminaries for the proofs of the main theorems. In §4 we prove Theorem 1.1 and in §5 we prove Theorem 1.2. The zero homology modules for the two-step boundary maps are instances of both theorems, but the proofs are independent and involve somewhat different ideas. In §6 we extend the arguments in §5 to prove Theorem 1.3. The final section §7 suggests four directions for future work inspired by Theorems 1.1 and 1.2. In particular Conjectures 7.5 and 7.6 give two attractive binomial identities that would be categorified by an extension of these results to odd characteristic.

2. Background

Exterior powers of the natural permutation module. Suppose that $\mathbb{F}$ has prime characteristic p and let $M = \langle e_1, \ldots, e_n \rangle_{\mathbb{F}}$ be the natural permutation module for $\mathbb{F}S_n$. The $\mathbb{F}S_n$-module $\bigwedge^k M$ has as an $\mathbb{F}$-basis all $(k-1)$-*simplices* $e_{i_1} \wedge \cdots \wedge e_{i_k}$ where $1 \leq i_1 < \cdots < i_k \leq n$. For $k \in \mathbb{N}$, the *boundary map* $\delta_k : \bigwedge^k M \to \bigwedge^{k-1} M$ is defined by

$$(e_{i_1} \wedge \cdots \wedge e_{i_k})\delta = \sum_{\ell=1}^{k} (-1)^{\ell-1} e_{i_1} \wedge \cdots \wedge \widehat{e_{i_\ell}} \wedge \cdots \wedge e_{i_k}$$

where $\widehat{e_{i_\ell}}$ indicates that this factor is omitted. A short calculation shows that $\delta_{k+1}\delta_k = 0$, and so $\operatorname{im}\delta_{k+1} \subseteq \ker \delta_k$, for all k. Thus

$$\bigwedge^n M \xrightarrow{\delta_n} \bigwedge^{n-1} M \xrightarrow{\delta_{n-1}} \cdots \xrightarrow{\delta_3} \bigwedge^2 M \xrightarrow{\delta_2} M \xrightarrow{\delta_1} \mathbb{F} \tag{5}$$

is a chain complex. Given $v \in \ker \delta_k$ a variation on the product rule for derivatives implies that

$$(e_1 \wedge v)\delta_{k+1} = v - e_1 \wedge (v\delta_k) = v, \tag{6}$$

and so (5) is exact. Correspondingly, as is very well known, the solid $(n-1)$-simplex has zero homology in all non-zero dimensions. (Note that the final map $M \xrightarrow{\delta_1} \mathbb{F}$, with domain spanned by the 0-simplices $e_1, \ldots, e_n$, has no geometric interpretation as a boundary map, and so is omitted when computing the geometric homology.) The identity (6) is the algebraic statement of the *suspension trick* showing that an arbitrary cycle $v \in \operatorname{im}\delta_{k+1}$ is a boundary lying in $\ker \delta_k$: see Figure 1 below. We adapt this trick in Lemma 3.6: this lemma is critical to the proof of Theorem 1.1, and is also used in the proof of Theorem 1.2(ii).

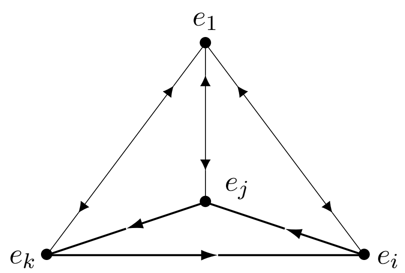


FIGURE 1. Suspension trick: the cycle $e_i \wedge e_j + e_j \wedge e_k + e_k \wedge e_i$ is equal to the boundary $(e_1 \wedge e_i \wedge e_j)\delta_3 + (e_1 \wedge e_j \wedge e_k)\delta_3 + (e_1 \wedge e_k \wedge e_i)\delta_3$.

Let $U = \langle e_i - e_1 : 1 < i \leq n \rangle$. Then U is a submodule of M isomorphic to the Specht module $S^{(n-1,1)}$ and $U = \ker \delta_1$. By (6), it easily follows that $\bigwedge^k U \subseteq \ker \delta_k$ for each k. On the other hand, since

$$(e_{i_1} - e_1) \wedge \cdots \wedge (e_{i_k} - e_1) = (e_1 \wedge e_{i_1} \wedge \cdots e_{i_k})\delta_{k+1} \in \operatorname{im}\delta_{k+1}$$

we have $\bigwedge^k U \supseteq \operatorname{im}\delta_{k+1}$. By exactness we deduce that $\bigwedge^k U = \ker \delta_k$. If p does not divide n then $M = U \oplus \langle e_1 + \cdots + e_n \rangle$ and so $\bigwedge^k M \cong \bigwedge^k U \oplus \bigwedge^{k-1} U \cong \ker \delta_k \oplus \operatorname{im}\delta_k$ and (5) is split exact.

To motivate a key step in the proofs of Theorems 1.2 and Theorem 1.3, we sketch an alternative proof of this decomposition, related to the suspension trick. For $k \in \mathbb{N}$, define $f_k : \bigwedge^{k-1} M \to \bigwedge^k M$ by $(e_{i_1} \wedge \cdots \wedge e_{i_{k-1}})f_k = e_1 \wedge e_{i_1} \wedge \cdots \wedge e_{i_{k-1}}$. Then $\delta_k f_k + f_{k+1}\delta_{k+1} = \mathrm{id}$ for each k. Hence the maps f_k define a chain homotopy between (5) and the zero complex. As it stands, f_k is not an $\mathbb{F}S_n$-homomorphism, but replacing f_k with the symmetrized map F_k defined by $(e_{i_1} \wedge \cdots \wedge e_{i_{k-1}})F_k = (e_1 + \cdots + e_n) \wedge (e_{i_1} \wedge \cdots \wedge e_{i_{k-1}})$, we get

$$\delta_k F_k + F_{k+1}\delta_{k+1} = n\,\mathrm{id}. \tag{7}$$

Since $F_k F_{k+1} = 0$, a basic argument from homotopy theory, which we repeat in the proof of Proposition 5.8, shows that if p does not divide n then $\bigwedge^k M = \mathrm{im}\, F_k \oplus \mathrm{im}\, \delta_{k+1}$ for every k and so (5) is split exact.

There is a canonical isomorphism

$$\ker \delta_k \cong S^{(n-k,1^k)} \tag{8}$$

first constructed by Hamernik [11] in the case $n = p$ and Peel [20, Proposition 2] in general. (For the definition of Specht modules and polytabloids see [16, Ch. 4].) The isomorphism is defined by sending $(e_{i_1} - e_1) \wedge \cdots \wedge (e_{i_k} - e_1)$ to the polytabloid e_t where t is the unique standard $(n-k, 1^k)$-tableau having first column entries $1, i_1, \ldots, i_r$. By the Standard Basis Theorem (see [16, Corollary 8.5]), this defines a linear isomorphism. It follows easily from the definition of polytabloids that it commutes with the permutations fixing 1; a short calculation with Garnir relations (see [18, Proposition 2.3] or [8, Proposition 5.1(b)]) shows that it commutes with $(1, 2)$.

The following result completely determines the structure of $\bigwedge^k M$ when p is odd. It was proved in the author's D. Phil thesis [22, §1.3] using the ideas in Hamernik [11], Peel [20] and James [16, Theorem 24.1].

Proposition 2.1. *Let p be odd. We have $\bigwedge^0 M \cong \mathbb{F}$ and $\bigwedge^n M \cong \mathrm{sgn}$.*

(i) *If p does not divide n and $k \in \{1, \ldots, n-1\}$ then $S^{(n-k,1^k)}$ is simple and $\bigwedge^k M \cong S^{(n-k,1^k)} \oplus S^{(n-k-1,1^{k-1})}$ is semisimple.*

(ii) *Suppose p divides n. Let $D = U/\langle e_1 + \cdots + e_n \rangle$ and let D_k denote $\bigwedge^k D$. Then D_k is simple and there is a non-split exact sequence $D_{k-1} \hookrightarrow S^{(n-k,1^k)} \twoheadrightarrow D_k$ for each $k \in \{1, \ldots, n-2\}$. For $k \in \{1, \ldots, n-1\}$, each $\bigwedge^k M$ is indecomposable with Loewy layers*

$$\begin{matrix} D_{k-1} \\ D_{k-2} \oplus D_k \\ D_{k-1} \end{matrix},$$

where D_{-1} and D_{n-1} should be ignored when $k = 1$ or $k = n-1$.

A corollary of this proposition, which may easily be proved directly by considering possible images of the generator $e_1 \wedge \cdots \wedge e_k$ of $\bigwedge^k M$, is that if p is odd and $|k - \ell| \geq 2$ then $\mathrm{Hom}_{\mathbb{F}S_n}(\bigwedge^k M, \bigwedge^\ell M) = 0$. This rules out a generalization to odd characteristic of the main theorems in which $\mathbb{F}\Omega_k$ is replaced with $\bigwedge^k M$. At the end of §7 we propose an alternative generalization.

Other related work. The maps $\varphi_k^{(t)}$ are critical to James' proof [15] of the decomposition numbers for Specht modules labelled by two-row partitions. (In [15], our map $\varphi_k^{(t)}$ is denoted ϑ_{k-t}^k.) James' Lemma 2.7 gives an inductive construction of generators for the module $\bigcap_{t=k-r}^k \ker \varphi_k^{(t)}$; his Lemma 3.6 shows that the intersection is the same when taken only over those t of the form 2^τ. James' Lemma 3.5 states that $\ker \varphi^{(k)}_{s+t}$ contains $\ker \varphi^{(k)}_s$ if and only if $\binom{s+t}{s}$ is odd; we adapt his proof to prove the related Proposition 5.3 below. The example following James' Lemma 2.7 describes some of the submodules in our Example 1.4. Later in [16, Chapter 17, 24], James revisited these ideas. His Theorem 17.13(i) implies that $\{2, 4, \ldots, 2k\} \sum_{\sigma \in G_\ell} \sigma$ generates the kernel of $\varphi^{(k-\ell+1)}_k$ *when* this map is restricted to the submodule of $\mathbb{F}\Omega_k$ generated by $\{2, 4, \ldots, 2k\} \sum_{\sigma \in G_{\ell-1}} \sigma$. (The full kernel is in general larger.) In particular, taking $\ell = k - 1$ shows that $v_k \in \ker \varepsilon_k$. Part of our Theorem 1.1 gives the stronger result that $v_k + \operatorname{im} \varepsilon_k$ generates the homology module $\ker \varepsilon_k / \operatorname{im} \varepsilon_{k+2}$; the proof uses somewhat different ideas to James. Conjecture 7.2 proposes a generalization of this result.

