PLETHYSMS OF SYMMETRIC FUNCTIONS AND POLYNOMIAL REPRESENTATIONS OF GENERAL LINEAR GROUPS: MFO AUGUST 2022

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Contents

1. Introduction

This is more a survey paper of my recent work than the summary that was asked for, but I hope it will still be of interest to some readers. If you skim the unnumbered theorems in §2 and §3 and Theorem 3.3 you'll get a flavour of my recent work.

1.1. **Outline.** §2 presents joint work on $\mathrm{SL}_2(\mathbf{C})$-plethysms with Paget [15] and with McDowell [13], replacing $\mathbf{C}$ with an arbitrary field. In §3 we present a theory on the maximal and minimal constituents of a general plethysm $s_\nu \circ s_\mu$ proved using highest-weight vectors in joint work with deBoeck and Paget [4], and give some related stability results. In §4 reports on recent joint work with Paget with a more combinatorial flavour. In §5 makes a connection with a recent paper of Law [9]. Finally §6 has some open problems. The appendix is a record, of most interest only to the author, of work in progress with Paget, which we hope to continue at the MFO meeting.

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1.2. **Plethysms and modules.** Let E be the natural representation of $\mathrm{GL}_d(\mathbf{C})$ and let $\mathrm{Par}(n)$ be the set of partitions of n. For $\lambda \in \mathrm{Par}(n)$, the Schur function s_λ, evaluated at d variables, is the formal character of the polynomial representation $\nabla^\lambda(E)$ of $\mathrm{GL}_d(\mathbf{C})$. Equivalently, $s_\lambda(\gamma_1, \ldots, \gamma_d)$ is the trace of the diagonal matrix $\mathrm{diag}(\gamma_1, \ldots, \gamma_d)$ acting on $\nabla^\lambda(E)$. For instance, $\nabla^{(n)}(E) = \mathrm{Sym}^n E$ and correspondingly

$$s_{(n)}(x_1, \ldots, x_d) = \sum_{1 \leq i_1 \leq \ldots \leq i_n \leq d} x^{i_1} \ldots x^{i_d}$$

is the complete symmetric function of degree n in d variables. To give another example, if $d = 3$ then $\nabla^{(2,1)}(E)$ restricts to the 8-dimensional adjoint representation of $\mathrm{SL}_3(\mathbf{C})$, famous from the eight-fold way. Its character $s_{(2,1)}(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + 2x_1 x_2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2$ enumerates the set $\mathrm{SSYT}_{\leq 3}(2,1)$ of the 8 semistandard tableaux of shape $(2,1)$ with entries from $\{1, 2, 3\}$. These tableaux are shown in Figure 1 labelling weight vectors on the $\mathsf{sl}_3(\mathbf{C})$-root lattice

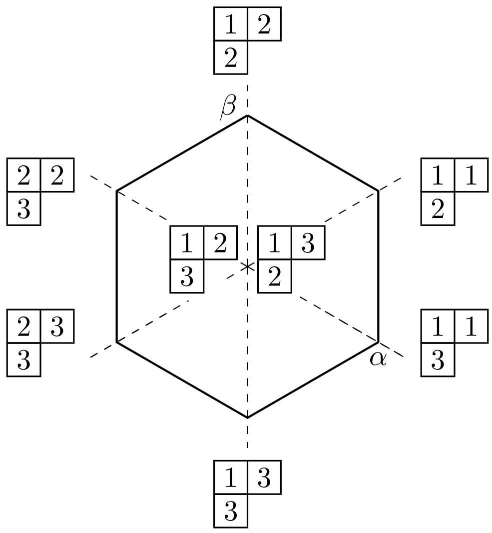


FIGURE 1. The eight-fold way representation of either $\mathrm{SL}_3(\mathbf{C})$ or $\mathsf{sl}_3(\mathbf{C})$ with weight spaces labelled by semistandard tableaux in $\mathrm{SSYT}_{\leq 3}(2,1)$: in the standard notation $\alpha = \varepsilon_1 - \varepsilon_2$, $\beta = \varepsilon_2 - \varepsilon_3$. Taking these as basic positive roots, the unique highest weight vector (up to scalars) spans the weight space for $\alpha + \beta$; in the construction used in [4], it is $e_1^2 \otimes e_2 - e_1 e_2 \otimes e_1 \in \mathrm{Sym}^2 E \otimes E$, where E has basis e_1, e_2, e_3. See §3 for an indication of the general definition of $\nabla^\lambda(E)$

The *plethysm product* $s_\nu \circ s_\mu$ may be defined as the formal character of the composite representation $\nabla^\nu\big(\nabla^\mu(E)\big)$. Decomposing $s_\nu \circ s_\mu$ as a sum of Schur functions, or equivalently, decomposing the representation $\nabla^\nu\big(\nabla^\mu(E)\big)$ into a direct sum of irreducible representations, is one of the main open problems in algebraic combinatorics.

Example 1.1. Let $d = 2$ and let E have basis e_1, e_2. There is a $\mathrm{GL}_2(\mathbf{C})$-equivariant map $\mathrm{Sym}^2\big(\mathrm{Sym}^2 E\big) \to \mathrm{Sym}^4 E$ defined by $(uv)(xy) \mapsto uvxy$.

