

AN INTRODUCTION TO THE PARTITION ALGEBRA: SCHUR–WEYL DUALITY, DIAGRAM BASES FOR STANDARD MODULES AND WALKS ON YOUNG’S LATTICE

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1. INTRODUCTION

In these notes we give a motivated introduction to the partition algebra by Schur–Weyl duality, finishing with a categorification (or ‘module-ification’) of the fact that the Bell number B_r , defined as the number of set partitions of $\{1, \dots, r\}$, is equal to the number of r -step walks in Young’s lattice starting and finishing at (r) .

The guiding seesaw. Throughout we are guided by the seesaw diagram below in which E is the n -dimensional natural representation of $\mathrm{GL}_n(\mathbb{F})$: top-left to bottom-right is classical Schur–Weyl duality; the up-ended seesaw from bottom-left to top-right is partition algebra duality between the symmetric group Per_n of $n \times n$ permutation matrices acting diagonally on $E^{\otimes r}$ by $\sigma(v_1 \otimes \dots \otimes v_r) = (\sigma v_1) \otimes \dots \otimes (\sigma v_r)$ and $\mathrm{Par}_r(n)$ acting on the right by the definition in §4.1, as thoroughly motivated by the examples in §3.

$$\begin{array}{ccccc}
 \mathrm{GL}(E) & & & & \mathrm{Par}_r(n) \\
 & \searrow & & \swarrow & \\
 & & E^{\otimes r} & & \\
 & \swarrow & & \searrow & \\
 \mathrm{Per}_n & & & & \mathfrak{S}_r
 \end{array}$$

$\uparrow$ (from Per_n to $\mathrm{GL}(E)$) $\uparrow$ (from $\mathfrak{S}_r$ to $\mathrm{Par}_r(n)$)

The main novel feature of these notes, besides the extensive motivation, is that we work whenever possible over a field $\mathbb{F}$ of arbitrary characteristic. As is often the case, this reveals some interesting dualities that are hidden when working only over the complex numbers.

I hope these notes will be useful to beginners or re-learners and will suggest new ideas. Comments to mark.wildon@rhul.ac.uk are very welcome.

We write S^λ for Specht modules for the left-acting group Per_n , and Sp^λ and Sp_λ for the Specht and dual Specht modules, respectively, for the right-acting place-permutation group $\mathfrak{S}_r$.

Outline.

- We begin in §2 by reviewing classical Schur–Weyl duality from the perspective of someone who has seen some $\mathrm{GL}_n(\mathbb{F})$ -representations but is perhaps less sure how they might embed in $E^{\otimes r}$. We show

that if λ is a partition of r then the maps from the standard module $\Delta^\lambda(E)$ into $E^{\otimes r}$ are acted on by the symmetric group $\mathfrak{S}_r$. This defines a representation of $\mathfrak{S}_r$ that is isomorphic to an ordinary Specht module: in fact, provided $n \geq r$ and $\mathbb{F}$ is infinite,

$$\mathrm{Hom}_{\mathrm{GL}(E)}(\Delta^\lambda(E), E^{\otimes r}) \cong \mathrm{Sp}^\lambda$$

as right $\mathfrak{S}_r$ -modules. In the case when $\mathbb{F}$ has characteristic zero, the right-hand side is simple, and all the non-zero homomorphisms are injective embeddings. This is a prototype for what happens for the partition algebra.

- In §3 we up-end the seesaw and give a very informal construction of the partition algebra $\mathrm{Par}_r(n)$ and some of its standard modules using the left-action of the group Per_n of permutation matrices and the right-action of the partition algebra on $E^{\otimes r}$.
- In §4 we give the general results behind the examples in §3, working as far as possible over an arbitrary field $\mathbb{F}$. Given a partition κ of k and $n \geq k + \kappa_1$, let $\kappa_{[n]}$ denote the partition $(n - k, \kappa_1, \dots, \kappa_{\ell(\kappa)})$. In this up-ended setting, we show that the maps from the Specht module $S^{\kappa_{[n]}}$ into $E^{\otimes r}$ are acted on by the partition algebra $\mathrm{Par}_r(n)$ on the right. This defines a representation of $\mathrm{Par}_r(n)$: in fact the standard cell module Δ_r^κ for $\mathrm{Par}_r(n)$ indexed by a partition κ with $|\kappa| \leq r$ may be *defined* by

$$\mathrm{Hom}_{\mathrm{Per}_n}(S^{\kappa_{[n]}}, E^{\otimes r}) \cong \Delta_r^\kappa$$

provided $n \geq r + \kappa_1$ and $\mathrm{char} \mathbb{F} \neq 2$. (See Definition 4.7 and Theorem 4.6.) This gives some flavour of how the partition algebra ‘forgets the first row’ of partitions. When $\mathbb{F}$ has characteristic zero, the right-hand side is simple, and all the non-zero homomorphisms are injective embeddings.

- In §5 we finish with the result on Bell numbers already mentioned: in fact this is the case $\kappa = \emptyset$ and $r = n$ of the more general result in Corollary 5.6.

2. MOTIVATION FROM CLASSICAL SCHUR–WEYL DUALITY

In classical Schur–Weyl duality we start with the n -dimensional natural representation E of the general linear group $\mathrm{GL}_n(\mathbb{F})$ over an infinite field $\mathbb{F}$, and take the diagonal action of this group on $E^{\otimes r}$, defined by linear extension of

$$g(v_1 \otimes \cdots \otimes v_r) = gv_1 \otimes \cdots \otimes gv_r.$$

This makes $E^{\otimes r}$ a polynomial representation of $\mathrm{GL}(E)$ of polynomial degree r . (Note that since $\mathbb{F}$ is infinite, ‘polynomial degree’ is a sensible notion.) In characteristic zero we can decompose $E^{\otimes r}$ as a direct sum of irreducible representations of $\mathrm{GL}_n(\mathbb{F})$, in which every polynomial representation of degree r appears.

2.1. Small degree examples. In the case $r = 2$, when $\text{char } \mathbb{F} \neq 2$, this becomes the familiar decomposition of the tensor square into its symmetric and antisymmetric parts:

$$\begin{aligned} E \otimes E &= \text{Sym}_2 E \oplus \bigwedge^2 E \\ &= \langle v \otimes w + w \otimes v : v, w \in E \rangle \oplus \langle v \otimes w - w \otimes v : v, w \in E \rangle. \end{aligned} \quad (2.1)$$

Notice that each summand is invariant under the right place-permutation action of the symmetric group $\mathfrak{S}_2$: since $\text{Sym}_2 E$ is the space of invariants for this action, the action on this summand is trivial, while on $\bigwedge^2 E$, the symmetric group acts as the sign representation. We have therefore shown that, when $\text{char } \mathbb{F} \neq 2$,

$$E^{\otimes 2} \cong (\text{Sym}_2 E \otimes \mathbb{F}) \oplus (\bigwedge^2 E \otimes \text{sgn}) \quad (2.2)$$

as a bimodule acted on by $\text{GL}(E)$ on the left and $\mathfrak{S}_2$ on the right.¹

Remark 2.1 (What is different in even characteristic). It is deliberate to write $\text{Sym}_2 E$ rather than $\text{Sym}^2 E$; the latter is the space of co-invariants, defined instead as the quotient of $E \otimes E$ by the subspace $\bigwedge^2 E$, and so not naturally a subspace at all. Indeed the canonical map

$$\text{Sym}^2 E \rightarrow \text{Sym}_2 E$$

defined for v and $w \in E$ by $vw \mapsto v \otimes w + w \otimes v$ is not injective in characteristic 2, because the squares v^2 for $v \in E$ span its kernel. In this case, the Frobenius twist $\langle v^2 : v \in E \rangle$ is the n -dimensional simple module of highest weight (2), usually denoted $L^{(2)}(E)$, and in the standard notation for standard and co-standard modules we have

$$\text{Sym}_2 E \cong \Delta^{(2)}(E) \cong \frac{L^{(2)}(E)}{L^{(1,1)}(E)}, \quad \text{Sym}^2 E \cong \nabla^{(2)}(E) \cong \frac{L^{(1,1)}(E)}{L^{(2)}(E)}$$

where $L^{(1,1)}(E) \cong \bigwedge^2 E$ is simple. Moreover the short exact sequence $0 \rightarrow \text{Sym}_2 E \rightarrow E \otimes E \rightarrow \bigwedge^2 E \rightarrow 0$ is non-split and

$$E \otimes E \cong \frac{\bigwedge^2 E}{\text{Sym}_2 E} \cong \frac{\text{Sym}^2 E}{\bigwedge^2 E} \cong \frac{L^{(1,1)}}{L^{(2)}} \cdot \frac{L^{(2)}}{L^{(1,1)}}.$$

More generally, $\Delta^\lambda(E)$ has a simple top, its polynomial dual $\nabla^\lambda(E)$ has a simple socle, each isomorphic to $L^\lambda(E)$, and all other composition factors of either module are certain $L^\mu(E)$ for μ strictly dominated by λ . This is one explanation for why, whether we construct $\bigwedge^r E$ as a submodule or as a quotient of $E^{\otimes r}$, we always get the same irreducible representation, in arbitrary characteristic. Even characteristic is no exception!

¹**Forgetful functors:** for example, if we forget the left $\text{GL}(E)$ action on $\text{Sym}_2 E \otimes \mathbb{F}$ leaving only the right $\mathfrak{S}_2$ action then we get $\mathbb{F} \oplus \dots \oplus \mathbb{F} \cong \mathbb{F}^{\dim \text{Sym}_2(E)}$, the direct sum of $\dim \text{Sym}_2 E$ copies of the trivial representation. If we forget the right $\mathfrak{S}_2$ action we get $\text{Sym}_2 E$ regarded just as a $\text{GL}(E)$ -representation, where we started in (2.1).

Assuming that $n \geq 3$, so that $\bigwedge^3 E$ appears in $E \otimes E \otimes E$, it is fairly easy to see that we still have unique copies of $\text{Sym}_3 E$ and $\bigwedge^3 E$ in $E^{\otimes 3}$, but since the sum of their dimensions is

$$\frac{n(n+1)(n+2)}{3!} + \frac{n(n-1)(n-2)}{3!} = \frac{n(n^2+2)}{3} = n^3 - \frac{2}{3}n(n-1)(n+1)$$

something is missing! In characteristic zero the missing representation is *two* copies of the standard module $\Delta^{(2,1)}(E)$, each of dimension $\frac{1}{3}n(n-1)(n+1)$. We show this by first constructing one copy, and then using the right place permutation action of $\mathfrak{S}_3$ to push it around in $E \otimes E \otimes E$, thus finding a second copy and constructing the Specht module $\text{Sp}^{(2,1)}$. In characteristic zero this is the third irreducible representation of $\mathfrak{S}_3$, after the trivial and sign representations. For the first copy, we use the construction in my note [8], later done in a much better way so as to lead to an explicit presentation of the module by McDowell in [6].

Example 2.2 (Construction of maps from $\Delta^{(2,1)}(E)$). Fix the canonical basis $e_1, \dots, e_n$ of E and consider

$$\bigwedge^2 E \otimes E \subseteq E \otimes E \otimes E$$

which contains the highest weight vector $e_1 \wedge e_2 \otimes e_1$, fixed by the upper-unitriangular subgroup, and hence annihilated by the Lie algebra action of strictly upper-triangular matrices.² Applying elements of $\text{GL}(E)$ we get further vectors, including all

$$e_i \wedge e_j \otimes e_i, \quad e_i \wedge e_j \otimes e_k + e_k \wedge e_j \otimes e_i,$$

$$\begin{array}{|c|c|} \hline i & k \\ \hline j & \\ \hline \end{array}$$

which span the standard module $\Delta^{(2,1)}(E)$ for $\text{GL}(E)$. In fact it suffices to take i, j and k with $i < j$ and $i \leq k$, corresponding to the semistandard condition on the $(2,1)$ -tableau shown in the margin. Thus $\Delta^{(2,1)}(E)$ is generated by its unique up-to-scalars highest-weight vector of weight $(2,1)$.³ Written out as tensors, we have constructed $\Delta^{(2,1)}(E)$ as the subspace of $E^{\otimes 3}$ spanned by all $(e_i \otimes e_j - e_j \otimes e_i) \otimes e_i$ and all $(e_i \otimes e_j - e_j \otimes e_i) \otimes e_k + (e_k \otimes e_j - e_j \otimes e_k) \otimes e_i$.