In [12], Henke determined the multiplicities of two-row Young modules in the two-row Young permutation modules (isomorphic to the $\mathbb{F}\Omega_k$) working in arbitrary characteristic. In [7], Doty, Erdmann and Henke used the Schur algebra in characteristic 2 to give an explicit construction of the primitive idempotents in $\operatorname{End}_{\mathbb{F}S_n}(\mathbb{F}\Omega_k)$. When (2) is split exact, each $\ker \varphi^{(t)}_k$ is a direct sum of Young modules, and the projection $\mathbb{F}\Omega_k \to \ker \varphi^{(t)}_k$ is the sum of the relevant idempotents. For instance, in Example 1.4, $\ker \varepsilon_4 \cong Y^{(6)}$ and $\ker \varepsilon_2 \cong Y^{(4,2)}$. In general multiple idempotents are required. For example, take $\tau \in \mathbb{N}_0$, $t = 2^\tau$, $k = 2^{\tau+1}$ and $n = (3 + 4r)2^\tau$ with $r \in \mathbb{N}$. By Theorem 1.3, $\ker \varphi^{(t)}_k$ is a direct summand of $\mathbb{F}\Omega_k$; an argument similar to Example 1.6 shows that the trivial module is a proper direct summand of $\ker \varphi^{(t)}_k$.

Earlier, in [19], Murphy proved a number of results on the endomorphism ring of $\ker \varphi^{(1)}_k \cong S^{(n-k,1^k)}$ when $p = 2$ and used them to determine when this hook-Specht module is decomposable. When n is odd an alternative proof of her criterion can be given using the results in [12], starting from the observation that $S^{(n-r,1^r)}$ is a direct summand of $\mathbb{F}\Omega_k$ containing $S^{(n-r,r)}$, and so is a direct sum of Young modules including $Y^{(n-r,r)}$.

The results on the restricted modules $D^{(m+1,m)}{\downarrow_{S_{2m}}}$ and $D^{(m+1,m-1)}{\downarrow_{S_{2m-1}}}$ in Theorem 1.1 were proved by Danz and Külshammer in [6, Proposition 3.3]; the authors' proof uses Kleshchev's very deep modular branching rule [17, Theorem 11.2.10]. The explicit construction of $D^{(m+1,m-1)}$ in [6], attributed to Uno, also implies these results. The proof here is self-contained and inductive. The generator for $D^{(m+1,m)}$ in Theorem 1.1 was first found by Benson (with a different description of the quotient module) in [3, Lemma 5.4].

Finally we note that there is an extensive theory of resolutions of (dual) Specht modules by Young permutation modules, beginning with [4]; the authors' conjectured resolution was proved to be exact in [21] using the Schur algebra. Even in the two-row case, the terms in these resolutions are sums of multiple Young permutation modules. Thus they do not appear to be closely connected to this work.

3. Preliminary results

From now until the final part of §7, let $\mathbb{F}$ be a field of characteristic 2.

Duality. Each $\mathbb{F}\Omega_r$ is isomorphic to its dual module $\mathbb{F}\Omega_r^\star$ by a canonical isomorphism sending $X \in \Omega_r$ to the corresponding element $X^\star$ of the dual basis of $\mathbb{F}\Omega_r^\star$. Under this identification, $\varphi_r^{(t)} : \mathbb{F}\Omega_r \rightarrow \mathbb{F}\Omega_{r-t}$ becomes the map $\varphi_r^{(t)\star} : \mathbb{F}\Omega_{r-t} \rightarrow \mathbb{F}\Omega_r$ defined by

$$Y\varphi_r^{(t)\star} = \sum_{\substack{Z \supseteq Y \\ |Z|=|Y|+t}} Z \tag{9}$$

for $Y \in \Omega_{r-t}$. (Note that the domain of $\varphi_r^{(t)\star}$ is defined to be $\mathbb{F}\Omega_{r-t}$, not $\mathbb{F}\Omega_r$ or $\mathbb{F}\Omega_{r-t}^\star$.) This duality explains the symmetry in the inequalities in Theorem 1.2.

Proposition 3.1.

(i) *For each r there is an isomorphism $\mathbb{F}\Omega_r \cong \mathbb{F}\Omega_{n-r}$.*
(ii) *The homology of*

$$\mathbb{F}\Omega_{k+t} \xrightarrow{\varphi_{k+t}^{(t)}} \mathbb{F}\Omega_k \xrightarrow{\varphi_k^{(t)}} \mathbb{F}\Omega_{k-t}$$

is dual to the homology of

$$\mathbb{F}\Omega_{n-k+t} \xrightarrow{\varphi_{n-k+t}^{(t)}} \mathbb{F}\Omega_{n-k} \xrightarrow{\varphi_{n-k}^{(t)}} \mathbb{F}\Omega_{n-k-t}.$$

Proof. Dualising the first sequence we obtain $\mathbb{F}\Omega_{k-t} \xrightarrow{\varphi_k^{(t)\star}} \mathbb{F}\Omega_k \xrightarrow{\varphi_{k+t}^{(t)\star}} \mathbb{F}\Omega_{k+t}$. Each $\mathbb{F}\Omega_r$ is isomorphic to $\mathbb{F}\Omega_{n-r}$ by the map sending each $Y \in \Omega_r$ to its complement $\{1,\ldots,n\}\backslash Y \in \Omega_{n-r}$. Applying this isomorphism we obtain the second sequence. In particular, the homology modules are dual. $\square$

Specht modules, Young permutation modules, simple modules. The Specht module S^λ canonically labelled by the partition λ of n is defined in [16, Ch. 4] as a submodule of the Young permutation module M^λ. There is a well-known canonical isomorphism $M^{(n-k,k)} \cong \mathbb{F}\Omega_k$ defined by sending a tabloid of shape $(n-k,k)$ to the set of entries in its bottom row. Let t be the $(n-k,k)$-tableau having $2,4,\ldots,2k$ in its bottom row. Then the corresponding polytabloid e_t generates $S^{(n-k,k)}$ and

$$e_t \mapsto \{2,4,\ldots,2k\} \sum_{\sigma \in G_k} \sigma. \tag{10}$$

The simple modules for $\mathbb{F}S_n$ are defined in [16, Theorem 11.5] as the top composition factors of certain Specht modules. For $2k < n$, let $D^{(n-k,k)}$ denote the simple $\mathbb{F}S_n$-module canonically labelled by the two-row partition $(n-k,k)$. We allow partitions to have zero parts: thus $D^{(n,0)}$ is the trivial $\mathbb{F}S_n$-module. By [16, Theorem 11.5] each simple $\mathbb{F}S_n$-module is self-dual.

Lemma 3.2.

(i) *If* $2k < n$ *then* $\mathbb{F}\Omega_k$ *has a composition series with factors* $D^{(n-r,r)}$ *for* $r \leq k$ *in which* $D^{(n-k,k)}$ *appears exactly once.*

(ii) *If* $n = 2m$ *then* $\mathbb{F}\Omega_m$ *has a composition series with factors* $D^{(2m-r,r)}$ *for* $r < m$.

(iii) *If* $n = 2m$ *then* $D^{(m+1,m-1)}$ *is a composition factor of* $\mathbb{F}\Omega_k$ *if and only if* $k = m-1$, $k = m$ *or* $k = m+1$.

(iv) *Let* $2k < n$ *and let* $2r < n-1$. *If* $D^{(n-1-r,r)}$ *is a composition factor of* $D^{(n-k,k)}\downarrow_{S_{n-1}}$ *then* $k \geq r$.

Proof. Parts (i) and (ii) are special cases of Theorem 12.1 in [16]. Using Proposition 3.1(i) to reduce to the case $2k \leq n$, part (iii) also follows from this theorem. The hypothesis for (iv) implies that $D^{(n-1-r,r)}$ appears in

$$\mathbb{F}\Omega_k\big\downarrow_{S_{n-1}} \cong \mathbb{F}\Omega_k^{[n-1]} \oplus \mathbb{F}\Omega_{k-1}^{[n-1]},$$

where each bracketed $n-1$ indicates that the summand is a module for $\mathbb{F}S_{n-1}$. By (i) and (ii) we deduce that $k \geq r$. □

The following consequence of Lemma 3.2 is used in both §4 and §5.

Proposition 3.3. *Let* $n \in \mathbb{N}$.

(i) *If* $n = 2m$ *then* $\mathbb{F}\Omega_m$ *has exactly two composition factors isomorphic to* $D^{(m+1,m-1)}$.

(ii) *If* $n = 2m+1$ *then* $\mathbb{F}\Omega_m$ *and* $\mathbb{F}\Omega_{m+1}$ *are isomorphic and each has a unique composition factor isomorphic to* $D^{(m+1,m)}$.

Proof. Recall that γ_k denotes $\varphi_k^{(1)}$. We use the one-step sequence

$$0 \to \mathbb{F}\Omega_n \xrightarrow{\gamma_n} \mathbb{F}\Omega_{n-1} \xrightarrow{\gamma_{n-1}} \cdots \xrightarrow{\gamma_2} \mathbb{F}\Omega_1 \xrightarrow{\gamma_1} \mathbb{F}\Omega_0 \to 0.$$

As seen after (5), this sequence is exact. If $n = 2m$ then, by Proposition 3.1(i) and Lemma 3.2(i), the isomorphic modules $\mathbb{F}\Omega_{m-1}$ and $\mathbb{F}\Omega_{m+1}$ each have $D^{(m+1,m-1)}$ as a composition factor. By Lemma 3.2(iii), $D^{(m+1,m-1)}$ is not a composition factor of $\mathbb{F}\Omega_{m-2} \cong \mathbb{F}\Omega_{m+2}$. Therefore $D^{(m+1,m-1)}$ must appear twice in $\mathbb{F}\Omega_m$. The proof is similar when $n = 2m+1$. □

Composing multistep maps. We need a generalization of the result $\varphi_k^{(t)}\varphi_{k-t}^{(t)} = 0$ proved in the introduction. Given s, $t \in \mathbb{N}_0$, we say that the *addition of* s *to* t *is carry free* if $\binom{s+t}{s}$ is odd. Abusing notation slightly, we may abbreviate this to '$s+t$ is carry free'. As motivation, we recall that if $s = \sum_{i=0}^{c} s_i 2^i$ and $t = \sum_{i=0}^{c} t_i 2^i$ where s_i, $t_i \in \{0,1\}$ for each i, then $s+t$ is carry free if and only if $s_i + t_i \leq 1$ for all i, and so s and t can be added in binary without carries. (This follows immediately from Lucas' Theorem: see for instance [16, Lemma 22.4].)

Lemma 3.4. *If* s, $t \in \mathbb{N}$ *then*

$$\varphi_k^{(s)}\varphi_{k-s}^{(t)} = \begin{cases} \varphi_k^{(s+t)} & \text{if the addition of } s \text{ to } t \text{ is carry free} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The argument in the introduction shows that $\varphi_k^{(s)}\varphi_{k-s}^{(t)} = \binom{s+t}{s}\varphi_k^{(s+t)}$. The lemma now follows from the definition of carry free. $\square$

Products of sets. Define the *support* of $v \in \mathbb{F}\Omega_k$ to be the union of the k-subsets that appear in v with a non-zero coefficient. The vector space $\bigoplus_{k=0}^n \mathbb{F}\Omega_k$ becomes a graded algebra with product defined by bilinear extension of

$$X \cdot Y = \begin{cases} X \cup Y & \text{if } X \cap Y = \varnothing \\ 0 & \text{otherwise.} \end{cases}$$

for $X \in \Omega_k$ and $Y \in \Omega_\ell$. We denote this product by concatenation. Except in the warning example following Lemma 3.5, we only take the product of $v \in \mathbb{F}\Omega_k$ and $w \in \mathbb{F}\Omega_\ell$ when v and w have disjoint support.

The Splitting Rule and the Suspension Lemma. The product rule for derivatives has the following analogue for the multistep boundary maps.

Lemma 3.5 (Splitting Rule)**.** *Let $v \in \mathbb{F}\Omega_k$ and let $w \in \mathbb{F}\Omega_\ell$. If v and w have disjoint support then*

$$(vw)\varphi_{k+\ell}^{(t)} = \sum_{s=0}^{t} (v\varphi_k^{(s)})(w\varphi_\ell^{(t-s)}).$$

Proof. By bilinearity of the product $\mathbb{F}\Omega_\ell \times \mathbb{F}\Omega_m \to \mathbb{F}\Omega_{k+\ell}$, it suffices to prove the lemma in the special case when v is an k-subset X and w is a disjoint ℓ-subset Y. It then holds since every $(k+\ell-t)$-subset Z of $X \cup Y$ splits uniquely as a union $(Z \cap X) \cup (Z \cap Y)$ of a subset of X and a subset of Y. $\square$

When $t > 1$ the assumption in Lemma 3.5 that v and w have disjoint support is essential. For example $(\{1,2\}\{2\})\varepsilon_2 = 0\varepsilon_2 = 0$, but $(\{1,2\}\varepsilon_2)\{2\} + (\{1,2\}\gamma_1)(\{2\}\gamma_1) + \{1,2\}(\{2\}\varepsilon_2) = \varnothing\{2\} + (\{1\}+\{2\})\varnothing = \{1\}$.

The following lemma is the analogue of (6) in §2.

Lemma 3.6 (Suspension Lemma)**.** *Let $t \in \mathbb{N}$ and let $0 \le \ell < t$. Let $v \in \mathbb{F}\Omega_k$. Suppose that $v \in \ker \varphi_k^{(s)}$ whenever $\ell < s \le t$ and that the support of v is disjoint from $X \in \Omega_{\ell+t}$. If the addition of ℓ to t is carry free and the addition of ℓ to $t-s$ is not carry free when $0 < s \le \ell$ then*

$$v = \big(v(X\varphi_{\ell+t}^{(\ell)})\big)\varphi_{k+t}^{(t)}.$$

Proof. By the Splitting Rule the right-hand side is

$$\sum_{s=0}^{t} (v\varphi^{(s)})(X\varphi^{(\ell)}\varphi^{(t-s)}). \tag{11}$$

(Here, and in the remainder of the proof, we omit the degrees of the maps to increase readability.) By hypothesis $v\varphi^{(s)} = 0$ if $\ell < s \le t$. When $0 < s \le \ell$ the addition of ℓ to $t-s$ is not carry free, again by hypothesis. Therefore, by Lemma 3.4, we have $X\varphi^{(\ell)}\varphi^{(t-s)} = 0$ for all such s. The only remaining summand in (11) occurs when $s = 0$, in which case another application of Lemma 3.4 shows that $v(X\varphi^{(\ell)}\varphi^{(t)}) = v\varnothing = v$. $\square$

For example, take $t = 2^\tau$ where $\tau \in \mathbb{N}_0$ and take $k < 2^\tau$. Then $k + 2^\tau$ is carry free, and if $0 < s \le k$ then $k + (2^\tau - s)$, is clearly not carry free, since it has 2^τ in its binary form. The sets $v = \{n-k+1,\ldots,n\}$ and $X = \{1,\ldots,k+2^\tau\}$ are disjoint whenever $n - k \ge k + 2^\tau$. Hence the hypotheses of the Suspension Lemma hold provided $n \ge 2k + 2^\tau$ and we get

$$\{n-k+1,\ldots,n\} = \big(\{n-k+1,\ldots,n\}(\{1,\ldots,k+2^\tau\}\varphi^{(k)}_{k+2^\tau})\big)\varphi^{(2^\tau)}_{k+2^\tau}.$$

Therefore $\varphi^{(2^\tau)}_{k+2^\tau} : \mathbb{F}\Omega_{k+2^\tau} \to \mathbb{F}\Omega_k$ is surjective. We use a small generalization this argument in the proof of part of Theorem 1.2(ii).

4. Two-step homology: proof of Theorem 1.1

Recall that $H_k = \ker \varepsilon_k / \operatorname{im} \varepsilon_{k+2}$. The outline of the proof is as follows: in Lemmas 4.1, 4.2 and 4.3 and Proposition 4.4 we show that $v_k + \operatorname{im}\varepsilon_{k+2}$ generates H_k. Using that v_k is supported on a set of size $2k-1$, it follows from the Suspension Lemma that $H_k = 0$ when $n \ge 2k+2$. By duality we get the same result when $n \le 2k-2$. We then identify the composition factors responsible for the non-zero homology modules, and find their structure by induction on n. Thus a large part of the proof is to show that $\ker\varepsilon_k$ has a generator of 'small' support: as motivation note that, conversely, if $\ker\varepsilon_k = \operatorname{im}\varepsilon_{k+2}$, then $\ker\varepsilon_k$ has a generator supported on $\{1,\ldots,k+2\}$.

Throughout γ_k denotes $\varphi^{(1)}_k$ and ε_k denotes $\varphi^{(2)}_k$.

Lemma 4.1. *Let $2 \le k \le n-2$. The homology module H_k is generated, as an $\mathbb{F}S_n$-module, by all $\{n\}v + \{n-1,n\}(v\gamma_{k-1}) + \operatorname{im}\varepsilon_{k+2}$ where $v \in \mathbb{F}\Omega_{k-1}$ has support disjoint from $\{n-1,n\}$ and satisfies $v\varepsilon_{k-1} = 0$.*

Proof. Given any $X \in \mathbb{F}\Omega_k$ with support disjoint from $\{n-1,n\}$, the Splitting Rule implies that

$$X = \big(\{n-1,n\}X\big)\varepsilon_{k+2} + \{n-1\}(X\gamma_k) + \{n\}(X\gamma_k) + \{n-1,n\}(X\varepsilon_k).$$

Since the first summand lies in $\operatorname{im}\varepsilon_{k+2}$, and X generates $\mathbb{F}\Omega_k$ as an $\mathbb{F}S_n$-module, it follows that $\mathbb{F}\Omega_k/\operatorname{im}\varepsilon_{k+2}$ is generated by all $\{n-1\}u + \{n\}v + \{n-1,n\}w + \operatorname{im}\varepsilon_{k+2}$ where $u \in \mathbb{F}\Omega_{k-1}$, $v \in \mathbb{F}\Omega_{k-1}$ and $w \in \mathbb{F}\Omega_{k-2}$ have support disjoint from $\{n-1,n\}$. Now, omitting indices on the maps for readability, we have

$$\begin{aligned}&(\{n-1\}u + \{n\}v + \{n-1,n\}w)\varepsilon\\ &\quad= (u\gamma + v\gamma + w) + \{n-1\}(u\varepsilon + w\gamma) + \{n\}(v\varepsilon + w\gamma) + \{n-1,n\}(w\varepsilon).\end{aligned}$$

The right-hand side is zero if and only if $u\gamma + v\gamma = w$, $u\varepsilon = v\varepsilon = w\gamma$ and $w\varepsilon = 0$. The first equation implies that $w \in \operatorname{im}\gamma$, and so $w\gamma = 0$; hence the three equations are equivalent to $u\gamma + v\gamma = w$ and $u\varepsilon = v\varepsilon = 0$. Thus H_k is generated by all

$$\{n-1\}u + \{n\}v + \{n-1,n\}(u\gamma + v\gamma) + \operatorname{im}\varepsilon_k$$

such that $u\varepsilon = v\varepsilon = 0$. Applying the transposition $(n-1,n)$ to $\{n\}v + \{n-1,n\}v\gamma$, we see that H_k is generated by elements of the required form. $\square$

Lemma 4.2. *If $2k \leq n$ then $v_k\gamma_k = \{2,4,\ldots,2(k-1)\}\sum_{\sigma\in G_{k-1}}\sigma$.*

Proof. Let w_k denote the right-hand side. We have

$$\begin{aligned} v_k\gamma_k &= \sum_{\sigma\in G_{k-1}} \{2,4,\ldots,2(k-1),2k\}\sigma\gamma_k \\ &= \sum_{\sigma\in G_{k-1}}\sum_{j=1}^{k-1}\{2,4,\ldots,2(k-1),2k\}\sigma\setminus\{(2j)\sigma\} + w_k. \end{aligned}$$

For each fixed j, the summands for σ and $\sigma(2j-1,2j)$ are equal, and so cancel. Therefore $v_k\gamma = w_k$, as required. □

Lemma 4.3. *If $v \in \ker\varepsilon_k$ has support of size at most $n-3$ then $v \in \operatorname{im}\varepsilon_{k+2}$.*

Proof. By hypothesis, there is a 3-subset Z of $\{1,\ldots,n\}$ disjoint from the support of v. By the argument seen in the example following the Suspension Lemma (Lemma 3.6), we have

$$\big(v(Z\gamma_3)\big)\varepsilon_{k+2} = v.$$

Therefore $v \in \operatorname{im}\varepsilon_{k+2}$ as required. □

Proposition 4.4. *Let $k \in \mathbb{N}_0$. If $2k \leq n$ then H_k is generated by $v_k + \operatorname{im}\varepsilon_{k+2}$.*

Proof. We work by induction on n dealing with all admissible k at once. The inductive step below is effective when $k \geq 2$ and $k+6 \leq n$. Since $v_0 = \varnothing$ and $v_1 = \{2\}$ generate $\mathbb{F}\Omega_0$ and $\mathbb{F}\Omega_1$, respectively, the result holds if $k < 2$. When $k = 2$, Lemma 4.1 implies that H_2 is generated by all $\{n\}\{j\} + \{n-1,n\} + \operatorname{im}\varepsilon_4$, where $j \in \{1,\ldots,n-2\}$. Therefore H_2 is generated by $v_2 = \{2,4\} + \{1,4\} + \operatorname{im}\varepsilon_4$ as required. When $k = 3$ and $n \in \{6,7,8\}$, or $k = 4$ and $n \in \{8,9\}$, or $k = 5$ and $n = 10$ the proposition has been checked using the computer algebra package Magma.[1]

For the inductive step we may suppose, by the previous paragraph, that $k \geq 2$ and $k+6 \leq n$. By Lemma 4.1, H_k is generated by the elements $\{n\}v + \{n-1,n\}(v\gamma_{k-1})$ for $v \in V$, where $V = \ker\varepsilon^{[n-2]}_{k-1} : \mathbb{F}\Omega^{[n-2]}_{k-1} \to \mathbb{F}\Omega^{[n-2]}_{k-3}$. (The bracketed $n-2$ emphasises that these are modules and module homomorphisms for $\mathbb{F}S_{n-2}$.) The map $\varepsilon^{[n-2]}_{k-1}$ is part of the sequence

$$\mathbb{F}\Omega^{[n-2]}_{k+1} \xrightarrow{\varepsilon^{[n-2]}_{k+1}} \mathbb{F}\Omega^{[n-2]}_{k-1} \xrightarrow{\varepsilon^{[n-2]}_{k-1}} \mathbb{F}\Omega^{[n-2]}_{k-3}.$$

Observe that $H^{[n-2]}_{k-1} = V/\operatorname{im}\varepsilon^{[n-2]}_{k+1}$. Since $2(k-1) \leq n-2$, the inductive hypothesis for $n-2$ implies that $V/\operatorname{im}\varepsilon^{[n-2]}_{k+1}$ is generated by $v_{k-1} + \operatorname{im}\varepsilon^{[n-2]}_{k+1}$. Since $\operatorname{im}\varepsilon^{[n-2]}_{k+1}$ is generated by $Y\varepsilon_{k+1}$, where $Y = \{1,\ldots,k+1\}$, it follows that H_k is generated by $\{n\}v_{k-1} + \{n-1,n\}(v_{k-1}\gamma_{k-1}) + \operatorname{im}\varepsilon_{k+2}$ together with $u + \operatorname{im}\varepsilon_{k+2}$, where

$$u = \{n\}(Y\varepsilon_{k+1}) + \{n-1,n\}(Y\varepsilon_{k+1}\gamma_{k-1}).$$

[1] Magma code for constructing the $\varphi^{(t)}_k$ homomorphisms and verifying these claims may be downloaded from the author's webpage: `www.rhul.ac.uk/~uvah099/`.