Using that the place permutation action of the symmetric group $\mathfrak{S}_3$ commutes with the action of $\text{GL}(E)$, we can get two further copies of $\Delta^{(2,1)}(E)$ by applying the cyclic place permutation $1 \mapsto 2 \mapsto 3 \mapsto 1$. Each copy of $\Delta^{(2,1)}(E)$ is isomorphic to the original one, and so has a unique up-to-scalars

²**Highest-weight vectors.** See [1, §2.5] or, in characteristic zero, [4, Part VI] for the wider theory. The vector is an eigenvector for the diagonal matrices in $\text{GL}_n(\mathbb{C})$ of weight $(2,1)$ and is fixed by every upper-unitriangular matrix. For example,

$$\begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix} (e_1 \wedge e_2 \otimes e_1) = e_1 \wedge (\gamma e_1 + e_2) \otimes e_1 = e_1 \wedge e_2 \otimes e_1.$$

In characteristic zero one can simplify calculations by using the Lie algebra action; see [7, §4.2, reduction] for a technical trick that permits the Lie algebra to be used even in prime characteristic.

³**More generally:** $\Delta^\lambda(E)$ is generated by a highest-weight vector of weight λ whatever the characteristic of $\mathbb{F}$. This was seen in Remark 2.1 for $\Delta^{(2)}(E) \cong \text{Sym}_2 E$.

highest-weight vector of weight $(2, 1)$, which is a generator in arbitrary characteristic, namely

$$\begin{aligned} e_1 \wedge e_2 \otimes e_1 &= e_1 \otimes e_2 \otimes e_1 - e_2 \otimes e_1 \otimes e_1, \\ e_1 \otimes e_1 \wedge e_2 &= e_1 \otimes e_1 \otimes e_2 - e_1 \otimes e_2 \otimes e_1, \\ &\quad - e_1 \otimes e_1 \otimes e_2 + e_2 \otimes e_1 \otimes e_1. \end{aligned} \quad (2.3)$$

The third, in which I deliberately introduced a negation, has no convenient wedge product notation because the alternating part is split over the first and third tensor factors. By our dimension count, we know that we want only one new copy of $\Delta^{(2,1)}(E)$, and correspondingly these three highest-weight vectors are *not* linearly independent: indeed their sum is zero. The space of highest weight vectors is isomorphic to the Specht module $\text{Sp}^{(2,1)}$ constructed in short-hand notation as $\{\alpha e_{121} + \beta e_{112} + \gamma e_{211} : \alpha + \beta + \gamma = 0\}$. (This is the usual Specht module as constructed by James as right $\mathfrak{S}_r$ -module in [5].) Moreover, writing ψ for the map from $\Delta^{(2,1)}(E)$ into $E \otimes E \otimes E$ defined by $e_1 \wedge e_2 \otimes e_1 \mapsto e_1 \otimes e_2 \otimes e_1 - e_2 \otimes e_1 \otimes e_1$, we have shown that

$$\text{Hom}_{\text{GL}(E)}(\Delta^{(2,1)}(E), E^{\otimes 3}) = \langle \psi, \psi(1, 2, 3) \rangle \cong \text{Sp}^{(2,1)}$$

is a 2-dimensional module for the right action of $\mathfrak{S}_3$, isomorphic to $\text{Sp}^{(2,1)}$.

We can now get the $r = 3$ analogue of (2.2). Indeed, arguing as in the case $r = 2$ for the summands $\text{Sym}_3 E$ and $\bigwedge^3 E$ that appear just once, we conclude that, when $\mathbb{F}$ has characteristic zero and so $E^{\otimes 3}$ is completely reducible as a $\text{GL}(E)$ -module, there is an isomorphism

$$\begin{aligned} E^{\otimes 3} &\cong (\Delta^{(3)}(E) \otimes \text{Sp}^{(3)}) \oplus (\Delta^{(2,1)}(E) \otimes \text{Sp}^{(2,1)}) \oplus (\Delta^{(1,1,1)}(E) \otimes \text{Sp}^{(1,1,1)}) \\ &= (\text{Sym}_3 E \otimes \mathbb{F}) \oplus (\Delta^{(2,1)}(E) \otimes \text{Sp}^{(2,1)}) \oplus (\bigwedge^3 E \otimes \text{sgn}_{\mathfrak{S}_3}) \end{aligned} \quad (2.4)$$

of $E^{\otimes 3}$ as a left $\text{GL}(E)$ - and right $\mathfrak{S}_3$ -bimodule. Each summand above is irreducible as a bimodule. In particular, we have recovered the trivial and sign representations of $\mathfrak{S}_3$.

Exercise 2.3. Assume that $\mathbb{F}$ is infinite and $n \geq 3$. In Example 2.2 we saw the map $\phi : \bigwedge^2 E \otimes E \rightarrow E^{\otimes 3}$ defined by

$$\phi(e_i \wedge e_j \otimes e_k) = (e_i \otimes e_j - e_j \otimes e_i) \otimes e_k,$$

and its restriction ϕ to $\Delta^{(2,1)}(E) \subseteq \bigwedge^2 E \otimes E$.

- (i) Show that the embeddings ϕ , $\phi(1, 2, 3)$, $\phi(1, 3, 2)$ are linearly independent and span $\text{Hom}_{\text{GL}(E)}(\bigwedge^2 E \otimes E, E \otimes E \otimes E)$.
- (ii) Let $M = \langle v_1, v_2, v_3 \rangle$ be the natural permutation representation of $\mathfrak{S}_3$. Show that $\text{Hom}_{\text{GL}(E)}(\bigwedge^2 E \otimes E, E \otimes E \otimes E) \cong M \otimes \text{sgn}_{\mathfrak{S}_3}$.
- (iii) Show that the restriction map $\text{Hom}_{\text{GL}(E)}(\bigwedge^2 E \otimes E, E \otimes E \otimes E) \rightarrow \text{Hom}_{\text{GL}(E)}(\Delta^{(2,1)}(E), E \otimes E \otimes E)$ is surjective with a 1-dimensional kernel isomorphic to $\text{sgn}_{\mathfrak{S}_3}$, and deduce that

$$\text{Hom}_{\text{GL}(E)}(\Delta^{(2,1)}(E), E \otimes E \otimes E) \cong \frac{\langle v_1, v_2, v_3 \rangle}{\langle v_1 + v_2 + v_3 \rangle} \otimes \text{sgn}_{\mathfrak{S}_3}$$

(iv) Identify the right-hand side with the Specht module $\text{Sp}^{(2,1)}$.⁴

2.2. The view from the general linear group side of the seesaw.

Imagine you are an expert on the general linear group (*known world*, left side of seesaw), but less familiar with the symmetric group (*unknown world*, right side of seesaw). Then this method for constructing symmetric group representations out of general linear group representations would seem very natural. Indeed, this is how Schur developed the subject in his thesis, although his Schur functor picks out the $(1, \dots, 1, 0, \dots, 0)$ -weight space for a particular $\Delta^\lambda(E)$, rather than, as above, the space of all λ -highest-weight vectors. (It is notable that our construction using highest weight vectors above *does not* require $n \geq r$; our construction is closer to the analogue for partition algebra.) When we up-end the seesaw, the analogous procedure is to construct representations of the partition algebra (*unknown world*, right-side of seesaw) as hom-spaces embedding representations of the symmetric group subgroup $\mathfrak{S}_n$ of the general linear group $\text{GL}(E)$ (*known world*, left-side of seesaw) $E^{\otimes r}$. This should I hope now seem more reasonable!

2.3. Embedding symmetric powers. We now consider a variation on this theme that motivates the construction of the partition algebra module $\Delta_r^{(2,1)}$ in §3.2. The natural embeddings of $\bigwedge^2 E \otimes E$ into $E^{\otimes 3}$ are controlled by the signed permutation module $M^{(2,1)} \otimes \text{sgn}$. For our application (see in particular Proposition 4.5) we instead need embeddings of $\text{Sym}_2 E \otimes E$. These have the similar pattern ‘two tensor positions symmetrized, the third distinguished’, but without the sign. The three natural embeddings are linearly independent in arbitrary characteristic: their values on $(e_1 \otimes e_2 + e_2 \otimes e_1) \otimes e_3$ are the disjointly supported tensors

$$\begin{aligned} (e_1 \otimes e_2 + e_2 \otimes e_1) \otimes e_3 &= e_{123} + e_{213}, \\ e_3 \otimes (e_1 \otimes e_2 + e_2 \otimes e_1) &= e_{312} + e_{321}, \\ e_2 \otimes e_3 \otimes e_1 + e_1 \otimes e_3 \otimes e_2 &= e_{231} + e_{132}. \end{aligned} \tag{2.5}$$

The maps themselves are permuted by the right action of $\mathfrak{S}_3$ as $M^{(2,1)}$. We now use these maps to embed the Young permutation module $M^{(n-3,2,1)}$ and the left Specht module $S^{(n-3,2,1)}$ into tensor space.

Notation. Note that $M^{(n-3,2,1)}$ and $S^{(n-3,2,1)}$ are modules for Per_n , the group of $n \times n$ permutation matrices. We write S rather than Sp for the Specht module to make this distinction between the left-action of Per_n and

⁴**Solution:** (i) is routine; for (ii) since $\phi(1, 2) = -\phi$, an explicit map is defined by $\mathfrak{S}_3$ -extension of $\phi \mapsto v_3 \otimes 1$. For (iii) we use that $\Delta^{(2,1)}(E)$ is generated by its highest weight vector $e_1 \wedge e_2 \otimes e_1$ and the linear dependency in (2.3) to see that $\phi + \phi(1, 2, 3) + \phi(1, 3, 2)$ is zero on restriction to $\Delta^{(2,1)}(E)$; equivalently $\phi + \phi(1, 2, 3) + \phi(1, 3, 2) = 0$. For (iv), write $S = \{\sum_i a_i v_i : \sum_i a_i = 0\} \cong \text{Sp}^{(2,1)}$. Using the bilinear form on M in which the v_i are orthonormal we have a surjective map $M \rightarrow S^*$ defined by linear extension $v_i \mapsto \langle v_i, - \rangle$; the kernel is $v_1 + v_2 + v_3$. This proves the required isomorphism in every characteristic. This isomorphism is the case $\lambda = (2, 1)$ of the more general result $(\text{Sp}^\lambda)^* \otimes \text{sgn} \cong \text{Sp}^{\lambda'}$: see [5, Theorem 8.15].

the right action of $\mathfrak{S}_r$. See Remark 3.1 for why it is helpful to write Per_n rather than the isomorphic group $\mathfrak{S}_n$ in this context.

Example 2.4. For $n \geq 3$, the Young permutation module $M^{(n-3,2,1)}$ can be defined as the permutation module for Per_n with permutation basis all ordered set partitions of $\{1, \dots, n\}$ into sets of sizes $n-3$, 2 and 1. In any such set partition, the first part of size $n-3$ is determined by the other two parts, so we can identify $M^{(n-3,2,1)}$ with ordered set partitions of a 3-subset of $\{1, 2, \dots, n\}$ into parts of sizes 2 and 1: thus a general element of the canonical basis is $(\{i, j\}, \{k\})$ with $i, j, k \in \{1, \dots, n\}$ distinct.⁵ Each of the three embeddings of $\text{Sym}_2 E \otimes E$ labelled by the vectors in (2.5) gives an embedding of $M^{(n-3,2,1)}$ into $E^{\otimes 3}$: for instance, the first with pattern ‘symmetrize the first two tensor positions’ defines

$$(\{i, j\}, \{k\}) \mapsto (e_i \otimes e_j + e_j \otimes e_i) \otimes e_k \quad (2.6)$$

Writing ϕ for this map, the three maps ϕ , $\phi(1, 2, 3)$ and $\phi(1, 3, 2)$ are a basis for a subspace of $\text{Hom}_{\text{Per}_n}(M^{(n-3,2,1)}, E^{\otimes 3})$ isomorphic to $M^{(2,1)}$. (There are other homomorphisms from $M^{(n-3,2,1)}$ into $E^{\otimes 3}$, for example $(\{i, j\}, \{k\}) \mapsto (e_i + e_j) \otimes (e_i + e_j) \otimes e_k$, and the non-injective $(\{i, j\}, \{k\}) \mapsto v \otimes v \otimes v$ where $v = e_1 + \dots + e_n$, but ϕ , $\phi(1, 2, 3)$ and $\phi(1, 3, 2)$ are the correct ones to consider for the next step.) Now suppose that $n \geq 5$ and we restrict these embeddings to the submodule $S^{(n-3,2,1)}$ of $M^{(n-3,2,1)}$. This is the module linearly spanned by all column antisymmetrizations

$$\sum_{\tau \in \mathfrak{S}_{\{a,i,k\}} \times \mathfrak{S}_{\{b,j\}}} \text{sgn}(\tau) \tau(\{i, j\}, \{k\}).$$

To remove some of the mystery, this is the polytabloid e_s where

$$s = \begin{array}{|c|c|c|} \hline a & b & c \\ \hline i & j & \\ \hline k & & \\ \hline \end{array} \quad \cdots \quad \begin{array}{|c|} \hline d \\ \hline \end{array} \quad (2.7)$$

The symmetric group Per_n acts on tableaux with entries from $\{1, \dots, n\}$ in the obvious way. Since $\sigma(e_s) = e_{\sigma(s)}$, under this action $S^{(n-3,2,1)}$ is cyclic, generated by any single polytabloid. Thus the image of any single e_t determines a map in $\text{Hom}_{\text{Per}_n}(S^{(n-3,2,1)}, E^{\otimes 3})$. To make things more concrete we shall fix the $(n-3, 2, 1)$ -tableau

$$t = \begin{array}{|c|c|c|} \hline 4 & 5 & 6 \\ \hline 1 & 2 & \\ \hline 3 & & \\ \hline \end{array} \quad \cdots \quad \begin{array}{|c|} \hline n \\ \hline \end{array}$$

Observe that e_t may be written by summing over coset representatives for $\mathfrak{S}_{\{1,3\}}$ in $\mathfrak{S}_{\{1,3,4\}} \times \mathfrak{S}_{\{2,5\}}$ as

$$e_t = f_{123} + f_{421} + f_{324} - f_{153} - f_{451} - f_{354} \quad (2.8)$$

where

$$f_{ijk} = (\{i, j\}, \{k\}) - (\{k, j\}, \{i\}). \quad (2.9)$$

⁵**Tabloid notation:** this corresponds to removing the redundant first row in a tabloid.