The support of u is $\{1,\dots,k+1\}\cup\{n-1,n\}$, of size $k+3$. Since $k+6\le n$, Lemma 4.3 implies that $u \in \operatorname{im}\varepsilon_{k+2}$.

The first summand in the other generator $\{n\}v_{k-1}+\{n-1,n\}(v_{k-1}\gamma_{k-1})+\operatorname{im}\varepsilon_{k+2}$ is $\sum_{\sigma\in G_{k-2}}\bigl(\{2,4,\dots,2(k-2)\}\sigma\cup\{2(k-1),n\}\bigr)$, and, by Lemma 4.2, the second summand is $\sum_{\sigma\in G_{k-2}}\bigl(\{2,4,\dots,2(k-2)\}\sigma\cup\{n-1,n\}\bigr)$. Relabelling so that $n-1$ becomes $2(k-1)-1$ and n becomes $2k$, their sum becomes v_k. Therefore $v_k+\operatorname{im}\varepsilon_{k+2}$ generates H_k. □

Corollary 4.5. *If $2k+2\le n$ then $H_k=0$.*

Proof. By Proposition 4.4, H_k is generated by $v_k+\operatorname{im}\varepsilon_{k+2}$. The support of v_k is $\{1,\dots,2k-2,2k\}$, of size $2k-1$. Since $2k+2\le n$, it follows from Lemma 4.3 that $v_k\in\operatorname{im}\varepsilon_{k+2}$. Hence $H_k=0$. □

By the duality in Proposition 3.1(i) we may assume that $2k\le n$. Therefore the previous corollary determines all the homology modules H_k except when $k=m$ and either $n=2m$ or $n=2m+1$. In these cases the non-zero homology reflects the obstruction to exactness identified in Proposition 3.3.

To complete the proof of Theorem 1.1 we show, by induction on n, that when $n=2m$ or $n=2m+1$, the module H_m is as claimed. The base case is $n=1$, in which case the chain complex $\mathbb{F}\Omega_2\to\mathbb{F}\Omega_0\to\mathbb{F}\Omega_{-2}$ has two zero modules and homology $H_0=\mathbb{F}\Omega_0\cong\mathbb{F}\cong D^{(1,0)}$, generated by $v_1=\varnothing$.

Inductive step even to odd. Suppose that $n=2m+1$ so $n-1=2m$. The restriction of the sequence $\mathbb{F}\Omega_{m+2}\xrightarrow{\varepsilon_{m+2}}\mathbb{F}\Omega_m\xrightarrow{\varepsilon_m}\mathbb{F}\Omega_{m-2}$ to S_{2m} is the direct sum of

$$\mathbb{F}\Omega^{[2m]}_{m+2}\xrightarrow{\varepsilon_{m+2}}\mathbb{F}\Omega^{[2m]}_{m}\xrightarrow{\varepsilon_{m}}\mathbb{F}\Omega^{[2m]}_{m-2},$$
$$\mathbb{F}\Omega^{[2m]}_{m+1}\xrightarrow{\varepsilon_{m+1}}\mathbb{F}\Omega^{[2m]}_{m-1}\xrightarrow{\varepsilon_{m-1}}\mathbb{F}\Omega^{[2m]}_{m-3}.$$

(For readability, and since the distinction is no longer so vital, we omit the $[2m]$ label on the two-step boundary maps.) By induction the second sequence is exact. Again by induction, the first has non-zero homology $E^{(m+1,m-1)}$ in degree m. Therefore

$$H_m\big\downarrow_{S_{2m}}\cong\frac{D^{(m+1,m-1)}}{D^{(m+1,m-1)}}.$$

By Lemma 3.2(iv), the two-row simple modules for $\mathbb{F}S_{2m+1}$ whose restrictions to S_{2m} may have $D^{(m+1,m-1)}$ as a composition factor are $D^{(m+1,m)}$ and $D^{(m+2,m-1)}$. By Proposition 3.3(ii), $D^{(m+1,m)}$ appears exactly once in H_m. By Nakayama's Conjecture (see [13, 6.1.21]), $D^{(m+2,m-1)}$ is in a different block to $D^{(m+1,m)}$. Since $H_m\downarrow_{S_{2m}}$ is indecomposable, we have $H_m\cong D^{(m+1,m)}$ as required. By Proposition 4.4, H_m is generated by $v_m+\operatorname{im}\varepsilon_{m+2}$.

Inductive step odd to even. Suppose that $n = 2m$ so $n - 1 = 2m - 1$. The restriction of the sequence $\mathbb{F}\Omega_{m+2} \to \mathbb{F}\Omega_m \to \mathbb{F}\Omega_{m-2}$ to S_{2m} is the direct sum of

$$\begin{aligned} \mathbb{F}\Omega_{m+2}^{[2m-1]} &\xrightarrow{\varepsilon_{m+2}} \mathbb{F}\Omega_{m}^{[2m-1]} \xrightarrow{\varepsilon_{m}} \mathbb{F}\Omega_{m-1}^{[2m-1]}, \\ \mathbb{F}\Omega_{m+1}^{[2m-1]} &\xrightarrow{\varepsilon_{m+1}} \mathbb{F}\Omega_{m-1}^{[2m-1]} \xrightarrow{\varepsilon_{m-1}} \mathbb{F}\Omega_{m-3}^{[2m-1]}. \end{aligned}$$

By Proposition 3.1 these sequences are dual to one another. By induction, each has homology $D^{(m,m-1)}$. Hence

$$H_m\big\downarrow_{S_{2m-1}} \cong D^{(m,m-1)} \oplus D^{(m,m-1)}.$$

By Lemma 3.2(iv), the only two-row simple module for $\mathbb{F}S_{2m}$ whose restriction to S_{2m-1} may have $D^{(m,m-1)}$ as a composition factor is $D^{(m+1,m-1)}$. By Proposition 3.3(i), $D^{(m+1,m-1)}$ appears exactly twice in H_m. Hence either $H_m \cong D^{(m+1,m-1)} \oplus D^{(m+1,m-1)}$ or H_m is a non-split extension of $D^{(m+1,m-1)}$ by itself. By Proposition 4.4, H_m is generated by $v_m + \operatorname{im}\varepsilon_{m+2}$. Therefore H_m is cyclic. Since the direct sum of two non-zero isomorphic modules is not cyclic, it follows that H_m is a non-split extension, as required.

This completes the proof of Theorem 1.1. As a corollary we get a new proof that $\dim D^{(m+1,m-1)} = 2^{m-1}$ and $\dim D^{(m+1,m)} = 2^m$. For this we need the binomial identity

$$\sum_j (-1)^j \binom{2m+1}{2j} = \begin{cases} (-1)^{m/2}2^m & \text{if } m \text{ is even} \\ (-1)^{(m+1)/2}2^m & \text{if } m \text{ is odd,} \end{cases} \tag{12}$$

which is most easily proved by taking real parts in

$$2^m\mathrm{i}^m + 2^m\mathrm{i}^{m+1} = (1+\mathrm{i})^{2m+1} = \sum_j (-1)^j \binom{2m+1}{2j} + \mathrm{i}\sum_j (-1)^j \binom{2m+1}{2j+1}.$$

Corollary 4.6. *We have* $\dim D^{(m+1,m-1)} = 2^{m-1}$ *and* $\dim D^{(m+1,m)} = 2^m$.

Proof. By part of Theorem 1.1, we have $D^{(m+1,m-1)}\big\downarrow_{S_{2m-1}} \cong D^{(m,m-1)}$. It therefore suffices to prove the second claim. Suppose that m is even. Consider the chain complex of $\mathbb{F}S_{2m+1}$-modules

$$0 \to \mathbb{F}\Omega_{2m} \xrightarrow{\varepsilon_{2m}} \cdots \xrightarrow{\varepsilon_{m+4}} \mathbb{F}\Omega_{m+2} \xrightarrow{\varepsilon_{m+2}} \mathbb{F}\Omega_m \xrightarrow{\varepsilon_m} \mathbb{F}\Omega_{m-2} \xrightarrow{\varepsilon_{m-2}} \cdots \xrightarrow{\varepsilon_2} \mathbb{F}\Omega_0 \to 0.$$

By Theorem 1.1 this chain complex has non-zero homology uniquely in degree m, where $H_m \cong D^{(m+1,m)}$. The alternating sum of the dimensions of the modules in a chain complex agrees with the alternating sum of the dimensions of the homology modules. Hence

$$\sum_{j=0}^{m} (-1)^j \dim \mathbb{F}\Omega_{2j} = \sum_{j=0}^{m} (-1)^j \dim H_{2j} = (-1)^{m/2} \dim D^{(m+1,m)}.$$

Since the left-hand side is $\sum_{j=0}^m (-1)^j \binom{2m+1}{2j}$, the result follows from (12). The proof is similar if m is odd. $\square$

We end by using the one-step boundary maps $\gamma_k : \mathbb{F}\Omega_k \to \mathbb{F}\Omega_{k-1}$ to give a more explicit description of the non-split extension in Theorem 1.1. The following calculation is required.

Lemma 4.7. *If* $0 \le k \le n-2$ *then* $(\operatorname{im}\varepsilon_{k+2})\gamma_k\gamma_k^\star \subseteq \ker\varepsilon_k$.

Proof. Fix $Z \in \Omega_{k+2}$. If $Y \in \Omega_k$ has a non-zero coefficient in $Z\varepsilon_{k+2}\gamma_k\gamma_k^\star$ then either $Y = Z\backslash\{i,i'\}$, for distinct $i,i' \in Z$ or $Y = Z \cup \{j\}\backslash\{i,i',i''\}$ for distinct $i,i',i'' \in Z$ and $j \notin Z$. In the former case the coefficient of Y is k and in the latter it is 1. Therefore $\varepsilon_{k+2}\gamma_k\gamma_k^\star = k\varepsilon_{k+2} + \psi$ where

$$Z\psi = \sum_{\substack{i,i',i''\in Z\\ j\notin Z}} \bigl(Z\cup\{j\}\backslash\{i,i',i''\}\bigr).$$

Since $\varepsilon_{k+2}\varepsilon_k = 0$, it suffices to prove that $\psi\varepsilon_k = 0$. We may suppose that $k \ge 2$. If $X \in \Omega_{k-2}$ has a non-zero coefficient in $Z\psi\varepsilon_k$ then either $X = Z\backslash D$ where $D \subseteq Z$ and $|D| = 4$ or $X = Z\cup\{j\}\backslash E$ where $E \subseteq Z$, $|E| = 5$ and $j \notin Z$. In both cases the coefficient is in fact zero: in the first there are $\binom{4}{3}$ choices for $\{i,i',i''\} \subseteq D$ and in the second there are $\binom{5}{3}$ choices for $\{i,i',i''\} \subseteq E$. □

Let $n = 2m$ be even and let U be the submodule of $\mathbb{F}\Omega_m$ generated by $v_m + v_m(2m-1, 2m)$.

Proposition 4.8. *Under the canonical isomorphism* $\mathbb{F}\Omega_m \cong M^{(m,m)}$*, the image of* U *is* $S^{(m,m)}$*. There is a chain*

$$\operatorname{rad} U + \operatorname{im}\varepsilon_{m+2} \subseteq U + \operatorname{im}\varepsilon_{m+2} \subseteq \ker\varepsilon_m$$

in which the two quotients are isomorphic to $D^{(m+1,m-1)}$.

Proof. By Theorem 1.1, $v_m \in \ker\varepsilon_m$. Therefore U is a submodule of $\ker\varepsilon_m$. By (10) in §3, under the canonical isomorphism $\mathbb{F}\Omega_m \cong M^{(m,m)}$, the image of $v_m + v_m(2m-1,2m)$ is the polytabloid e_t, where t is the standard tableau of shape (m,m) having $\{2,4,\ldots,2m\}$ in its bottom row; this polytabloid generates the Specht module $S^{(m,m)}$. Therefore $U \cong S^{(m,m)}$.

By the Branching Rule (see [16, Theorem 9.3]) the restriction of $S^{(m,m)}$ to S_{2m-1} is $S^{(m,m-1)}$; this module has $D^{(m,m-1)}$ as its unique top composition factor. By Lemma 3.2(iv), the only two-row simple module for $\mathbb{F}S_{2m}$ whose restriction to S_{2m-1} may have $D^{(m,m-1)}$ as a composition factor is $D^{(m+1,m-1)}$. Therefore, as noted by Benson in [2, Lemma 5.2], $S^{(m,m)}$ has $D^{(m+1,m-1)}$ as its unique top composition factor, and the multiplicity of $D^{(m+1,m-1)}$ in $S^{(m,m)}$ is 1. Hence $U/\operatorname{rad} U \cong D^{(m+1,m-1)}$. By Lemma 3.2(iii), $D^{(m+1,m-1)}$ is not a composition factor of $\operatorname{im}\varepsilon_{m+2}$. Since $\ker\varepsilon_m/\operatorname{im}\varepsilon_{m+2}$ has two composition factors of $D^{(m+1,m-1)}$, it follows that the chain has the claimed quotients. □

Proposition 4.9. *Let* $n = 2m$ *be even. The endomorphism* $\gamma_m\gamma_m^\star$ *of* $\mathbb{F}\Omega_k$ *restricts to an endomorphism of* $\ker\varepsilon_m$ *satisfying*

(i) $v_m\gamma_m\gamma_m^\star = v_m + v_m(2m-1, 2m)$*;*

(ii) $U\gamma_m\gamma_m^\star = 0$;
(iii) $(\operatorname{im}\varepsilon_{m+2})\gamma_m\gamma_m^\star \subseteq \operatorname{im}\varepsilon_{m+2}$.

Proof. By Lemma 4.2, $v_m\gamma_m = \{2,4,\ldots,2(m-1)\}\sum_{\sigma\in G_{m-1}}\sigma$. Hence

$$v_m\gamma_m\gamma_m^\star = \sum_{\sigma\in G_{m-1}} \sum_{\substack{1\le i\le 2m \\ i\not\in\{2,4,\ldots,2(m-1)\}\sigma}} \big(\{2,4,\ldots,2(m-1)\}\cup\{i\}\big).$$

There are summands corresponding to the pairs $(\sigma, 2j)$ and $(\sigma(2j-1,2j), 2j-1)$ if and only if $(2j)\sigma = 2j-1$; when present, these summands are equal are so cancel. The summands for $i = 2m$ give v_m and the summands for $i = 2m-1$ give $v_m(2m-1,2m)$. Hence $v_m\gamma_m\gamma_m^\star = v_m + v_m(2m-1,2m)$, proving (i). Moreover, since $\big(1+(2m-1,2m)\big)^2 = 0$, we have $\big(v_m+v_m(2m-1,2m)\big)\gamma_m\gamma_m^\star = 0$. Hence $U\gamma_m\gamma_m^\star = 0$, proving (ii).

By Lemma 4.7, $(\operatorname{im}\varepsilon_{m+2})\gamma_m\gamma_m^\star \subseteq \ker\varepsilon_{m+2}$. By Lemma 3.2(iii), $\operatorname{im}\varepsilon_{m+2}$ does not have $D^{(m+1,m-1)}$ as a composition factor. It therefore follows from Proposition 4.8 and the Jordan–Hölder Theorem that $(\operatorname{im}\varepsilon_{m+2})\gamma_m\gamma_m^\star \subseteq \operatorname{im}\varepsilon_{m+2}$ as required for (iii). □

Corollary 4.10. *Let $n = 2m$. The map $\vartheta : H_m \to H_m$ induced by restricting $\gamma_m\gamma_m^\star$ to $\ker\vartheta_m$ is a well-defined $\mathbb{F}S_n$-endomorphism of H_m such that $\vartheta \neq 0$ and $\vartheta^2 = 0$.*

Proof. By Proposition 4.9, ϑ is well-defined. By Theorem 1.1, H_m is generated by $v_m + \operatorname{im}\varepsilon_{m+2}$. Therefore $H_m\vartheta$ is generated by $v_m + v_m(2m-1,2m) + \operatorname{im}\varepsilon_{m+2}$; by Propositions 4.8 and 4.9(ii) this is a non-zero element of H_m lying in $\ker\vartheta$. □

5. Proof of Theorem 1.2

In this section we prove the characterization in Theorem 1.2 of when

$$\mathbb{F}\Omega_{k+t} \xrightarrow{\varphi_{k+t}^{(t)}} \mathbb{F}\Omega_k \xrightarrow{\varphi_k^{(t)}} \mathbb{F}\Omega_{k-t} \tag{4}$$

is exact. We showed in §2 that (4) is always exact when $t = 1$. Thus Theorem 1.2(i) is a sufficient condition. Clearly (4) is not exact when both $k+t > n$ and $k-t < 0$ and so only the middle module is non-zero. In §5.1 we deal with the case when there is exactly one zero module. This leaves the most interesting case of three non-zero modules, described by (i) and (iii). We show these conditions are necessary in §5.2 and sufficient in §5.3.

The following lemma indicates the obstruction to exactness removed by the condition $k+t \le n-k$.

Lemma 5.1. *Suppose that $t > 1$ and $k \le n-k < k+t$. Then $\mathbb{F}\Omega_k$ has a composition factor not present in either $\mathbb{F}\Omega_{k+t}$ or $\mathbb{F}\Omega_{k-t}$.*

Proof. By Proposition 3.1(i) we have $\mathbb{F}\Omega_{k+t} \cong \mathbb{F}\Omega_{n-(k+t)}$. By hypothesis, $n-(k+t) < k$. If $2k < n$ then Lemma 3.2(i) implies that $D^{(n-k,k)}$ is a composition factor of $\mathbb{F}\Omega_k$ not present in either $\mathbb{F}\Omega_{n-(k+t)}$ or $\mathbb{F}\Omega_{k-t}$. In the remaining case $2k = n$ and $\mathbb{F}\Omega_{k+t} \cong \mathbb{F}\Omega_{k-t}$. Since $k - t < k - 1$, Lemma 3.2(iii) implies that $D^{(k+1,k-1)}$ is a composition factor of $\mathbb{F}\Omega_k$ not present in $\mathbb{F}\Omega_{k-t}$. □

5.1. **Surjective and injective maps: Theorem 1.2(ii).** There is exactly one zero module in (4) if and only if $k < t \leq n - k$ or $n - k < t \leq k$. By Proposition 3.1(i) we can reduce to the first case, when the sequence is

$$\mathbb{F}\Omega_{k+t} \xrightarrow{\varphi^{(t)}_{k+t}} \mathbb{F}\Omega_k \longrightarrow 0.$$

It then suffices to prove the following proposition.

Proposition 5.2. *Let $k < t \leq n - k$ and let 2^τ be the least two-power appearing in the binary form of t. Then $\varphi^{(t)}_{k+t} : \mathbb{F}\Omega_{k+t} \to \mathbb{F}\Omega_k$ is surjective if and only if $k < 2^\tau$ and $k + t \leq n - k$.*

Proof. Suppose that $k + t > n - k$. Then, by Lemma 5.1, $\mathbb{F}\Omega_k$ has a composition factor $D^{(n-k,k)}$ not present in $\mathbb{F}\Omega_{k+t}$, and so $\varphi^{(t)}_{k+t}$ is not surjective. Suppose that $k \geq 2^\tau$. Since the addition of 2^τ to $t - 2^\tau$ is carry free, Lemma 3.4 implies that $\varphi^{(t)}_{k+t}$ factorizes as $\varphi^{(t-2^\tau)}_{k+t}\varphi^{(2^\tau)}_{k+2^\tau}$. In the sequence

$$\mathbb{F}\Omega_{k+2^\tau} \xrightarrow{\varphi^{(2^\tau)}_{k+2^\tau}} \mathbb{F}\Omega_k \xrightarrow{\varphi^{(2^\tau)}_{k}} \mathbb{F}\Omega_{k-2^\tau}$$

the map $\varphi^{(2^\tau)}_k$ is non-zero. Since $\operatorname{im} \varphi^{(2^\tau)}_{k+2^\tau} \subseteq \ker \varphi^{(2^\tau)}_k$, it follows that $\varphi^{(2^\tau)}_{k+2^\tau}$ is not surjective. Therefore $\varphi^{(t)}_{k+t}$ is not surjective.