When we map f_{ijk} into $E^{\otimes 3}$ using the first embedding $\phi : M^{(n-3,2,1)} \rightarrow E^{\otimes 3}$ we get

$$\begin{aligned}\phi(f_{ijk}) &= (e_i \otimes e_j + e_j \otimes e_i) \otimes e_k - (e_k \otimes e_j + e_j \otimes e_k) \otimes e_i \\ &= e_{ijk} + e_{jik} - e_{kji} - e_{jki}.\end{aligned}$$

As seen earlier for $\Delta^{(2,1)}(E)$, the sum of the three embeddings ϕ , $\phi(1, 2, 3)$, $\phi(1, 3, 2)$ applied to f_{ijk} vanishes, this time because

$$(e_{ijk} + e_{jik} - e_{kji} - e_{jki}) + (e_{kij} + e_{kji} - e_{ikj} - e_{ijk}) + (e_{jki} + e_{ikj} - e_{jik} - e_{kij}) = 0.$$

The same relation holds for all the other summands of e_t in (2.8): this can be seen as an application of the left action of Per_n commuting with the right action of $\mathfrak{S}_3$. The other summands have distinct summands, so are linearly independent. *Thus homomorphisms from $S^{(n-3,2,1)}$ are determined by the ‘gadget’ f_{ijk} defined in (2.9).* Exactly as seen earlier in Exercise 2.3 for the $\Delta^{(2,1)}(E)$ embeddings, where we restricted from $\bigwedge^2 E \otimes E$ to $\Delta^{(2,1)}(E)$, restricting the domain from $M^{(n-3,2,1)}$ to $S^{(n-3,2,1)}$ defines a map

$$\text{Hom}_{\text{Per}_n}(M^{(n-3,2,1)}, E^{\otimes 3}) \rightarrow \text{Hom}_{\text{Per}_n}(S^{(n-3,2,1)}, E^{\otimes 3})$$

that is not injective. We have just seen that $\phi + \phi(1, 2, 3) + \phi(1, 3, 2)$ is in its kernel. (The kernel also contains other maps from $M^{(n-3,2,1)}$ into $E^{\otimes 3}$, for example, the embedding $(\{i, j\}, \{k\}) \mapsto (e_i + e_j) \otimes (e_i + e_j) \otimes e_k$ mentioned above.) Except that there is no tensor factor of $\text{sgn}_{\mathfrak{S}_3}$, we get the same description of the hom-space from $S^{(n-3,2,1)}$ as we did before for $\Delta^{(2,1)}(E)$ in Exercise 2.3(iv):

$$\text{Hom}_{\text{Per}_n}(S^{(n-3,2,1)}, E^{\otimes 3}) = \frac{\langle \phi, \phi(1, 2, 3), \phi(1, 3, 2) \rangle}{\langle \phi + \phi(1, 2, 3) + \phi(1, 3, 2) \rangle} \cong \frac{M^{(2,1)}}{\langle v_1 + v_2 + v_3 \rangle} \quad (2.10)$$

This is a standard construction of the dual right Specht module $\text{Sp}_{(2,1)}$. We conclude that $\text{Hom}_{\text{Per}_n}(S^{(n-3,2,1)}, E^{\otimes 3}) \cong \text{Sp}_{(2,1)}$ as a right $\mathfrak{S}_3$ -module, and that this holds in *arbitrary* characteristic.

In Proposition 4.5 we prove the generalizations: if κ is a partition of k and $n \geq k + \kappa_1$, then $\text{Hom}_{\text{Per}_n}(M^{\kappa[n]}, E^{\otimes k})$ has a submodule isomorphic to M^κ as a right $\mathfrak{S}_k$ -module, and $\text{Hom}_{\text{Per}_n}(S^{\kappa[n]}, E^{\otimes k}) \cong \text{Sp}_\kappa$ as right $\mathfrak{S}_k$ -modules.

Exercise 2.5. Observe that the embeddings of $M^{(n-3,2,1)}$ into $E^{\otimes 3}$ used in Example 2.4 are into its anti-diagonal subspace $\langle e_i \otimes e_j \otimes e_k : i, j, k \text{ distinct} \rangle$. Show that the embedding defined by

$$(\{i, j\}, \{k\}) \mapsto (e_i + e_j) \otimes (e_i + e_j) \otimes e_k$$

can be decomposed as the sum of an anti-diagonal embedding and the embedding $(\{i, j\}, \{k\}) \mapsto (e_i \otimes e_i + e_j \otimes e_j) \otimes e_k$, and that this second embedding becomes zero on restriction to $S^{(n-3,2,1)}$.⁶

⁶**Outline solution:** the relevant anti-diagonal embedding is ϕ since $(e_i + e_j) \otimes (e_i + e_j) \otimes e_k = (e_i \otimes e_j + e_j \otimes e_i) \otimes e_k + (e_i \otimes e_i + e_j \otimes e_j) \otimes e_k$ where the first summand is $(\{i, j\}, \{k\})\phi$. Writing f'_{ijk} for $(e_i \otimes e_i + e_j \otimes e_j) \otimes e_k$, it follows from the form of (2.8)

3. DISCOVERING THE PARTITION ALGEBRA BY SCHUR–WEYL DUALITY

Up-ending the seesaw we now restrict from the general linear group $\mathrm{GL}(E)$ to its subgroup of permutation matrices Per_n . The action of the group remains diagonal and since $\sigma(e_i) = e_{\sigma(i)}$, it is given explicitly on the canonical basis by

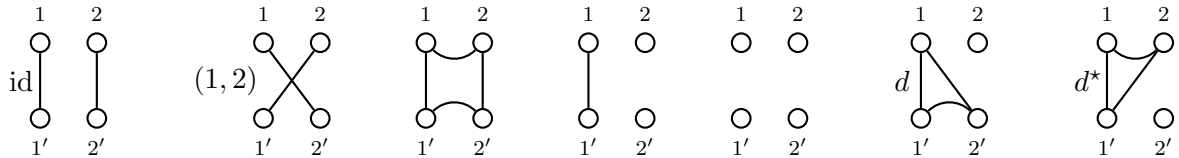
$$\sigma(e_{i_1} \otimes \cdots \otimes e_{i_r}) = e_{\sigma(i_1)} \otimes \cdots \otimes e_{\sigma(i_r)} \quad (3.1)$$

After this restriction $E = \langle e_1, \dots, e_n \rangle$ is the natural permutation representation for Per_n .

Remark 3.1. Please compare (3.1) with the place permutation action of $\mathfrak{S}_r$ on the right, exemplified by $(e_i \otimes e_j \otimes e_k)(1, 2, 3) = e_k \otimes e_i \otimes e_j$. To make the distinction clearer, we have chosen to write Per_n for the symmetric group of permutation matrices. In Per_n , which acts on the left on $E^{\otimes r}$, permutations compose right-to-left as for matrix multiplication; in $\mathfrak{S}_r$, which acts on the right on $E^{\otimes r}$, permutations compose left-to-right. Thus in Per_n , σ and τ stand for permutation matrices and $\sigma\tau$ means ‘do τ then do σ ’, the opposite order to $\mathfrak{S}_r$.

In classical Schur–Weyl duality over an infinite field, the centralizer of $\mathrm{GL}(E)$ on $E^{\otimes r}$ is the image of $\mathbb{F}\mathfrak{S}_r$ acting on the right; this group-algebra action is faithful when $n \geq r$. Now we have replaced $\mathrm{GL}(E)$ with its much smaller finite subgroup Per_n , the centralizer will become a bigger algebra, which must contain $\mathfrak{S}_r$. It is the partition algebra $\mathrm{Par}_r(n)$. In this section I am going to be somewhat ambitious and attempt to define the action of the partition algebra $\mathrm{Par}_r(n)$ on $E^{\otimes r}$ and some of its standard modules all at the same time. (See Proposition 4.5 in §4 for a concise statement of the results we motivate here.) If all this is mostly new to you, I suggest you see the example below; see also the solutions to Exercise 3.4 and Exercise 3.5 for a briefer reminder.

Example 3.2. Let δ be an element of the field $\mathbb{F}$. By definition, $\mathrm{Par}_r(\delta)$ is the algebra with basis all set partitions of the set $\{1, \dots, r\} \cup \{1', \dots, r'\}$ and the multiplication defined below. We refer to the canonical basis elements as *diagrams* because of the diagrammatic interpretation shown below for six basis elements of $\mathrm{Par}_2(\delta)$. The notation d^* indicates the reflection of d in the horizontal axis that swaps primed and unprimed elements.

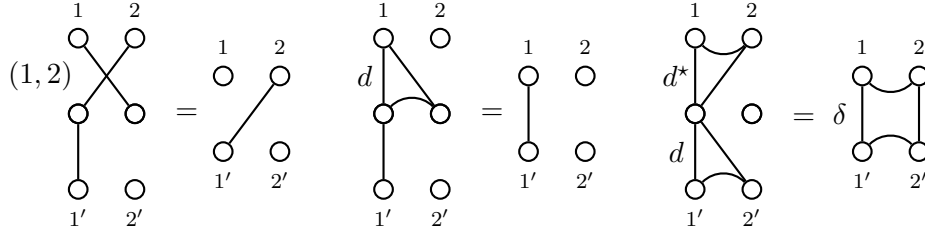


that the image of e_t under the second embedding

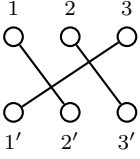
$$(f'_{123} - f'_{153}) + (f'_{421} - f'_{451}) + (f'_{324} - f'_{354})$$

in which the coefficient of $e_1 \otimes e_1 \otimes e_3$ is $1 - 1 + 0 + 0 + 0 + 0$ vanishes. Similarly rebracketing, or a symmetry argument, shows that the image of e_t is in fact zero. The proof of Proposition 4.5 outlines a more conceptual reason for this cancellation.

The product de of diagrams d and e is, up to a scalar, the diagram defined by concatenating d above e so that the primed numbers in d are identified with unprimed numbers in e , as shown in the products below. We then put labels $a, b \in \{1, \dots, r, 1', \dots, r'\}$ in the same part if and only if there is a route from a to b in the concatenated diagram. Note that here either or both a and b may be primed numbers. The first example below in $\text{Par}_2(n)$ shows the product $\{\{1, 2'\}, \{2, 1'\}\} \{\{1, 1'\}, \{2\}, \{2'\}\} = \{\{1\}, \{2, 1'\}, \{2'\}\}$; already I hope you are preferring to think with diagrams!



The scalar is always a power δ^c of the parameter δ ; the exponent c is the number of components in the concatenated diagram not connected to any top or bottom dot. For instance, in the final product above, the concatenated diagram for d^*d has a single isolated component, and so the scalar is δ .



Note that by our convention diagrams should be read top-to-bottom: for example the diagram in the margin shows the 3-cycle $(1, 2, 3)$ in the subalgebra of $\text{Par}_3(n)$ isomorphic to $\mathbb{C}\mathfrak{S}_3$.