Conversely, suppose that $k + t \leq n - k$ and $k < 2^\tau$. Generalizing the example following the Suspension Lemma (Lemma 3.6), take $\ell = k$, $v = \{n - k + 1, \ldots, n\} \in \ker \varphi^{(t)}_k$ and $X = \{1, \ldots, k + t\}$. By hypothesis these sets are disjoint. The least two-power appearing in the binary form of t is 2^τ, hence $k + t$ is carry free. Moreover if $0 < s \leq k$ then $k + (t - s)$ is not carry free, since it has 2^τ in its binary form while $t - s$ does not. Hence

$$\{n - k + 1, \ldots, n\} = \big(\{n - k + 1, \ldots, n\}(\{1, \ldots, k + t\}\varphi^{(k)}_{k+t})\big)\varphi^{(t)}_{k+t}$$

where the left-hand side generates $\mathbb{F}\Omega_k$. Therefore $\varphi^{(t)}_{k+t}$ is surjective. □

5.2. **Necessity: Theorem 1.2(iii).** We now suppose that the sequence (4) has three non-zero modules and that $t > 1$ and show that the condition in (iii) is necessary for it to be exact.

By Proposition 3.1 we may assume that $2k \leq n$. Suppose that $n < 2k + t$. Then $k \leq n - k < k + t$, so by Lemma 5.1, $\mathbb{F}\Omega_k$ has a composition factor not present in $\mathbb{F}\Omega_{k+t}$ or $\mathbb{F}\Omega_{k-t}$. Therefore (4) is not exact.

It remains to show that if t is not a two-power then (4) is not exact. The proof of the following proposition uses the same idea as Lemma 3.5 in [15].

Proposition 5.3. *Suppose that $t > s$ and that the addition of s to t is carry free. If $k \geq s$ then* $\ker \varphi^{(t)}_k$ *properly contains* $\ker \varphi^{(s)}_k$.

Proof. Since $s+t$ is carry free, Lemma 3.4 implies that $\varphi_k^{(t)} = \varphi_k^{(s)}\varphi_{k-s}^{(t-s)}$. Therefore $\ker\varphi_k^{(t)}$ contains $\ker\varphi_k^{(s)}$. Since $t>s$, there exists β such that 2^β appears in the binary form of t but not in the binary form of s. Let $v = \{1,\ldots,k+2^\beta\}\varphi_{k+2^\beta}^{(2^\beta)}$. Since $t+2^\beta$ is not carry free, while $s+2^\beta$ is carry free, Lemma 3.4 implies that $v\varphi_k^{(t)} = 0$ and $v\varphi_k^{(s)} \neq 0$. □

Corollary 5.4. *Suppose that t is not a two-power. Then* (4) *is not exact.*

Proof. Choose 2^β such that 2^β appears in the binary form of t and set $s = t-2^\beta$. By Lemma 3.4 we have $\varphi_k^{(t)} = \varphi_k^{(s)}\varphi_{k-s}^{(2^\beta)}$ and $\varphi_{k+t}^{(t)} = \varphi_{k+t}^{(2^\beta)}\varphi_{k+s}^{(s)}$. Hence

$$\ker\varphi_k^{(t)} \supseteq \ker\varphi_k^{(s)} \supseteq \operatorname{im}\varphi_{k+s}^{(s)} \supseteq \operatorname{im}\varphi_{k+t}^{(t)}$$

where the first containment is strict by Proposition 5.3. Hence (4) is not exact. □

5.3. **Sufficiency: Theorem 1.2(iii).** By Proposition 3.1 we may assume that $2k \leq n$. Thus (iii) holds if and only if $n \geq 2k+t$ and $t = 2^\tau$ is a two-power. We shall show by induction on n that this condition implies that (4) is exact. Perhaps surprisingly, most of the work comes in the base case when $n = 2k+t$, where we prove in Proposition 5.8 the stronger result that (4) is split exact, that is, $\mathbb{F}\Omega_k = \ker\varphi_k^{(t)} \oplus C_k$ for an $\mathbb{F}S_n$-module C_k. In this case (4) is part of the chain complex

$$\cdots \xrightarrow{\varphi_{k+3t}^{(t)}} \mathbb{F}\Omega_{k+2t} \xrightarrow{\varphi_{k+2t}^{(t)}} \mathbb{F}\Omega_{k+t} \xrightarrow{\varphi_{k+t}^{(t)}} \mathbb{F}\Omega_k \xrightarrow{\varphi_k^{(t)}} \mathbb{F}\Omega_{k-t} \xrightarrow{\varphi_{k-t}^{(t)}} \cdots . \tag{13}$$

Since $n = 2k+t$, this chain complex is invariant under the duality in Proposition 3.1; the case $n=6$, $t=2$ and $k=2$ can be seen in Example 1.4.

Splitting of (13). Motivated by (7) in §2, we show that the dual maps $\varphi_r^{(t)\star}$ defined in (9) at the start of §3 define a chain homotopy between (13) and the zero chain complex. The first of the two lemmas below can also be deduced from (2.9) and (2.10) in [19]. In it $X \mathbin{\triangle} Y$ denotes the symmetric difference of sets X and Y.

Lemma 5.5. *If $Y \in \Omega_k$ then*

$$Y\varphi_k^{(t)}\varphi_k^{(t)\star} = \sum_{d=0}^{t} \binom{k-d}{t-d} \sum_{\substack{X\in\Omega_k \\ |X\triangle Y|=2d}} X,$$

$$Y\varphi_{k+t}^{(t)\star}\varphi_{k+t}^{(t)} = \sum_{d=0}^{t} \binom{n-k-d}{t-d} \sum_{\substack{X\in\Omega_k \\ |X\triangle Y|=2d}} X.$$

Proof. If $X \in \Omega_k$ is a summand of $Y\varphi_k^{(t)}\varphi_k^{(t)\star}$ then $X = (Y\backslash D)\cup A$ for unique sets $D \subseteq Y$ and $A \subseteq \{1,\ldots,n\}\backslash Y$. Clearly $|D| = |A|$. If their common size is d then $|X \mathbin{\triangle} Y| = 2d$. If R is a t-subset of Y such that $R \supseteq D$, we may obtain X by removing R from Y and then inserting the elements of

$A \cup (R\backslash D)$. Therefore the coefficient of X is the number of choices for R, namely $\binom{k-d}{t-d}$. The proof for $Y\varphi^{(t)\star}_{k+t}\varphi^{(t)}_{k+t}$ is similar. $\square$

Lemma 5.6. *Let $\tau \in \mathbb{N}_0$. The following are equivalent*

(i) $\binom{k-d}{2^\tau-d} + \binom{n-k-d}{2^\tau-d} \equiv 0 \bmod 2$ *for* $1 \le d \le 2^\tau$*;*

(ii) $\binom{k+e}{e} + \binom{n-k+e}{e} \equiv 0 \bmod 2$ *for* $0 \le e < 2^\tau$*;*

(iii) $\binom{k+2^\rho}{2^\rho} + \binom{n-k+2^\rho}{2^\rho} \equiv 0 \bmod 2$ *for* $0 \le \rho < \tau$*;*

(iv) $n \equiv 2k \bmod 2^\tau$*.*

Proof. Observe that if $\ell < 2^\tau$ and $k \equiv k' \bmod 2^\tau$ then

$$k + \ell \text{ is carry free} \iff k' + \ell \text{ is carry free.} \tag{†}$$

Replacing d with $2^\tau - e$ in (i) shows that (i) is equivalent to $\binom{k-2^\tau+e}{e} + \binom{n-k-2^\tau+e}{e} \equiv 0 \bmod 2$ for $0 \le e < 2^\tau$. From (†) we see that $(k - 2^\tau) + e$ is carry free if and only if $k+e$ is carry free. Therefore (i) is equivalent to (ii). Clearly (ii) implies (iii). We show that (iii) implies (iv) by induction on τ. If $\tau = 0$ then (iii) is vacuous and (iv) obviously holds. Suppose that (iii) holds as stated, so by induction $n \equiv 2k \bmod 2^\tau$. Either $n-k \equiv k \bmod 2^{\tau+1}$, in which case (†) implies that $\binom{k+2^\tau}{2^\tau} \equiv \binom{n-k+2^\tau}{2^\tau} \bmod 2$, or $n - k \equiv k + 2^\tau \bmod 2^{\tau+1}$ and similarly (†) implies that $\binom{k+2^\tau}{2^\tau} + \binom{n-k+2^\tau}{2^\tau} \equiv 1 \bmod 2$. This completes the inductive step. Finally if (iv) holds then $k - d \equiv n - k - d \bmod 2^\tau$ for all $d \in \mathbb{N}$. By (†) this implies (i). $\square$

Lemma 5.7. *Let $\tau \in \mathbb{N}_0$. We have*

$$\binom{k-d}{2^\tau-d} + \binom{n-k-d}{2^\tau-d} \equiv 0 \bmod 2 \quad \textit{for } 1 \le d \le 2^\tau$$

and $\binom{k}{2^\tau} + \binom{n-k}{2^\tau} \equiv 1 \bmod 2$ *if and only if* $n \equiv 2k + 2^\tau \bmod 2^{\tau+1}$*.*

Proof. By Lemma 5.6, the first condition holds if and only if $n \equiv 2k \bmod 2^\tau$. As in the proof of this lemma, the second condition then holds if and only if exactly one of $k + 2^\tau$ and $(n-k) + 2^\tau$ is carry free; equivalently $n \equiv 2k + 2^\tau \bmod 2^{\tau+1}$. $\square$

Proposition 5.8. *If $t = 2^\tau$ and $n \equiv 2k+t \bmod 2^{\tau+1}$ then* $\ker \varphi^{(t)}_k = \operatorname{im} \varphi^{(t)}_{k+t}$ *and* $\mathbb{F}\Omega_k = \ker \varphi^{(t)}_k \oplus \operatorname{im} \varphi^{(t)\star}_k$*.*

Proof. By Lemmas 5.5 and 5.7,

$$\varphi^{(t)}_k \varphi^{(t)\star}_k + \varphi^{(t)\star}_{k+t}\varphi^{(t)}_{k+t} = \mathrm{id}. \tag{14}$$

Hence, repeating part of a basic argument from homotopy theory, we have $\mathbb{F}\Omega_k = \operatorname{im} \varphi^{(t)\star}_k + \operatorname{im} \varphi^{(t)}_{k+t}$. If $v \in \operatorname{im} \varphi^{(t)\star}_k \cap \ker \varphi^{(t)}_k$ then $v\varphi^{(t)}_k = 0$ and, since $\varphi^{(t)\star}_k \varphi^{(t)\star}_{k+t} = 0$, we also have $v\varphi^{(t)\star}_{k+t} = 0$. Evaluating (14) at v implies that $v = 0$. Since $\operatorname{im} \varphi^{(t)}_{k+t} \subseteq \ker \varphi^{(t)}_k$ it follows that $\mathbb{F}\Omega_k = \operatorname{im} \varphi^{(t)\star}_k \oplus \ker \varphi^{(t)}_k$ and $\operatorname{im} \varphi^{(t)}_{k+t} = \ker \varphi^{(t)}_k$, as required. $\square$

We are now ready to show that Theorem 1.2(iii) is a sufficient condition for (4) to be exact.