Exercise 3.3. Let $\mathfrak{S}_r$ act on $\{1', \dots, r'\}$ in the obvious way. Show that the diagrams in $\text{Par}_r(n)$ of the special form

$$\{\{1, 1'\sigma\}, \dots, \{r, r'\sigma\}\}$$

for $\sigma \in \mathfrak{S}_r$ are a basis for a subalgebra of $\text{Par}_r(n)$ isomorphic to $\mathbb{F}\mathfrak{S}_r$.⁷

3.1. The modules $\text{Hom}_{\text{Per}_n}(S^{(n-1,1)}, E^{\otimes 2})$ and $\Delta_2^{(1)}$. Throughout this subsection assume that $\mathbb{F}$ has characteristic zero and $n \geq 2$. Let $v = e_1 + \dots + e_n$ so that $\langle v \rangle$ affords the trivial representation of Per_n . Following the method from §2.1 of trying to understand all embeddings of an important module into tensor space, we begin with the embedding ϕ of the Specht module $S^{(n-1,1)}$ into $E \otimes E$ defined by restricting

$$e_i \xrightarrow{\phi} e_i \otimes (e_1 + \dots + e_n) = e_i \otimes v$$

from E to its submodule $S^{(n-1,1)} = \langle e_i - e_j : 1 \leq i < j \leq n \rangle$. Thus the image of the generator $e_1 - e_2$ of $S^{(n-1,1)}$ is $(e_1 - e_2) \otimes v$. Whatever bigger algebra we can use to ‘push around’ the embedding ϕ , it clearly contains $\mathfrak{S}_2$ acting by place permutation; this gives one obvious further embedding, namely $\phi(1, 2)$. If you solve the next exercise for yourself you will get some idea of what further elements exist in $\text{Par}_2(n)$.

⁷**Solution:** We check the multiplication. Writing d_σ for the diagram for σ , the labels i and j' are in the same part of $d_\sigma d_\tau$ if and only if there exists k such that $\{i, k'\}$ is a part of d_σ and $\{k, j'\}$ is a part of d_τ ; equivalently $k = i\sigma$ and $j = k\tau$, so $j = i\sigma\tau$ and $d_\sigma d_\tau = d_{\sigma\tau}$.

Exercise 3.4. Assume that $n \geq 3$ and that $\mathbb{F}$ has characteristic zero.

- (i) Find another embedding ψ of E into $E \otimes E$ in which the image of each e_i is a pure tensor product. Restrict it to $S^{(n-1,1)}$ to get a third linearly independent embedding of $S^{(n-1,1)}$.
- (ii) What is the image of $e_1 - e_2 \in S^{(n-1,1)}$ under this new embedding?
- (iii) Is $\phi, \phi(1, 2), \psi$ a basis for $\text{Hom}_{\text{Per}_n}(S^{(n-1,1)}, E^{\otimes 2})$?
- (iv) By the hoped for Schur–Weyl duality between Per_n and $\text{Par}_r(n)$, the image of $\text{Par}_2(n)$ in the centralizer should act irreducibly on $\text{Hom}_{\text{Per}_n}(S^{(n-1,1)}, E \otimes E)$. In particular, by density, there should exist an element of the partition algebra $\text{Par}_2(n)$ sending the first embedding $e_1 - e_2 \mapsto (e_1 - e_2) \otimes v$ to your new embedding and another element going back. Find such elements. [*Hint: several single diagrams work!*]⁸

To avoid confusion in the following exercise, please remember we compose partition algebra elements left-to-right, consistent with the right action on the homomorphism space; correspondingly as seen in Example 3.2 and Exercise 3.3, to compute a product dd' , the diagram d is put on top of d' .

⁸**Solution:** Another embedding of E into $E \otimes E$ is defined by $e_i \mapsto e_i \otimes e_i$, restricting to an embedding $S^{(n-1,1)} \rightarrow E \otimes E$ defined on the generator $e_1 - e_2$ by

$$e_1 - e_2 \mapsto e_1 \otimes e_1 - e_2 \otimes e_2$$

answering (i) and (ii). Provided $n \geq 3$, this vector is linearly independent of the images $(e_1 - e_2) \otimes v$ and $v \otimes (e_1 - e_2)$ already considered; for instance look at the coefficients of $e_2 \otimes e_3$ and $e_3 \otimes e_2$ in the three images. (See Exercise 3.6 for what goes wrong if $n = 2$.) A direct calculation, or an application of Proposition 5.4 below, shows that

$$E \otimes E \cong \mathbb{F} \oplus \mathbb{F} \oplus S^{(n-1,1)} \oplus S^{(n-1,1)} \oplus S^{(n-1,1)} \oplus S^{(n-2,2)} \oplus S^{(n-2,1,1)}$$

where $S^{(n-2,2)}$ should be omitted if $n = 3$. In particular $S^{(n-1,1)}$ has multiplicity 3 in $E^{\otimes 2}$. Therefore the answer to (iii) is ‘yes’: we have a basis for $\text{Hom}_{\text{Per}_n}(S^{(n-1,1)}, E \otimes E)$. For (iv), comparing the embeddings defined by $e_i \mapsto e_i \otimes v$ and $e_i \mapsto e_i \otimes e_i$, it seems we want a linear combination of diagrams $x \in \text{Par}_2(n)$ satisfying

$$(e_i \otimes e_j)x = e_i \otimes e_i. \quad (3.2)$$

In fact we can take x to be the single diagram $\{\{1, 1', 2'\}, \{2\}\}$ shown top right in the margin, where I read the first as ‘copy position 1 to positions 1’ and 2’, ignoring position 2’. (See §4.1 for the general formulation.) This x satisfies (3.2) for all $1 \leq i, j \leq n$, and so $(e_i \otimes v)x = ne_i \otimes e_i$ and

$$((e_i - e_j) \otimes v)x = n(e_i \otimes e_i - e_j \otimes e_j).$$

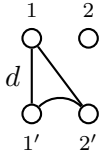
Thus up to a scalar, we hit the third embedding on the nose, answering (iv). Moreover $x^* = \{\{1, 2, 1'\}, \{2'\}\}$ takes us back. Another solution for x is the diagram $q = \{\{1, 2, 1', 2'\}\}$ shown bottom right in the margin which can be read as ‘if positions 1 and 2 are equal, copy to equal new positions 1’ and 2’; we now have

$$((e_1 - e_2) \otimes v)q = e_1 \otimes e_1 - e_2 \otimes e_2,$$

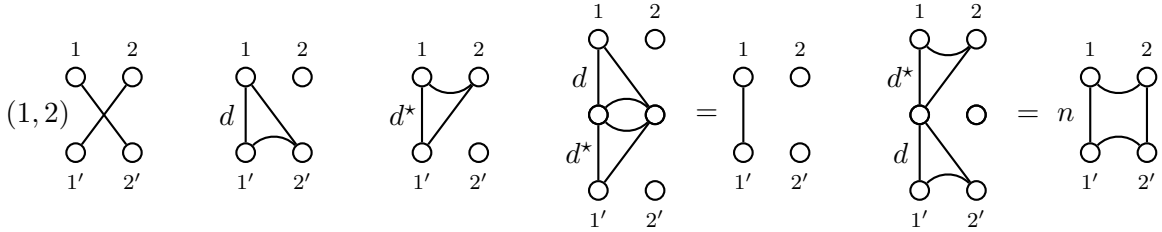
so no scalar is necessary. (My apologies if you found this arguably better solution, and were slightly put out not to find it first!) Noting that the n in $S^{(n-1,1)}$ now appears only as a parameter, we have constructed the 3-dimensional standard module for $\text{Par}_2(n)$ labelled by the partition (1) as the span of the three embeddings. It is $\Delta_2^{(1)}$ in the notation of Definition 4.7, taking the parameter δ to be $n \geq 3$.

Exercise 3.5. We have found three embeddings of $S^{(n-1,1)}$ into $E \otimes E$ defined by

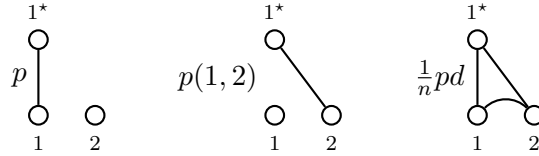
$$\begin{aligned} e_1 - e_2 &\xrightarrow{\phi} (e_1 - e_2) \otimes v \\ e_1 - e_2 &\xrightarrow{\phi(1,2)} v \otimes (e_1 - e_2) \\ e_1 - e_2 &\xrightarrow{\frac{1}{n}\phi d} e_1 \otimes e_1 - e_2 \otimes e_2 \end{aligned}$$



that form a basis for $\text{Hom}_{\text{Per}_n}(S^{(n-1,1)}, E^{\otimes 2})$, where d is the diagram in the margin. Compute the action of $(1, 2)$ and d in this basis and verify that the second and third embeddings are indeed $\phi(1, 2)$ and $\frac{1}{n}\phi d$. [Hint: you can safely identify the embedding, i.e. the element of $\text{Hom}_{\text{Per}_n}(S^{(n-1,1)}, E \otimes E)$, with the image of $e_1 - e_2$.] Verify the products dd^* and d^*d shown right below act in a consistent way, i.e. we have a module action.



Now show that if we represent the three embeddings by the *semi-diagrams* below (this term is not official),



the action of Par_2 is given by concatenating a full diagram below a semi-diagram, and this gives the second and third labels $p(1, 2)$ and $\frac{1}{n}pd$ above. In the action of semi-diagrams, if after concatenating the top dot is in a non-propagating part (that is, a part not meeting both top and bottom dots, here necessarily an isolated singleton in the top row) then the result is the zero homomorphism. Check for instance that $p(1, 2)d = 0$ and correspondingly $\phi(1, 2)d = 0$.⁹

By Theorem 4.6 and Definition 4.7, $\text{Hom}_{\text{Per}_n}(S^{(n-1,1)}, E^{\otimes 2})$ is the standard module $\Delta_2^{(1)}$ for Par_2 whenever $n \geq 3$.

⁹**Solution.** We have already seen that ϕ , $\phi(1, 2)$ and $\frac{1}{n}\phi d$ form a basis for $\text{Hom}_{\text{Per}_n}(S^{(n-1,1)}, E^{\otimes 2})$. The algebraic statement of the ‘copy rules’ above is that $(e_i \otimes e_j)d = e_i \otimes e_i$ and, dually $(e_i \otimes e_j)d^* = [i = j]e_i \otimes v$. (See §4.1 for the general case.) Thus the matrices for the right action, with each row recording the image of the corresponding basis homomorphism, are

$$(1, 2) \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad d \mapsto \begin{pmatrix} 0 & 0 & n \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad d^* \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

For instance d kills the embedding $e_1 - e_2 \mapsto v \otimes (e_1 - e_2)$ because j does not appear in the right-hand side of $(e_i \otimes e_j)d = e_i \otimes e_i$, and so the second row of the matrix for d is all-zeros. (More generally, there is the informal rule that a diagram with a component containing only dots in the top row kills all embeddings of a non-trivial Specht module defined using a subset of the tensor positions in this component.) Computing the products of matrices

Example 3.6. We consider what changes in the previous exercise if $n = 2$. There are still three embeddings ϕ , $\phi(1, 2)$ and $\frac{1}{2}\phi d$ of $S^{(1,1)} = \langle e_1 - e_2 \rangle \cong \text{sgn}$ into $E \otimes E$. The images of $e_1 - e_2$ are, in shorthand notation $e_{11} - e_{21} + e_{12} - e_{22}$, $e_{11} + e_{21} - e_{12} - e_{22}$ and $e_{11} - e_{22}$; these are now linearly dependent. The hom-space is *still* a simple module working over the complex numbers (as it must be by Schur–Weyl duality, since for any n , $E^{\otimes 2}$ is a semisimple Per_n -module — see Proposition 4.8 for the more general result) but it is *not* the 3-dimensional standard module $\Delta_2^{(1)}$ (see Definition 4.7) for the partition algebra $\text{Par}_2(2)$ with parameter 2 spanned by the abstract semi-diagrams p , $p(1, 2)$, $\frac{1}{n}pd$, but instead its 2-dimensional simple top, which would usually be denoted $L_2^{(1)}$. The kernel of the action map

$$\Delta_2^{(1)} \rightarrow \text{Hom}_{\text{Per}_2}(S^{(1,1)}, E^{\otimes 2}) \cong L_2^{(1)}$$

comes from the relation $\phi + \phi(1, 2) - 2\frac{1}{2}\phi d = 0$ corresponding to the combination $v = p + p(1, 2) - 2\frac{1}{2}pd$ of abstract semi-diagrams shown below:

$$v = \begin{array}{c} 1^* \\ \circ \\ | \\ \circ \\ 1 \end{array} \quad \begin{array}{c} \circ \\ 2 \end{array} \quad + \quad \begin{array}{c} 1^* \\ \circ \\ \diagdown \\ \circ \quad \circ \\ 1 \quad 2 \end{array} \quad - \quad \begin{array}{c} 1^* \\ \circ \\ | \quad \diagdown \\ \circ \quad \circ \\ 1 \quad 2 \end{array}$$

(Recall that p is as shown in the margin. Note that $pd = 2d$ hence the extra factor of 2.) This vector v spans the trivial representation of the permutation subalgebra $\mathfrak{S}_2$. It is routine to check that $vp = 2p + 0 - 2p = 0$;

we *should* have

$$dd^* \mapsto \begin{pmatrix} n & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad d^*d \mapsto \begin{pmatrix} 0 & 0 & n \\ 0 & 0 & n \\ 0 & 0 & n \end{pmatrix}$$

and this can be verified directly. For instance the final diagram above shows that, in $\text{Par}_2(n)$, $d^*d = n\{\{1, 2, 1', 2'\}\}$ and since this scaled diagram acts by $e_i \otimes e_j \mapsto n[i = j]e_i \otimes e_i$, its action on $\text{Hom}_{\text{Per}_n}(S^{(n-1,1)}, E^{\otimes 2})$ is given, identifying each homomorphism ϕ , $\phi(1, 2)$, $\frac{1}{n}\phi d$ with its image of $e_1 - e_2$, by

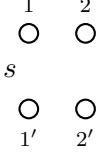
$$\begin{aligned} (e_1 - e_2) \otimes v &= e_1 \otimes v - e_2 \otimes v \mapsto n(e_1 \otimes e_1 - e_2 \otimes e_2) \\ v \otimes (e_1 - e_2) &= v \otimes e_1 - v \otimes e_2 \mapsto n(e_1 \otimes e_1 - e_2 \otimes e_2) \\ e_1 \otimes e_1 - e_2 \otimes e_2 &\mapsto n(e_1 \otimes e_1 - e_2 \otimes e_2) \end{aligned}$$

and this agrees with the matrix for d^*d computed by a matrix product above. For the final part we check $\frac{1}{n}pd$: computing pd by concatenating the semi-diagram for p above d we get

$$\begin{array}{c} 1^* \\ \circ \\ | \\ \circ \\ p \\ | \\ \circ \\ d \\ \diagdown \quad \diagup \\ \circ \quad \circ \\ 1' \quad 2' \end{array} \quad \begin{array}{c} \circ \\ 2 \end{array} \quad = \quad n \quad \begin{array}{c} 1^* \\ \circ \\ \diagdown \quad \diagup \\ \circ \quad \circ \\ 1' \quad 2' \end{array} \quad \begin{array}{c} \circ \\ 2 \end{array} \quad \begin{array}{c} 1^* \\ \circ \\ | \\ \circ \\ p \\ | \\ \circ \\ d \\ \diagdown \quad \diagup \\ \circ \quad \circ \\ 1' \quad 2' \end{array} \quad \begin{array}{c} \circ \\ 2 \end{array} \quad = 0$$

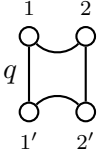
where the n comes from the isolated component from the singleton part $\{2\}$ in d , and so $\frac{1}{n}pd$ is indeed the half-diagram for the third embedding. The right-hand diagram above shows that in $p(1, 2)d$ the top dot 1^* does not propagate and so $p(1, 2)d = 0$. Correspondingly, since $(e_1 - e_2)\phi(1, 2) = v \otimes (e_1 - e_2)$ and $(v \otimes e_j)d = \sum_i e_i \otimes e_i$ for any j , we have $\phi(1, 2)d = 0$.

$$\begin{array}{c} 1^* \\ \circ \\ \diagdown \\ \circ \quad \circ \\ (1, 2)p \quad d \\ \diagdown \quad \diagup \\ \circ \quad \circ \\ 1' \quad 2' \end{array} = 0$$

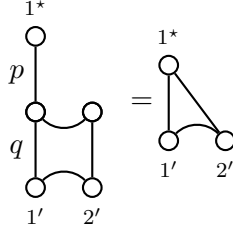


here the second summand vanishes because, by the rule stated at the end of Exercise 3.5, the top dot in the concatenation $(1, 2)pd$ shown in the margin is not propagating. Similarly if $s = \{\{1\}, \{2\}, \{1'\}, \{2'\}\}$ is the ‘all singletons’ diagram shown in the margin then $vs = 0$. Thus $\langle v \rangle$ is acted on trivially by the ideal of the partition algebra of all diagrams having propagation number 0 or 1 and so is the inflation of the trivial module from $\mathfrak{S}_2$ to $\text{Par}_2(n)$; this identifies it with the simple module $L_2^{(2)}$ for $\text{Par}_2(n)$. We therefore have a case where the map

$$\text{Par}_2(n) \rightarrow \text{End}_{\text{Per}_n}(E \otimes E)$$



is not injective. As I think our examples so far strongly suggest, this map is always surjective: see (4.1). To give another calculation, we remark that if $q = \{\{1, 2, 1', 2'\}\}$ is the ‘box’ shown in the margin, then $vq = pd + pd - 2pd = 0$ also vanishes. The calculation $pq = pd$ is shown diagrammatically below.



$$\Delta_2^{(1)} = \begin{matrix} L_2^{(1)} \\ L_2^{(2)} \end{matrix}$$

Thus we have shown that if $n \geq 3$ then $\Delta_2^{(1)}$ is simple, while if $n = 2$ then it is reducible with Loewy series as shown in the margin.

The previous example shows that, even in characteristic zero, a standard module for the partition algebra $\text{Par}_r(n)$ may be reducible if the parameter n is not sufficiently large.

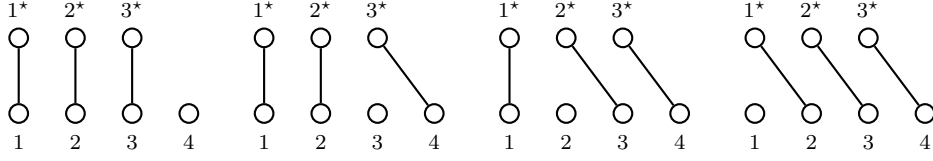
3.2. The module $\Delta_4^{(2,1)}$. We now extend Example 2.4 on the Specht module $S^{(n-3,2,1)}$ to describe the partition algebra module $\Delta_4^{(2,1)}$. This is, I think, sufficiently general to show how to define a general standard module for the partition algebra. Throughout this subsection assume that $\text{char } \mathbb{F} \neq 2$ and $n \geq 5$. We saw in this example that $S^{(n-3,2,1)}$ is generated by

$$e_t = f_{123} + f_{421} + f_{324} - f_{153} - f_{451} - f_{354}$$

where $f_{ijk} = (\{i, j\}, \{k\}) - (\{k, j\}, \{i\}) \in M^{(n-3,2,1)}$ was defined in (2.9), and that one embedding $\psi : S^{(n-3,2,1)} \rightarrow E^{\otimes 3}$ is defined on the larger domain $M^{(n-3,2,1)}$ by

$$\psi(\{i, j\}, \{k\}) = (e_i \otimes e_j + e_j \otimes e_i) \otimes e_k = e_{ijk} + e_{jik}.$$

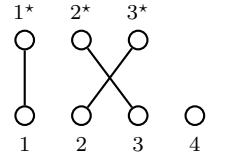
At the end of Example 2.4 we saw that $\text{Hom}_{\text{Per}_n}(S^{(n-3,2,1)}, E^{\otimes 3})$ is isomorphic to $\text{Sp}_{(2,1)}$ as a right module for $\mathfrak{S}_3$ and that ψ and $\psi(1, 2, 3)$ form a basis for this hom-space. We therefore already understand how to embed $S^{(n-3,2,1)}$ using any *chosen* three tensor positions from $E^{\otimes 4}$. These embeddings correspond to the basis elements of $\text{Hom}_{\text{Per}_n}(S^{(n-3,2,1)}, E^{\otimes 4})$ represented by semi-diagrams having a singleton in the bottom row shown below, annotated with either of the homomorphisms ψ and $\psi(1, 2, 3)$.



Thus the leftmost diagram, when annotated with ψ , is the homomorphism

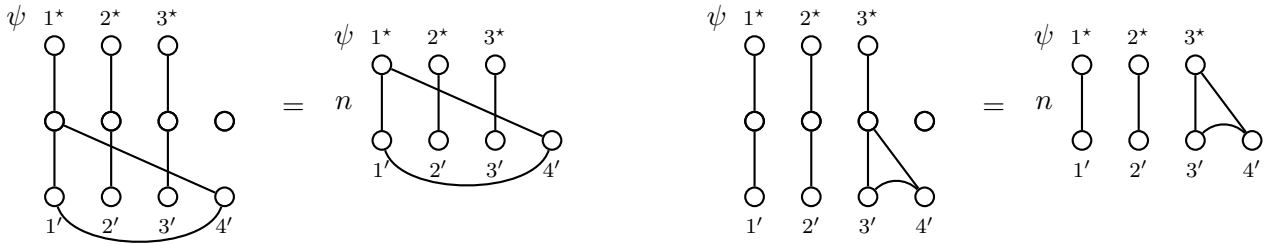
$$e_t \mapsto e_t \psi \otimes v$$

and the rightmost diagram, when annotated with $\psi(1, 2, 3)$, is the homomorphism defined by $e_t \mapsto v \otimes e_t \psi(1, 2, 3)$. Note that by the assumption $n \geq 6$, each embedding has summands in the ‘anti-diagonal’ of tensors $e_{i_1} \otimes e_{i_2} \otimes e_{i_3} \otimes e_{i_4}$ in which i_1, i_2, i_3, i_4 are distinct: in each example we use up 5 dimensions to accommodate e_t , and then we need one more for the v factor. (Experts will recognise the use of the ‘orbit’ basis here.) It follows that the 8 embeddings are linearly independent, avoiding the problem in Exercise 3.6. Note also that the lines from 1^* , 2^* and 3^* are uncrossed: for instance, the crossing semi-diagram in the margin corresponds to $e_t \mapsto e_t \psi(2, 3) \otimes v$ but since $\psi(2, 3)$ is a linear combination of ψ and $\psi(1, 2, 3)$, this homomorphism is a linear combination of the homomorphisms for the leftmost diagram displayed above.



The diagram below shows the action of two partition algebra elements giving, up to a scalar n , the new homomorphisms $S^{(n-3,2,1)} \rightarrow E^{\otimes 4}$ defined on the larger domain $M^{(n-3,2,1)}$ by

$$\begin{aligned} (\{i, j\}, \{k\}) &\mapsto e_i \otimes e_j \otimes e_k \otimes e_i + e_j \otimes e_i \otimes e_k \otimes e_j \\ (\{i, j\}, \{k\}) &\mapsto (e_i \otimes e_j + e_j \otimes e_i) \otimes e_k \otimes e_k. \end{aligned}$$



In these diagrams we record the homomorphism $S^{(n-3,2,1)} \rightarrow E^{\otimes 3}$ by an annotation on the top dots. Thus $\dim \text{Hom}_{\text{Per}_n}(S^{(n-3,2,1)}, E^{\otimes 4}) = 20$, and the hom-space has a basis indexed by

- four uncrossed semi-diagrams having a bottom-row singleton shown above, *and*
- six further uncrossed semi-diagrams in which one top dot is connected to two bottom dots,

in which each semi-diagram is annotated either with ψ or $\psi(1, 2, 3)$. (Here ψ and $\psi(1, 2, 3)$ can be replaced with any convenient elements indexing a basis for $\text{Hom}_{\text{Per}_n}(S^{(n-3,2,1)}, E^{\otimes 3})$, for example standard tableaux of shape $(2, 1)$: see Proposition 4.5(v).) Partition algebra elements that decrease the propagation number below the maximum of 3 (i.e. all three dots propagate to

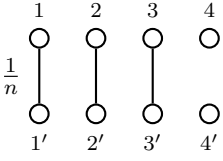
distinct parts at the bottom) act as zero. The assumption $n \geq 6$ is a special case of the general condition $n \geq r + \kappa_1$ in Theorem 4.6.

Exercise 3.7. Let H be the left Par_3 - and right Par_4 -module spanned by all semi-diagrams with three top dots and four bottom dots in which the left action of Par_3 is inflated from $\mathfrak{S}_3$, and so partition algebra elements whose semi-diagram action decrease the number of propagating parts below 3 act as zero. Show that

$$\text{Hom}_{\text{Per}_n}(S^{(n-3,2,1)}, E^{\otimes 4}) \cong \text{Inf}_{\mathfrak{S}_3}^{\text{Par}_3} \text{Sp}_{(2,1)} \otimes_{\text{Par}_3(n)} H.^{10} \quad (3.3)$$

Remark 3.8. Observe that the right-hand side in (3.3) does not depend on n , except as the parameter in the partition algebra Par_4 . In the case when n is invertible, we may define the idempotent

$$e_3 = \frac{1}{n} \{ \{1, 1'\}, \{2, 2'\}, \{3, 3'\}, \{4\}, \{4'\} \}$$



shown in the margin. The algebra $e_3 \text{Par}_4 e_3$ is isomorphic to Par_3 . If I is its ideal of diagrams with fewer than three propagating parts, then the module of semi-diagrams H satisfies $H \cong e_3 \text{Par}_4 / I e_3 \text{Par}_4$ as a left Par_3 - and right Par_4 -bimodule. Tensoring with the inflated $\text{Sp}_{(2,1)}$ kills I . Thus (3.3) implies that, provided $n \geq 6$,

$$\text{Hom}_{\text{Per}_n}(S^{(n-3,2,1)}, E^{\otimes 4}) \cong \text{Inf}_{\mathfrak{S}_3}^{\text{Par}_3} \text{Sp}_{(2,1)} \otimes_{e_3 \text{Par}_4 e_3} e_3 \text{Par}_4$$

This is a special case of the usual general definition for standard modules: see Remark 4.11.