Proposition 5.9. *Let t be a two-power. If $n \geq 2k+t$ then (4) is exact.*

Proof. We work by induction on n dealing with all admissible k at once. If $n = 2k+t$ then Proposition 5.8 shows that (4) is split exact. Now suppose that $n > 2k+t$ and, inductively, that the sequence of $\mathbb{F}S_{n-1}$-modules

$$\mathbb{F}\Omega_{k+t}^{[n-1]} \xrightarrow{\varphi_{k+t}^{(t)[n-1]}} \mathbb{F}\Omega_{k}^{[n-1]} \xrightarrow{\varphi_{k}^{(t)[n-1]}} \mathbb{F}\Omega_{k-t}^{[n-1]}$$

is exact. (As usual the bracketed $n-1$ indicates that these are modules, and importantly, module homomorphisms, for $\mathbb{F}S_{n-1}$.) Using the product operation on sets defined in §3, each element of $\mathbb{F}\Omega_k$ has a unique expression in the form $U + u\{n\}$ where $U \in \mathbb{F}\Omega_k^{[n-1]}$ and $u \in \mathbb{F}\Omega_{k-1}^{[n-1]}$. Suppose that $U + u\{n\} \in \ker \varphi_k^{(t)}$. By the Splitting Rule (Lemma 3.5),

$$(U + u\{n\})\varphi_k^{(t)} = U\varphi_k^{(t)} + u\varphi_{k-1}^{(t-1)} + u\varphi_{k-1}^{(t)}\{n\}. \tag{15}$$

Hence $U\varphi_k^{(t)} + u\varphi_{k-1}^{(t-1)} = 0$ and $u\varphi_{k-1}^{(t)} = 0$. Since $u \in \mathbb{F}\Omega_{k-1}^{[n-1]}$ and $n-1 \geq 2(k-1)+t$, applying the inductive hypothesis to

$$\varphi_{k-1}^{(t)[n-1]} : \Omega_{k-1}^{[n-1]} \longrightarrow \Omega_{k-1-t}^{[n-1]}$$

gives

$$u = v\varphi_{k-1+t}^{(t)[n-1]} \tag{16}$$

for some $v \in \mathbb{F}\Omega_{k-1+t}^{[n-1]}$. Substituting (16) into $U\varphi_k^{(t)} + u\varphi_{k-1}^{(t-1)} = 0$ we obtain

$$U\varphi_k^{(t)} + v\varphi_{k-1+t}^{(t)}\varphi_{k-1}^{(t-1)} = 0.$$

Since $t + (t-1)$ is carry free, Lemma 3.4 implies that $\varphi_{k-1+t}^{(t)}\varphi_{k-1}^{(t-1)} = \varphi_{k-1+t}^{(t-1)}\varphi_k^{(t)}$. Hence $\big(U + v\varphi_{k-1+t}^{(t-1)}\big)\varphi_k^{(t)} = 0$. Since $U + v\varphi_{k-1+t}^{(t-1)} \in \mathbb{F}\Omega_k^{[n-1]}$ and $n-1 \geq 2k+t$, applying the inductive hypothesis to

$$\varphi_{k}^{(t)[n-1]} : \Omega_{k}^{[n-1]} \longrightarrow \Omega_{k-t}^{[n-1]}$$

gives

$$U + v\varphi_{k-1+t}^{(t-1)} = W\varphi_{k+t}^{(t)[n-1]} \tag{17}$$

for some $W \in \mathbb{F}\Omega_{k+t}^{[n-1]}$. Substituting for U and u using (16) and (17) we find

$$\begin{aligned} U + u\{n\} &= v\varphi_{k-1+t}^{(t-1)} + W\varphi_{k+t}^{(t)} + v\varphi_{k-1+t}^{(t)}\{n\} \\ &= \big(W + v\{n\}\big)\varphi_{k+t}^{(t)}, \end{aligned}$$

hence $U + u\{n\} \in \operatorname{im} \varphi_{k+t}^{(t)} : \mathbb{F}\Omega_{k+t} \longrightarrow \mathbb{F}\Omega_k$, as required. □

6. Split exactness

In this section we prove Theorem 1.3, characterizing when the sequence

$$(2)\quad 0 \to \mathbb{F}\Omega_{a+ct} \xrightarrow{\varphi^{(t)}_{a+ct}} \mathbb{F}\Omega_{a+(c-1)t} \xrightarrow{\varphi^{(t)}_{a+(c-1)t}} \cdots \xrightarrow{\varphi^{(t)}_{a+2t}} \mathbb{F}\Omega_{a+t} \xrightarrow{\varphi^{(t)}_{a+t}} \mathbb{F}\Omega_a \to 0$$

is split exact. Suppose that there are just two non-zero modules. Then (2) is

$$0 \to \mathbb{F}\Omega_{a+t} \xrightarrow{\varphi^{(t)}_{a+t}} \mathbb{F}\Omega_a \to 0.$$

Comparing $\dim \mathbb{F}\Omega_{a+t} = \binom{n}{a+t}$ and $\dim \mathbb{F}\Omega_a = \binom{n}{a}$ shows that if $\varphi^{(t)}_{a+t}$ is an isomorphism then $n-(a+t)=a$, and so $n=2a+t$, as required in condition (a). Since the chain complex is then self-dual, Proposition 5.2 implies that $\varphi^{(t)}_{a+t}$ is an isomorphism if and only if $a<2^\tau$, where 2^τ is the least two-power appearing in the binary form of a. Hence condition (a) is necessary and sufficient for (2) to be split exact.

Now suppose (2) has at least three non-zero modules and is split exact. Therefore condition (a) does not hold. If condition (b) holds then $t=2^\tau$ for some $\tau \in \mathbb{N}_0$ and $n=2a+(2s+1)2^\tau$ for some $s \in \mathbb{N}_0$. By maximality of c, we have $c=2s+1$ and $n=2a+ct$. By Proposition 5.2, $\varphi^{(t)}_{a+t}$ is surjective and, dually, $\varphi^{(t)}_{a+ct}$ is injective. If $k=a+j2^\tau$ where $1 \le j < c$ then, since $n \equiv 2k+2^\tau \bmod 2^{\tau+1}$, Proposition 5.8 implies that $\mathbb{F}\Omega_k = \ker \varphi^{(t)}_k \oplus \operatorname{im} \varphi^{(t)\star}_k$. Hence (2) is split exact. Conversely, suppose that (2) has at least three non-zero modules and is split exact. Since it is then exact. Theorem 1.2 implies that t is a two-power. Take s maximal such that $2a+(2s+1)t \le n$ and set $k=a+(s+1)t$. The exact sequence

$$\mathbb{F}\Omega_{k+t} \xrightarrow{\varphi^{(t)}_{k+t}} \mathbb{F}\Omega_k \xrightarrow{\varphi^{(t)}_k} \mathbb{F}\Omega_{k-t}$$

is then part of (2). By Theorem 1.2, either $k+t \le n-k$ or $n-k+t \le k$. By choice of s the first condition does not hold. Therefore $n-\bigl(a+(s+1)t\bigr)+t \le a+(s+1)t$ and so $n \le 2a+(2s+1)t$. Hence $n=2a+(2s+1)t$ and so $n \equiv 2a+t \bmod 2t$, as required in (b). This completes the proof.

7. Further directions

Recall that γ_k denotes $\varphi^{(1)}_k$ and ε_k denotes $\varphi^{(2)}_k$.

Split exactness. The sequence $\mathbb{F}\Omega_{k+t} \xrightarrow{\varphi^{(t)}_{k+t}} \mathbb{F}\Omega_k \xrightarrow{\varphi^{(t)}_k} \mathbb{F}\Omega_{k-t}$ in (4) was shown in Proposition 5.8 to be split exact when $t=2^\tau$ is a two-power and $n \equiv 2k+2^\tau \bmod 2^{\tau+1}$; call this condition (A). By Propositions 3.1 and 5.2 it is also split exact when $k<t$ or $k>n-t$; call this condition (B).

If $t=1$ then the combined condition (A) or (B), namely that n is odd or $k=0$ or $k=n$, is necessary and sufficient for (4) to be split exact. We outline a proof using that the ordinary character $\chi^{(n)}+\chi^{(n-1,1)}+\cdots+\chi^{(n-k,k)}$ of $\mathbb{F}\Omega_k$ is multiplicity-free, and so, by the results of [2, §3.11], $\operatorname{End}_{\mathbb{F}S_n}(\mathbb{F}\Omega_k)$ is abelian. It follows, by composing the projection maps, that if V and W are distinct direct summands of $\mathbb{F}\Omega_k$ then $\operatorname{Hom}_{\mathbb{F}S_n}(V,W)=0$. Hence the decomposition of $\mathbb{F}\Omega_k$ into direct summands is unique and each direct summand is self-dual. If $0 < k < n$ and (4) splits then $\mathbb{F}\Omega_k \cong \ker\gamma_k \oplus C_k$ for some non-zero complement C_k. We have $\operatorname{im}\gamma_k^\star \cong \operatorname{Ann}(\ker\gamma_k) \cong C_k^\star \cong C_k$. Therefore there is an endomorphism of $\mathbb{F}\Omega_k$ having $\ker\gamma_k$ in its kernel, and restricting to an isomorphism $C_k \cong \operatorname{im}\gamma_k^\star$. The uniqueness of the decomposition now shows that $\mathbb{F}\Omega_k = \ker\gamma_k \oplus \operatorname{im}\gamma_k^\star$. However, by Lemma 5.5, $\gamma_k\gamma_k^\star \neq 0$ and $\gamma_k\gamma_k^\star + \gamma_{k+1}^\star\gamma_{k+1} = n\mathrm{id}$, hence $\gamma_k\gamma_k^\star\gamma_k = n\gamma_k$. Therefore $\ker\gamma_k \cap \operatorname{im}\gamma_k^\star \neq \{0\}$ whenever n is even, showing that (4) is not split in this case.

This argument can be adapted to show that, when $t = 2$, (4) is split if and only if either (A) or (B) holds. Considerable calculation is required: for example, using only the γ and ε maps and their duals, the simplest obstruction to exactness when $n \equiv 1 \bmod 4$ and k is odd known to the author is $\gamma_k^\star\varepsilon_k\varepsilon_k^\star \neq 0$ and $\gamma_k^\star\varepsilon_k\varepsilon_k^\star\varepsilon_k = 0$. On the other hand, Example 1.6 shows that, when $t = 4$, (4) may be split in cases when neither (A) nor (B) holds. The following problem therefore appears to be quite deep.

Problem 7.1. *Find a necessary and sufficient condition for* (4) *to be split exact.*

Generators for homology modules. Recall that $G_\ell = \langle (1,2), \ldots, (2\ell-1, 2\ell)\rangle$. Generalizing the elements v_k defined before Theorem 1.1, we define $v_k^{(t)} = \{2,4,\ldots,2k\}\sum_{\sigma\in G_{k-t+1}}\sigma$. By [16, Theorem 17.13(i)], or a direct calculation similar to Lemma 4.2, $v_k^{(t)}$ generates a submodule of $\ker\varphi_k^{(t)}$.