As a final exercise it is instructive to go back to §3.1 and give a similar description of $\text{Hom}_{\text{Per}_n}(S^{(n-1,1)}, E^{\otimes 2})$ as a tensor product of the inflation of the trivial module from $\mathfrak{S}_1$ to Par_1 with a module of half-diagrams, each having a single dot in the top row.

4. GENERAL THEORY

We now give an account of the general results seen in special cases in the previous section. As ever $E = \langle e_1, \dots, e_n \rangle$ is the natural permutation representation of Per_n defined over the field $\mathbb{F}$. For technical reasons first seen in the proof of Proposition 4.2, we typically have to assume that $\text{char } \mathbb{F} \neq 2$.

4.1. Action on $E^{\otimes r}$. A diagram d in the partition algebra $\text{Par}_r(n)$ acts on $E^{\otimes r}$ by the rule

$$e_{i_1} \otimes \dots \otimes e_{i_r} d = \sum_{j_1, \dots, j_r} e_{j_1} \otimes \dots \otimes e_{j_r}$$

¹⁰**Outline solution:** The key point is that every semi-diagram d can be converted to an uncrossed diagram by the *left* action of some $\sigma \in \mathfrak{S}_3 \subseteq \text{Par}_3$. (Compare that one choice of coset representatives for the cosets of $\mathfrak{S}_3$ in $\mathfrak{S}_4$ are all $\sigma \in \mathfrak{S}_4$ such that $1 \leq 1\sigma < 2\sigma \leq 3\sigma \leq 4$.) Thus the tensor product $\text{Sp}_{(2,1)} \otimes_{\text{Par}_3(n)} H$ has a basis of the 10 uncrossed semi-diagrams already seen, each annotated with one of the two chosen basis elements of $\text{Sp}_{(2,1)}$; this annotation corresponds to a chosen homomorphism from $S^{(n-3,2,1)}$ to $E^{\otimes 3}$.

where the sum is over all $j_1, \dots, j_r$ such that if P is a part of d satisfying $\{1, \dots, r\} \cap P = A$ and $\{1', \dots, r'\} \cap P = B$ then, dropping primes from the elements of B , the set $\{i_a : a \in A\} \cup \{j_b : b \in B\}$ is a singleton. This unpacks into three cases:

- (a) if $i_a \neq i_{a^*}$ for some $a, a^* \in A$ then the sum is empty and the right-hand side is 0;
- (b) if P does not meet $\{1, \dots, r\}$ then $\{j_b : b \in B\}$ may be any singleton set, and so in the right-hand side the basis elements e_{j_b} in tensor positions $b \in B$ are constant, but may take any value;
- (c) if P meets both $\{1, \dots, r\}$ and $\{1', \dots, r'\}$ then the constant value of j_b for $b \in B$ is the constant value of i_a for $a \in A$, and so the rule ‘copy from any position in A to the positions in B ’ holds.

We have already seen many particular cases of this rule. For instance, if d is the partition algebra diagram in the margin then (b) and (c) imply that $(e_i \otimes e_j)d = e_i \otimes e_i$, as already seen in (3.2). Dually, (a) and (c) imply that

$$(e_i \otimes e_j)d^* = \begin{cases} e_i \otimes v & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

It can be shown using the orbit basis (see [2, Theorem 1.2] and [3]) that the action map

$$\text{Par}_r(n) \longrightarrow \text{End}_{\text{Per}_n}(E^{\otimes r}) \quad (4.1)$$

is always surjective. (It follows from Theorem 4.6 that it is injective when $n \geq 2r$: Example 3.6 shows it is not injective when $n = 2$ and $r = 2$.) Thus, for small n , the abstract partition algebra may act with a kernel, but its image is still the full centralizer algebra;

4.2. Specht homomorphisms. Recall that if λ is a partition of n then M^λ is the Young permutation module for Per_n (acting on the left) and S^λ is its Specht submodule. The modules $M^{(n-1,1)} \cong E$ and $S^{(n-1,1)} = \langle e_i - e_j : 1 \leq i < j \leq n \rangle$ were seen in §3.1 and $M^{(n-3,2,1)}$ and $S^{(n-3,2,1)}$ were seen in Example 2.4; in this example we saw embeddings of $M^{(n-3,2,1)}$ and $S^{(n-3,2,1)}$ into the subspace $\langle e_i \otimes e_j \otimes e_k : i, j, k \text{ distinct} \rangle$ of $E^{\otimes 3}$. More generally, we define

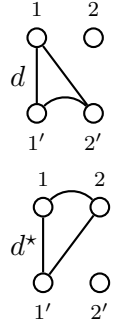
$$A_k = \langle e_{i_1} \otimes \dots \otimes e_{i_k} : i_1, \dots, i_k \text{ distinct} \rangle$$

to be the *anti-diagonal* subspace of $E^{\otimes k}$.

Exercise 4.1. This exercise introduces the main idea in the proof of Proposition 4.2(i).

- (i) Show that $e_1 \otimes e_2 - e_2 \otimes e_1 + e_2 \otimes e_3 - e_3 \otimes e_2 + e_3 \otimes e_1 - e_1 \otimes e_3 \in A_2$ affords the sign representation $\text{sgn}_{\text{Per}_3}$ of Per_3 .
- (ii) Show more generally, there is a copy of $\text{sgn}_{\text{Per}_n}$ in A_{n-1} .
- (iii) Assume $\text{char } \mathbb{F} \neq 2$. Show that there is no non-zero homomorphism from $\text{sgn}_{\text{Per}_n}$ into A_ℓ if $\ell < n - 1$.¹¹

¹¹**Solution:** for (iii), suppose that $v = \sum_{i_1, \dots, i_\ell} c_{i_1, \dots, i_\ell} e_{i_1} \otimes \dots \otimes e_{i_\ell}$ affords the sign representation of Per_n . Given any $i_1, \dots, i_\ell$ there exist $j, k \notin \{i_1, \dots, i_\ell\}$. Hence the coefficients of $e_{i_1} \otimes \dots \otimes e_{i_\ell}$ in v and $(jk)v$ are equal, where $(j, k) \in \text{Per}_n$ is the transposition



The notation $\kappa_{[n]} = (n - |\kappa|, \kappa_1, \dots, \kappa_{\ell(\kappa)})$ was defined in §1.

Proposition 4.2. *Assume that $\text{char } \mathbb{F} \neq 2$. Let κ be a partition of k .*

- (i) *There are no non-zero homomorphisms from $S^{\kappa_{[n]}}$ into $E^{\otimes \ell}$ for $\ell < k$.*
- (ii) *There are no non-zero homomorphisms from $S^{\kappa_{[n]}}$ into $A_k^\perp$.*

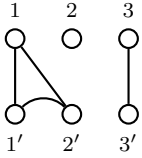
Proof. For (i), observe that for each set partition $\mathcal{P}$ of $\{1, \dots, \ell\}$ there is a corresponding subspace of $E^{\otimes \ell}$ spanned by all tensors equal precisely on the positions in the same part of $\mathcal{P}$. (This could be the zero space.) For example, if $\ell = 2$, then

$$E \otimes E = \langle e_i \otimes e_j : 1 \leq i \neq j \leq n \rangle \oplus \langle e_1 \otimes e_1, \dots, e_n \otimes e_n \rangle \cong A_2 \oplus A_1$$

corresponding to the set partitions $\{\{1\}, \{2\}\}$ and $\{\{1, 2\}\}$. Each subspace is a Per_n module isomorphic to A_j , where j is the number of parts of the set partition. It is therefore enough to rule out homomorphisms into every such A_j .

Let $\phi : S^{\kappa_{[n]}} \rightarrow A_j$ be a homomorphism, where $j \leq \ell < k$. Choose a $\kappa_{[n]}$ -tableau s with columns $C_1, \dots, C_a$ and observe that $\phi(e_s)$ affords the sign representation of the Young subgroup with these orbits. Since $j < (|C_1| - 1) + \dots + (|C_a| - 1) = k$, each canonical basis element of A_j omits two elements from at least one set C_i . As in the solution to Exercise 4.1(iii), it is therefore invariant under a transposition, and so cannot appear in the support of $\phi(e_s)$. Hence $\phi(e_s) = 0$.

For (ii), observe that $A_k^\perp$ is the direct sum of the subspaces corresponding to set partitions of $\{1, \dots, k\}$ having $j < k$ blocks. Each such subspace is isomorphic to A_j , so part (ii) follows immediately from part (i). Alternatively, using that the right action of Par_r commutes with Per_n , use that these summands may be obtained from the A_j by ‘duplicating positions’ with partition diagrams such as the one shown in the margin. $\square$



Remark 4.3. In characteristic zero the proposition can also be proved by the logically independent Corollary 5.5, by arguing that any walk on Young’s Lattice from (n) to $\kappa_{[n]}$ moves at most one box down from the top row in each step, and so must have at least k steps. This also shows that the copy of the sign representation of Per_n found in A_{n-1} in Exercise 4.1(ii) is unique.

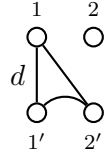
Exercise 4.4. Show that the assumption that $\text{char } \mathbb{F} \neq 2$ is essential in both parts of Proposition 4.2.¹²

The next proposition generalizes Example 2.4 and Exercise 2.5. In (iii), the perpendicular space is taken with respect to the canonical bilinear form on M^κ that makes the permutation basis of set partitions of type κ orthonormal. Note that we use only the action of $\mathfrak{S}_k$ on the homomorphism

swapping j and k . But since v affords the sign representation, $(j, k)v = -v$. Therefore, since $\text{char } \mathbb{F} \neq 2$ we have $c_{i_1 \dots i_\ell} = 0$.

¹²**Solution:** We take $\kappa = (1^{n-1})$ to get the situation in Exercise 4.1 when $S^{\kappa_{[n]}} \cong \text{sgn}_{\text{Per}_n}$. Now $\text{sgn}_{\text{Per}_n}$ is the trivial representation and this embeds into A_ℓ for any ℓ , for instance by symmetrizing $e_1 \otimes \dots \otimes e_\ell$; the special case $\ell = 1$ is that $e_1 + \dots + e_n \in A_1$ affords the trivial representation. For (ii), take $n = 3$ and $\kappa = (1, 1)$, so $k = 2$, and note that $e_1 \otimes e_1 + e_2 \otimes e_2 + e_3 \otimes e_3 \in A_2^\perp$ affords the trivial representation.

spaces. (They are not in general partition algebra modules: for instance $A_2 = \langle e_i \otimes e_j : 1 \leq i \neq j \leq n \rangle$ is not closed under the action of the partition algebra element d shown in the margin.) In (iv), Sp_κ is the dual of the usual Specht module: $\text{Sp}_\kappa \cong (\text{Sp}^\kappa)^*$; it can be constructed in other ways, but for this proposition, thinking of it as the dual of Sp^κ is ideal. It is a right module for $\mathfrak{S}_k$.



Proposition 4.5. *Assume that $\text{char } \mathbb{F} \neq 2$. Let κ be a partition of k and let $n \geq k + \kappa_1$. Let P be the ordered set partition*

$$(\{1, \dots, \kappa_1\}, \{\kappa_1 + 1, \dots, \kappa_1 + \kappa_2\}, \dots, \{k - \kappa_{\ell(\kappa)} + 1, \dots, k\})$$

generating a permutation module isomorphic to $M^{\kappa[n]}$; let Per_κ be the Young subgroup of $\text{Per}_{\{1, \dots, k\}}$ with these orbits, and let $\mathfrak{S}_\kappa$ be the corresponding place-permutation subgroup of $\mathfrak{S}_k$.

(i) *There is a homomorphism $\phi : M^{\kappa[n]} \rightarrow A_k$ defined by*

$$\phi(P) = \sum_{\sigma \in \mathfrak{S}_\kappa} (e_1 \otimes \dots \otimes e_k) \sigma.$$

(ii) *The submodule of $\text{Hom}_{\text{Per}_n}(M^{\kappa[n]}, A_k)$ generated by ϕ under the right action of $\mathfrak{S}_k$ is isomorphic to M^κ as a right $\mathfrak{S}_k$ -module.*

(iii) *Thinking of the left-hand side as a space of homomorphisms by (ii), the restriction map $M^\kappa \rightarrow \text{Hom}_{\text{Per}_n}(S^{\kappa[n]}, A_k)$ has kernel $(\text{Sp}^\kappa)^\perp$.*

(iv) *As a right $\mathfrak{S}_k$ -module, $\text{Hom}_{\text{Per}_n}(S^{\kappa[n]}, A_k)$ has a submodule isomorphic to Sp_κ .*

(v) $\text{Hom}_{\text{Per}_n}(S^{\kappa[n]}, A_k) \cong \text{Sp}_\kappa$.

Proof. For (i) all we have to check (using for instance the characterization of $M^{\kappa[n]}$ as an induced module) is that $\phi(P)$ is invariant under $\text{Per}_\kappa \times \text{Per}_{\{k+1, \dots, n\}}$; this is clear. Now (ii) is routine generalization from Example 2.4: for instance we saw after (2.6) that if $\kappa = (2, 1)$ then $\phi(P) = (e_1 \otimes e_2 + e_2 \otimes e_1) \otimes e_3$ affords, under the right action of $\mathfrak{S}_3$ a copy of $M^{(2,1)}$.

For (iii), let $t(P)$ be the $\kappa[n]$ -tableau with first row entries $k+1, \dots, n$ whose remaining rows, read left-to-right, are the parts of P . Thus $t(P)$ generalizes the tableau t in (2.7) in Example 2.4 for the case $\kappa = (2, 1)$, repeated in the margin. Let $C_{t(P)}$ be the Young subgroup of Per_n whose orbits are the columns of $t(P)$ and let H be the Young subgroup of $C_{t(P)}$ fixing $\{k+1, \dots, n\}$ that permutes $\{1, \dots, k\}$ with the same orbits as $C_{t(P)}$.

The Specht module $S^{\kappa[n]}$ is cyclic, and generated as a Per_n -module by the polytabloid $e_{t(P)}$; in our construction of $M^{\kappa[n]}$ by ordered set partitions, we have $e_{t(P)} = \sum_{\sigma \in C_{t(P)}} \text{sgn}(\sigma) \sigma P$. By taking coset representatives for H in $C_{t(P)}$ we have

$$e_{t(P)} \phi = \sum_{\tau \in C_{t(P)}/H} \text{sgn}(\tau) \tau \sum_{\vartheta \in H} \text{sgn}(\vartheta) \phi(P) = \sum_{\tau \in C_{t(P)}/H} \text{sgn}(\tau) \tau f$$

where $f = \sum_{\vartheta \in H} \text{sgn}(\vartheta) \phi(P) \in A_k$. Observe that τf is supported on tensor products of the canonical basis elements $e_{\tau(1)}, \dots, e_{\tau(k)}$. Therefore the summands in the right-hand side are linearly independent and we have $e_{t(P)} \phi \neq 0$ if and only if $f \neq 0$. Moreover, under the right action of $\mathfrak{S}_k$, f generates

$$\begin{array}{|c|c|c|c|} \hline 4 & 5 & 6 & \cdots \\ \hline 1 & 2 & & \\ \hline 3 & & & \\ \hline \end{array} \quad \begin{array}{l} t(P), C_t, H \text{ if } \kappa = (2, 1) \\ \cdots \\ n \end{array}$$

$$\begin{array}{l} C_t = \text{Per}_{\{4,1,3\}} \times \text{Per}_{\{5,2\}} \\ H = \text{Per}_{\{1,3\}} \times \text{Per}_{\{2\}} \end{array}$$

a copy of the Specht module Sp^κ ; in the earlier Example 2.4, each τf is an ‘embedding gadget’ f_{ijk} .

By (ii), an arbitrary element of the copy of M^κ generated by ϕ is of the form $\sum_{\pi \in \mathfrak{S}_\kappa \setminus \mathfrak{S}_k} c_\pi \phi \pi$ and so is zero on restriction to $S^{\kappa[n]}$ if and only if $\sum_{\pi \in \mathfrak{S}_\kappa \setminus \mathfrak{S}_k} c_\pi f \pi = 0$. This holds if and only if $\langle v, \sum_{\pi \in \mathfrak{S}_\kappa \setminus \mathfrak{S}_k} c_\pi f \pi \rangle = 0$ for all $v \in M^\kappa$. Taking $v = P\rho$ for $\rho \in \mathfrak{S}_k$ and using $\langle P\rho, w \rangle = \langle P, w\rho^{-1} \rangle$, this is equivalent to

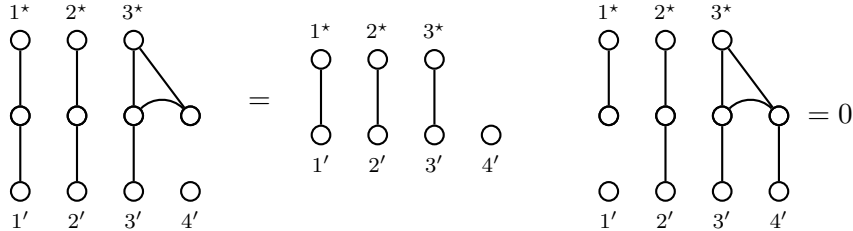
$$\langle P \sum_{\pi \in \mathfrak{S}_\kappa \setminus \mathfrak{S}_k} c_\pi \pi^{-1}, f\rho^{-1} \rangle = 0$$

for all $\rho \in \mathfrak{S}_k$. Each set partition Q of $\{1, \dots, k\}$ into parts of sizes $\kappa_1, \dots, \kappa_{\ell(\kappa)}$ is obtained as $P\pi^{-1}$ for a unique coset representative $\pi \in \mathfrak{S}_\kappa \setminus \mathfrak{S}_k$. Therefore we can identify the kernel of restriction with those vectors perpendicular to all $f\rho^{-1}$, i.e. with the perpendicular space to Sp^κ .

Part (iv) now follows from part (iii) and the first isomorphism theorem, since $M^\kappa / (\text{Sp}^\kappa)^\perp \cong \text{Sp}_\kappa$.

For (v) we use dimension counting. By [5, Corollary 13.15] and the isomorphism $A_k \cong M^{(n-k, 1^k)}$ just mentioned, $\dim \text{Hom}_{\text{Per}_n}(S^{\kappa[n]}, A_k)$ is the number of semistandard Young tableaux of shape $\kappa[n]$ and content $(n-k, 1^k)$. Any such tableau has $n-k$ 1s in its top row, and so these tableaux are in bijection with the set $\text{SYT}(\kappa)$ of standard Young tableaux of shape κ . But by (iv), $\text{Hom}_{\text{Per}_n}(S^{\kappa[n]}, A_k)$ has a subspace Sp_κ whose dimension is $|\text{SYT}(\kappa)|$. This completes the proof. $\square$

Generalizing §3.2, we define a *semi-diagram* with k top dots and r bottom dots to be a diagram with the maximum number of k propagating parts. Labelling the top dots $1^*, \dots, k^*$ and the bottom dots $1, \dots, r$ we say that a semi-diagram in which top dot i^* propagates to dots P_i at the bottom is *uncrossed* if $\min P_1 < \dots < \min P_k$. (We prefer ‘uncrossed’ to ‘non-crossing’ because an uncrossed diagram may still be impossible to draw in the plane.) Let $H_r(k)$ be the left $\text{Par}_k(n)$ - and right $\text{Par}_r(n)$ -module of all semi-diagrams, crossings permitted; the action is defined by diagram concatenation as usual, recording the number of isolated components as a power of n (or generally, the parameter δ), with the extra condition that the product is zero if fewer than k top dots are propagating. An example was seen in Exercise 3.5. A larger example in the context of the module $\Delta_4^{(2,1)}$ and the homomorphism $S^{(n-3,2,1)} \rightarrow E^{\otimes 4}$ seen in §3.2 is shown below.



Note that we do *not* annotate the semi-diagram with a homomorphism $S^{(n-3,2,1)} \rightarrow E^{\otimes 3}$ as we did, being deliberately a bit informal, in §3.2: instead this extra data comes from the tensor factor Sp_κ in the theorem below.

Theorem 4.6. *Assume that $\text{char } \mathbb{F} \neq 2$. Let κ be a partition of k . Provided $n \geq r + \kappa_1$,*

$$\text{Hom}_{\text{Per}_n}(S^{\kappa_{[n]}}, E^{\otimes r}) \cong \text{Inf}_{\mathfrak{S}_k}^{\text{Par}_k(n)} \text{Sp}_{\kappa} \otimes_{\text{Par}_k(n)} H_r(k).$$

The right-hand side has a canonical basis indexed by pairs (t, d) where $t \in \text{SYT}(\kappa)$ and d is an uncrossed semi-diagram.

Sketch of proof. This is a routine generalization of Exercise 3.7. The condition on n comes from using $k + \kappa_1$ basis vectors to write down the image in $E^{\otimes k}$ of a polytabloid in $S^{\kappa_{[n]}}$. This uses up k tensor factors, and the remaining $r - k$ tensor positions require $r - k$ further linearly independent basis vectors; thus we require $n \geq (k + \kappa_1) + (r - k) = r + \kappa_1$. $\square$

4.3. Standard modules for the generic partition algebra. The following definition has now been extensively motivated.

Definition 4.7. Let κ be a partition of k with $k \leq r$. The standard module for the generic partition algebra $\text{Par}_r(\delta)$ labelled by a partition κ is

$$\Delta_r^{\kappa} = \text{Inf}_{\mathfrak{S}_k}^{\text{Par}_k} \text{Sp}_{\kappa} \otimes_{\text{Par}_k(\delta)} H_r(k).$$

Observe that Δ_r^{κ} has a basis indexed by pairs (b, d) where d is an uncrossed semi-diagram having k top dots and r bottom dots and b is in a chosen set indexing a basis of Sp_{κ} . For instance, one suitable indexing set is all standard Young tableaux of shape κ . When $\delta = n$ there is a canonical map

$$\Delta_r^{\kappa} \rightarrow \text{Hom}_{\text{Per}_n}(S^{\kappa_{[n]}}, E^{\otimes r}) \quad (4.2)$$

defined by mapping the basis element (t, d) where $t \in \text{SYT}(\kappa)$ corresponds to the homomorphism $\psi_t : S^{\kappa_{[n]}} \rightarrow E^{\otimes k}$ to the homomorphism $\psi_t d$, where the right-action of d is on $E^{\otimes r}$. By dimension counting we conclude from Theorem 4.6 that

$$\Delta_r^{\kappa} \cong \text{Hom}_{\text{Per}_n}(S^{\kappa_{[n]}}, E^{\otimes r})$$

when $n \geq r + \kappa_1$. When $n < r + \kappa_1$, the canonical map need no longer be injective. In characteristic zero it is nevertheless surjective; this will follow immediately from the simplicity result below and is recorded after its proof.

Proposition 4.8. *Assume that $\text{char } \mathbb{F} = 0$. Let κ be a partition of k and let $n \geq k + \kappa_1$. Then*

$$\text{Hom}_{\text{Per}_n}(S^{\kappa_{[n]}}, E^{\otimes r})$$

is non-zero if and only if $r \geq k$ and in this case the homomorphism space is a simple right $\text{Par}_r(n)$ -module.

Proof. Since $\text{char } \mathbb{F} = 0$ we may take characters to get

$$\dim \text{Hom}_{\text{Per}_n}(S^{\kappa_{[n]}}, E^{\otimes r}) = \langle \chi^{\kappa_{[n]}}, \pi^r \rangle,$$

where π is the natural permutation character of Per_n . Corollary 5.5 therefore shows that this homomorphism space is non-zero if and only if $r \geq k$. Again

because $\text{char } \mathbb{F} = 0$, the Per_n -module $E^{\otimes r}$ is completely reducible. Put $C = \text{End}_{\text{Per}_n}(E^{\otimes r})$. The isotypic decomposition of $E^{\otimes r}$ gives

$$C \cong \bigoplus_{j=0}^r \bigoplus_{\substack{\nu \vdash j \\ n-j \geq \nu_1}} \text{End}_{\mathbb{F}}(H_\nu),$$

where $H_\nu = \text{Hom}_{\text{Per}_n}(S^{\nu[n]}, E^{\otimes r})$. Hence every non-zero H_κ is a simple right C -module. Since the action map $\text{Par}_r(n) \rightarrow C$ is surjective for every n , the same space is simple as a right $\text{Par}_r(n)$ -module. $\square$

Corollary 4.9. *Let κ be a partition of k . Assume that $\text{char } \mathbb{F} = 0$ and $n \geq k + \kappa_1$. Then the canonical map in (4.2) is surjective.*

Proof. There is nothing to prove if the target is zero. Otherwise $r \geq k$, and the canonical map is non-zero: take a basis element whose semi-diagram has k propagating parts. Its image is therefore a non-zero $\text{Par}_r(n)$ -submodule of the simple module in Proposition 4.8, and hence is the whole target. $\square$

Remark 4.10. Whenever $k + \kappa_1 \leq n < r + \kappa_1$, the homomorphism space $\text{Hom}_{\text{Per}_n}(S^{\kappa[n]}, E^{\otimes r})$ is still simple in characteristic zero, but the standard module Δ_r^κ may be reducible. Example 3.6, with $\kappa = (1)$, $r = 2$ and $n = 2$, is the smallest instance: the canonical map sends the reducible standard module onto the simple multiplicity space seen by the full centralizer algebra, and its kernel is the extra composition factor.