Conjecture 7.2. *If t is a two-power and $k \leq 2n$ then the homology module* $\ker\varphi_k^{(t)}/\operatorname{im}\varphi_{k+t}^{(t)}$ *is generated by* $v_k^{(t)} + \operatorname{im}\varphi_{k+t}^{(t)}$.

When $t = 1$ the conjecture holds trivially because all the homology modules are zero. When $t = 2$ it is implied by Theorem 1.1. It has been checked for all $n \leq 16$ using MAGMA and the code available from the author's webpage.

Restricted homology. Fix $s \in \mathbb{N}$. If $u \in \ker\varphi_k^{(s)}$ then, by Lemma 3.4, $u\varphi_k^{(t)}\varphi_{k-t}^{(s)} = u\varphi_k^{(s)}\varphi_{k-s}^{(t)} = 0$. Therefore $\varphi_k^{(t)} : \mathbb{F}\Omega_k \to \mathbb{F}\Omega_{k-t}$ restricts to a map $\ker\varphi_k^{(s)} \to \ker\varphi_{k-t}^{(s)}$ and we may ask for the homology of the sequence

$$\ker\varphi_{k+t}^{(s)} \xrightarrow{\varphi_{k+t}^{(t)}} \ker\varphi_k^{(s)} \xrightarrow{\varphi_k^{(t)}} \ker\varphi_{k-t}^{(s)}. \tag{18}$$

The following conjectures suggest that these restricted homology modules, denoted $\bar{H}_k$, are surprisingly well behaved. They have been checked for all $n \leq 12$ using MAGMA and the code available from the author's webpage.

Conjecture 7.3. *Let $n = 2m$.*

(i) *The sequence* $\ker\gamma_{k+2} \xrightarrow{\varepsilon_{k+2}} \ker\gamma_k \xrightarrow{\varepsilon_k} \ker\gamma_{k-2}$ *has non-zero homology if and only if $k \in \{m-1, m\}$. Moreover $\bar{H}_{m-1} \cong \bar{H}_m \cong D^{(m+1,m-1)}$.*

(ii) *The sequence* $\ker\varepsilon_{k+1} \xrightarrow{\gamma_{k+1}} \ker\varepsilon_k \xrightarrow{\gamma_k} \ker\varepsilon_{k-1}$ *has non-zero homology if and only if $k = m$. Moreover $\bar{H}_m \cong D^{(m+1,m-1)}$.*

Conjecture 7.4. *Let $n = 2m + 1$.*

(i) *The sequence* $\ker \gamma_{k+2} \xrightarrow{\varepsilon_{k+2}} \ker \gamma_k \xrightarrow{\varepsilon_k} \ker \gamma_{k-2}$ *has non-zero homology if and only if $k = m$. Moreover $\bar{H}_m \cong D^{(m+1,m)}$.*

(ii) *The sequence* $\ker \varepsilon_{k+1} \xrightarrow{\gamma_{k+1}} \ker \varepsilon_k \xrightarrow{\gamma_k} \ker \varepsilon_{k-1}$ *is exact.*

For example, taking $n = 6$ as in Example 1.4, the chain complex with restricted maps $0 \to \ker \gamma_6 \xrightarrow{\varepsilon_6} \ker \gamma_4 \xrightarrow{\varepsilon_4} \ker \gamma_2 \xrightarrow{\varepsilon_2} \ker \gamma_0 \to 0$ is

$$0 \to 0 \xrightarrow{\varepsilon_6} \begin{array}{c} \mathbb{F} \\ D^{(5,1)} \end{array} \xrightarrow{\varepsilon_4} \begin{array}{c} \mathbb{F} \\ D^{(4,2)} \\ \mathbb{F} \\ D^{(5,1)} \end{array} \xrightarrow{\varepsilon_2} \mathbb{F} \to 0$$

which has non-zero homology of $D^{(4,2)}$ uniquely in degree 2. This chain complex is dual to the chain complex $0 \to \ker \gamma_5 \xrightarrow{\varepsilon_5} \ker \gamma_3 \xrightarrow{\varepsilon_3} \ker \gamma_1 \xrightarrow{\varepsilon_1} 0$ which has non-zero homology of $D^{(4,2)}$ uniquely in degree 3. The chain complex $0 \to \ker \varepsilon_6 \xrightarrow{\gamma_6} \ker \varepsilon_5 \xrightarrow{\gamma_5} \cdots \xrightarrow{\gamma_2} \ker \varepsilon_1 \xrightarrow{\gamma_1} \ker \varepsilon_0 \to 0$ is

$$0 \to 0 \xrightarrow{\gamma_6} 0 \xrightarrow{\gamma_5} \mathbb{F} \xrightarrow{\gamma_4} \begin{array}{c} D^{(4,2)} \\ \mathbb{F} \\ D^{(5,1)} \oplus \boxed{\begin{array}{c} D^{(4,2)} \\ \mathbb{F} \end{array}} \end{array} \xrightarrow{\gamma_3} \begin{array}{c} D^{(5,1)} \\ \mathbb{F} \\ \boxed{\begin{array}{c} D^{(4,2)} \\ \mathbb{F} \\ D^{(5,1)} \end{array}} \end{array} \xrightarrow{\gamma_2} \begin{array}{c} \mathbb{F} \\ \boxed{\begin{array}{c} D^{(5,1)} \\ \mathbb{F} \end{array}} \end{array} \xrightarrow{\gamma_1} \boxed{\mathbb{F}} \to 0$$

where the boxes show the kernels of the maps γ_k, now each restricted to $\ker \varepsilon_k$. It has non-zero homology of $D^{(4,2)}$ uniquely in degree 3.

Multistep maps in odd characteristic. Now suppose that $\mathbb{F}$ has odd prime characteristic p. Lemma 3.4 generalizes to show that $\varphi^{(s)}_{k+s}\varphi^{(t)}_k = 0$ whenever p divides $\binom{s+t}{s}$. (Equivalently, a carry arises when s and t are added in base p.) Generalizing the usual definition, we may ask for the homology $H_k = \ker \varphi^{(t)}_k / \operatorname{im} \varphi^{(s)}_{k+s}$ of the sequence

$$\mathbb{F}\Omega_{k+s} \xrightarrow{\varphi^{(s)}_{k+s}} \mathbb{F}\Omega_k \xrightarrow{\varphi^{(t)}_k} \mathbb{F}\Omega_{k-t}. \tag{19}$$

The following two conjectures have been checked for all $n \leq 12$ using MAGMA and the code available from the author's webpage.

Conjecture 7.5. *If $p = 3$ then $\mathbb{F}\Omega_{k+2} \xrightarrow{\varepsilon_{k+2}} \mathbb{F}\Omega_k \xrightarrow{\gamma_k} \mathbb{F}\Omega_{k-1}$ has non-zero homology if and only if $k = \lfloor n/2 \rfloor$. Moreover in the exceptional case H_k is isomorphic to the sign module.*

Taking $n = 2m$, James' p-regularization theorem (see [14]) implies that $\mathrm{sgn} \cong D^{(m,m)}$ when $\mathbb{F}$ has characteristic 3. The analogue of Proposition 3.3 then implies that sgn is a composition factor of $\mathbb{F}\Omega_m$, but not of either $\mathbb{F}\Omega_{m+1}$ or $\mathbb{F}\Omega_{m-2}$. Hence H_m has the sign module as a composition factor. By the argument seen in the proof of Corollary 4.6, a proof of Conjecture 7.5

will categorify the binomial identity

$$(20)\qquad \sum_j \binom{n}{3j} - \sum_j \binom{n}{3j+1} = \begin{cases} (-1)^n & \text{if } n \equiv 0 \bmod 3 \\ 0 & \text{if } n \equiv 1 \bmod 3 \\ (-1)^{n-1} & \text{if } n \equiv 2 \bmod 3. \end{cases}$$

(This identity follows at once from (6.14) and (6.22) in [9], or by adapting the proof of (12) in §4, or most easily, by induction on n.) For example, when $n = 10$ the identity is categorified by the chain complex

$$0 \to \mathbb{F}\Omega_{10} \xrightarrow{\gamma_{10}} \mathbb{F}\Omega_9 \xrightarrow{\varepsilon_9} \mathbb{F}\Omega_7 \xrightarrow{\gamma_7} \mathbb{F}\Omega_6 \xrightarrow{\varepsilon_6} \mathbb{F}\Omega_4 \xrightarrow{\gamma_4} \mathbb{F}\Omega_3 \xrightarrow{\varepsilon_3} \mathbb{F}\Omega_1 \xrightarrow{\gamma_1} \mathbb{F}\Omega_0 \to 0,$$

which is exact in every degree.

Conjecture 7.6. *If $p = 5$ then $\mathbb{F}\Omega_{k+4} \xrightarrow{\varphi^{(4)}_{k+4}} \mathbb{F}\Omega_k \xrightarrow{\gamma_k} \mathbb{F}\Omega_{k-1}$ has non-zero homology if and only if $k \in \{\lfloor n/2 \rfloor, \lfloor n/2 \rfloor - 1\}$. Moreover, if $n = 2m$ is even then $H_{m-1} \cong D^{(m+1,m-1)}$ and $H_m \cong D^{(m,m)}$, and if $n = 2m+1$ is odd then $H_{m-1} \cong D^{(m+2,m-1)}$ and $H_m \cong D^{(m+1,m)}$.*

Again it is straightforward to show that the homology modules have the specified simple modules as composition factors. Somewhat remarkably, the dimensions of these simple modules appear to be certain Fibonacci numbers, as defined by $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. A proof of Conjecture 7.6 will imply that $\dim D^{(m,m)} = F_{2m-1}$ and $\dim D^{(m+1,m-1)} = \dim D^{(m+2,m-1)} = F_{2m}$, and categorify a family of binomial identities including

$$(21)\qquad \sum_j \binom{5m}{5j} - \sum_j \binom{5m}{5j+1} = (-1)^m F_{5m-1}$$

and $\sum_j \binom{5m+2}{5j} - \sum_j \binom{5m+2}{5j+1} = (-1)^{m-1} F_{5m+1}$. These identities are somewhat deeper than (20). Taken together they are equivalent to the identity

$$(22)\qquad F_n = \sum_k (-1)^k \binom{n}{\lfloor \frac{n-1-5k}{2} \rfloor}$$

proved by Andrews in [1] and later, with a simpler inductive proof, by Gupta in [10]. For example, since $\lfloor \frac{10r-2-5k}{2} \rfloor \equiv (-1)^{k-1} \bmod 5$, Andrews' identity implies that $F_{10r-1} = \sum_j \binom{10r-1}{5j-1} - \sum_j \binom{10r-1}{5j+1}$. Since $\binom{5m}{5j} = \binom{5m-1}{5j} + \binom{5m-1}{5j-1}$ and $\binom{5m}{5j+1} = \binom{5m-1}{5j+1} + \binom{5m-1}{5j}$, this is equivalent to (21) when m is even.

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