Remark 4.11. Suppose that $\delta \in \mathbb{F}$ is non-zero. Let $e_k \in \text{Par}_r(\delta)$ be the idempotent defined by

$$e_k = \frac{1}{\delta^{r-k}} \{ \{1, 1'\}, \dots, \{k, k'\}, \{k+1\}, \dots, \{r\}, \{(k+1)'\}, \dots, \{r'\} \}$$

as drawn below.

$$e_k = \frac{1}{\delta^{r-k}} \begin{array}{ccccc} & 1 & 2 & & k & k+1 & & r \\ & \circ & \circ & & \circ & \circ & & \circ \\ & | & | & & | & & & \\ & \circ & \circ & & \circ & \circ & & \circ \\ & 1' & 2' & & k' & (k+1)' & & r' \end{array}$$

Then $e_k \text{Par}_r$ is the right Par_r -module spanned by all diagrams with singleton top parts $\{k+1\}, \dots, \{r\}$. Note that all these diagrams have at most k propagating parts. When we tensor this with $\text{Inf}_{\mathfrak{S}_k}^{\text{Par}_k} \text{Sp}_\kappa$, diagrams with strictly fewer than k propagating parts become 0, because the ideal of Par_k spanned by diagrams with fewer than k propagating parts acts trivially on the inflated $\mathfrak{S}_k$ -module. The surviving diagrams are in bijection with semi-diagrams, and can, as before, be made uncrossed by the action of $\mathfrak{S}_k$. Now observe that $e_k \text{Par}_r e_k$ has an obvious basis of diagrams in which $\{k+1\}, \dots, \{r\}$ and $\{(k+1)'\}, \dots, \{r'\}$ are singleton parts, and so we have $e_k \text{Par}_r e_k \cong \text{Par}_k$. Therefore there is an isomorphism

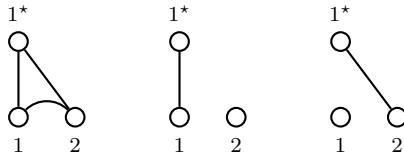
$$\text{Inf}_{\mathfrak{S}_k}^{\text{Par}_k} \text{Sp}_\kappa \otimes_{\text{Par}_k(\delta)} H_r(k) \cong \text{Inf}_{\mathfrak{S}_k}^{\text{Par}_k(\delta)} \text{Sp}_\kappa \otimes_{\text{Par}_k(\delta)} e_k \text{Par}_r(\delta)$$

showing that $\Delta_r^\kappa \cong \text{Inf}_{\mathfrak{S}_k}^{\text{Par}_k(\delta)} \text{Sp}_\kappa \otimes_{\text{Par}_k(\delta)} e_k \text{Par}_r(\delta)$. This is the conventional definition of the standard module.

5. WALKS AND MARKED SET PARTITIONS

In this final section we give a family of combinatorial results that have proofs using the partition algebra.

5.1. Marked partitions. We define a k -marked set partition of $\{1, \dots, r\}$ to be a set partition of $\{1, \dots, r\}$ in which k distinct parts are distinguished by *marks*. Let $B_r^{(k)}$ be the number of k -marked set partitions of $\{1, \dots, r\}$; thus $B_r^{(0)}$ is the usual Bell number B_r . Consistent with the notation $\{1^*, \dots, k^*\}$ for the top dots, we indicate marked parts by a star. For example, there are 3 1-marked set partitions of $\{1, 2\}$, namely $\{\{1, 2\}_*\}$, $\{\{1\}_*, \{2\}\}$, $\{\{1\}, \{2\}_*\}$, corresponding to the semi-diagrams shown below.



Correspondingly, by taking $\kappa = (1)$ in the following corollary, we obtain $\dim \text{Hom}_{\text{Per}_n}(S^{(n-1,1)}, E^{\otimes 2}) = 3$. This was already seen by a direct argument in §3.1.

Corollary 5.1. *Let κ be a partition of k . Assume that $\text{char } \mathbb{F} \neq 2$ and $n \geq r + \kappa_1$.*

$$\dim \text{Hom}_{\text{Per}_n}(S^{\kappa_{[n]}}, E^{\otimes r}) = |\text{SYT}(\kappa)| B_r^{(k)}.$$

Proof. This is immediate from Theorem 4.6. $\square$

5.2. Induction and restriction. For the remainder of this section we work in characteristic zero. Given a partition λ let $\lambda - \square$ and $\lambda + \square$ denote the sets of all partitions obtained by removing or adding a box from the Young diagram of λ . The *Branching Rule* states that if λ is a partition of n then $\chi^\lambda \downarrow_{\text{Per}_{n-1}} = \sum_{\mu \in \lambda - \square} \chi^\mu$.

Exercise 5.2. Assuming the Branching Rule just stated for restriction, deduce the dual rule for induction: $\chi^\lambda \uparrow^{\text{Per}_{n+1}} = \sum_{\nu \in \lambda + \square} \chi^\nu$ by Frobenius reciprocity.¹³

Lemma 5.3. *Given a partition λ of n we have*

$$\chi^\lambda \downarrow_{\text{Per}_{n-1}} \uparrow^{\text{Per}_n} = \sum_{\substack{\mu \in \lambda - \square \\ \nu \in \mu + \square}} \chi^\nu.$$

Proof. This is immediate from the branching rules. $\square$

5.3. Walks. We define a *walk* on partitions of n to be a sequence of *steps* $\lambda(1) \mapsto \dots \mapsto \lambda(m)$ in which each $\lambda(i+1)$ is obtained from $\lambda(i)$ by removing and then adding a box, as in Lemma 5.3. Note that a step depends on the intermediate partition of $n-1$, and so the number of steps from a partition λ to itself is the number of removable boxes of λ . Let π denote the Young permutation character $\pi^{(n-1,1)}$ of the natural representation E of Per_n .

¹³**Solution:** by Frobenius reciprocity $\langle \chi^\lambda \uparrow^{\text{Per}_{n+1}}, \chi^\nu \rangle = \langle \chi^\lambda, \chi^\nu \downarrow_{\text{Per}_n} \rangle$; now use the branching rule for restriction and observe that $\lambda \in \nu - \square$ if and only if $\nu \in \lambda + \square$.

Proposition 5.4. *For any partition λ of n , the multiplicity $\langle \pi^\ell, \chi^\lambda \rangle$ is the number of walks from (n) to λ .*

Proof. Using $\langle \pi^\ell, \chi^\lambda \rangle = \langle \mathbb{1}, \chi^\lambda \pi^\ell \rangle$ this follows by ℓ successive applications of Lemma 5.3. $\square$

Corollary 5.5. *Let κ be a partition of k . If $\ell < k$ then $\langle \chi^{\kappa_{[n]}}, \pi^\ell \rangle = 0$ and $\langle \chi^{\kappa_{[n]}}, \pi^k \rangle = |\text{SYT}(\kappa)|$.*

Proof. Any walk from (n) to $\kappa_{[n]}$ requires at least k steps, since each step can move at most one box down from the first part (n) , and the first part of $\kappa_{[n]}$ is $n - k$. Ways to obtain κ by a k -step walk are in bijection with standard Young tableaux of shape κ : the box containing j indicates which box was moved down from the first part at step j . $\square$

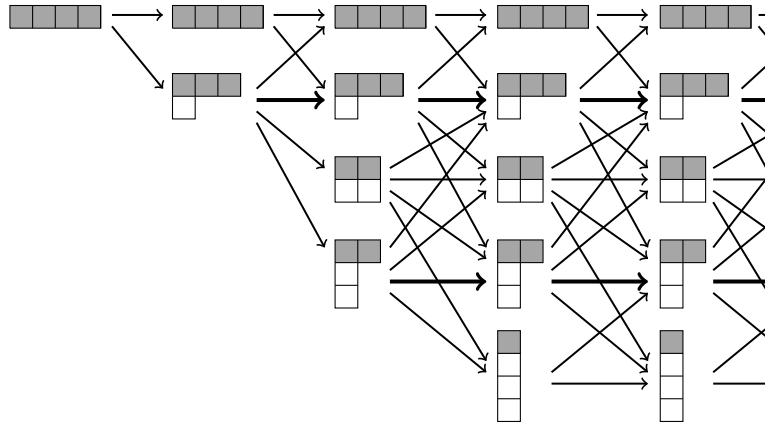
5.4. Walks and set partitions. We end by showing how the partition algebra can be used to count walks. I think it is striking that this corollary is stated entirely combinatorially, and yet the proof we have given goes quite deeply into the theory of the partition algebra.

Corollary 5.6. *Let κ be a partition of k and let $n \geq r + \kappa_1$. The number of r -step walks from (n) to $\kappa_{[n]}$ is independent of n for $n \geq r + \kappa_1$ and is equal to the product of $|\text{SYT}(\kappa)|$ and the number of k -marked set partitions of $\{1, \dots, r\}$.*

Proof. Combine Theorem 4.6 and Proposition 5.4. $\square$

In this final example we consider what happens if n is below the ‘stable’ value of $r + \kappa_1$ in Corollary 5.6.

Example 5.7. The diagram below shows walks on the partitions of 4, using bold arrows to indicate steps from a partition with 2 removable boxes to itself; these should be counted with multiplicity 2.



For example, there are 6 paths from (4) to $(2, 1, 1)$ in three steps, and so $\langle \pi^3, \chi^{(2,1,1)} \rangle_{\text{Per}_4} = 6$. By Corollary 5.1 taking $r = 3$, $n = 4$ and $\kappa = (1, 1)$, the condition $n \geq r + \kappa_1$ holds, and so this is the ‘stable value’ of the multiplicity: indeed, the dimension of $\Delta_3^{(1,1)}(\delta)$ is the number of 2-marked set partitions of $\{1, 2, 3\}$: these fall into two orbits for the action of $\mathfrak{S}_3$, represented by

$\{\{1\}_*, \{2, 3\}_*\}$ and $\{\{1\}_*, \{2\}_*, \{3\}\}$. If instead we take $\kappa = (2)$ then the bound $n \geq r + \kappa_1$ no longer holds when $n = 4$, and correspondingly the number of three-step paths from (4) to $(2, 2)$ is 5, one less than the stable value of 6, attained for $n \geq 5$. In this case the $\text{Par}_3(4)$ -homomorphism in (4.2)

$$\Delta_3^{(2)} \rightarrow \text{Hom}_{\text{Per}_4}(S^{(2)[4]}, E^{\otimes 3})$$

has a 1-dimensional kernel.

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Partition algebra diagrams. The partition diagrams were drawn in TikZ by adapting code produced by the Sage packages `PartitionDiagram` and `diagram_latex`. Example Sage code is below

```
from sage.combinat.diagram_algebras import
    PartitionDiagrams, diagram_latex
P = PartitionDiagrams(3)
D1 = P([[1, 2, -2, -3], [3, -1]])
print(diagram_latex(D1))
```

REFERENCES

- [1] M. de Boeck, R. Paget and M. Wildon, *Plethysms of symmetric functions and highest weight representations*, Trans. Amer. Math. Soc. **374** (2021), no. 11, 8013–8043.
- [2] Z. Daugherty and R. Orellana, *The quasi-partition algebra*, J. Algebra **404** (2014), 124–151.
- [3] T. Halverson and A. Ram, *Partition algebras*, European J. Combin. **26** (2005), 869–921.
- [4] J. E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, Graduate Texts in Mathematics, vol. 9, Springer, New York, 1972.
- [5] G. D. James, *The Representation Theory of the Symmetric Groups*, Lecture Notes in Mathematics, vol. 682, Springer, Berlin, 1978.
- [6] E. McDowell, *An explicit construction of the Weyl module as a quotient of symmetric tensors by dual Garnir relations*, arXiv:2508.14788, 2025. <https://arxiv.org/abs/2508.14788>.
- [7] E. McDowell and M. Wildon, *Modular plethystic isomorphisms for two-dimensional linear groups*, Journal of Algebra, 602:441–483, 2022.
- [8] M. Wildon, *A construction of the Carter–Lusztig Weyl module*, unpublished notes, version of 23 April 2021 available from <https://www.ma.rhul.ac.uk/~uvah099/Maths/WeylModule.pdf>.