The kernel of 'multiply-out' is spanned by $e_1^2e_2^2-(e_1e_2)^2$. This vector affords the one-dimensional determinant representation of $\mathrm{GL}(E)$, isomorphic to $\nabla^{(2,2)}(E)$. Therefore, by complete reducibility,

$$\mathrm{Sym}^2\bigl(\mathrm{Sym}^2 E\bigr) \cong \mathrm{Sym}^4 E \oplus \nabla^{(2,2)}(E). \tag{$\star$}$$

The proof reveals explicit highest weight vectors $(e_1^2)^2$ and $e_1^2e_2^2-(e_1e_2)^2$ generating the two summands (see §3 for why this is significant). The decomposition ($\star$) can of course also be obtained working only with symmetric functions. By definition $s_{(2)}(x_1,x_2)=x_1^2+x_1x_2+x_2^2$, $s_{(2)}(y_1,y_2,y_3)=y_1^2+y_1y_2+y_2^2+y_1y_3+y_2y_3+y_3^2$. Hence, by the definition of plethysm as 'substitute monomials for variables', we have

$$\begin{aligned}(s_{(2)}\circ s_{(2)})(x_1,x_2,x_3) &= s_{(2)}(x_1^2,x_1x_2,x_2^2)\\ &= (x_1^2)^2+(x_1^2)(x_1x_2)+(x_1x_2)(x_1x_2)+(x_1^2)(x_2^2)+(x_1x_2)x_2^2+(x_2^2)^2\\ &= x_1^4+x_1^3x_2+2x_1^2x_2^2+x_1x_2^3+x_2^4\\ &= s_{(4)}(x_1,x_2)+s_{(2,2)}(x_1,x_2).\end{aligned}$$

The six summands above correspond to the canonical basis $(e_1^2)^2$, $(e_1^2)(e_1e_2)$, $(e_1e_2)^2$, $(e_1^2)(e_2^2)$, $(e_1e_2)(e_2^2)$, $(e_2^2)^2$ of $\mathrm{Sym}^2\,\mathrm{Sym}^2 E$ and to the 6 tableaux below, in which each outer tableaux has two entries from $\mathrm{SSYT}_{\leq 2}(2)$.

$$\boxed{\boxed{1\,|\,1}\ \boxed{1\,|\,1}}\,,\ \boxed{\boxed{1\,|\,1}\ \boxed{1\,|\,2}}\,,\ \boxed{\boxed{1\,|\,1}\ \boxed{2\,|\,2}}\,,\ \boxed{\boxed{1\,|\,2}\ \boxed{1\,|\,2}}\,,\ \boxed{\boxed{1\,|\,2}\ \boxed{2\,|\,2}}\,,\ \boxed{\boxed{2\,|\,2}\ \boxed{2\,|\,2}}\,.$$

We return to these 'plethystic semistandard tableaux' in §3 and §4 below. We refer the reader to [4, Examples 1.7 ,1.10] for a geometric interpretation of this example and a generalization that gives explicit decompositions by highest weight vectors of $\mathrm{Sym}^2\,\mathrm{Sym}^n E$ and $\mathrm{Sym}^n\,\mathrm{Sym}^2 E$ for all n and d.

As this example suggests, the decomposition problem can profitably by attacked by both algebraic and combinatorial methods.

For background on polynomial representations of $\mathrm{GL}_d(\mathbf{C})$ see [7]. The modules $\nabla^\lambda(E)$ are constructed in [4] and also in [7, Ch. 4]. For the general definition of plethysm and further background I recommend [10].

2. Plethysms for special linear groups

2.1. $\mathrm{SL}_2(\mathbf{C})$ **plethysms and Stanley's Hook Content Formula.** Classical Hermite reciprocity states that $\mathrm{Sym}^n\,\mathrm{Sym}^m E \cong \mathrm{Sym}^m\,\mathrm{Sym}^n E$, where E is the natural representation of $\mathrm{SL}_2(\mathbf{C})$. In the setting of plethysms this is equivalent to

$$(s_n\circ s_m)(q,q^{-1}) = (s_m\circ s_n)(q,q^{-1})$$

and so to

$$s_n(1,q,\ldots,q^m) = s_m(1,q,\ldots,q^n)$$

(The two variables x_1 and x_2 become q and q^{-1} to account for the determinant condition on $\mathrm{SL}_2(\mathbf{C})$.) The left-hand side above is the generating function for partitions contained in the $m \times n$ box and similarly the right-hand side is the generating function for partitions in the $n \times m$ box. Therefore the two sides are equal. An attractive feature of this proof is that it shows the connections between three areas of mathematics:

$$\boxed{\begin{array}{l}\text{representations}\\ \text{of } \mathrm{SL}_2(\mathbf{C})\end{array}} \longrightarrow \boxed{\begin{array}{l}\text{plethysms of sym-}\\ \text{metric functions}\end{array}} \longrightarrow \boxed{\begin{array}{l}\text{combinatorial}\\ \text{enumeration}\end{array}}.$$

More generally, $s_\lambda(1, q, \ldots, q^m)$ is the generating function enumerating by weight (i.e. the sum of all the entries) the set of semistandard Young tableau of shape λ with entries from $0, 1, \ldots, m$. It has a beautiful closed form given by Stanley's Hook Content Formula:

$$s_\lambda(1, q, \ldots, q^m) = q^{b(\lambda)} \frac{\prod_{(i,j)\in[\lambda]} [j - i + m + 1]_q}{\prod_{(i,j)\in[\lambda]} [h_{(i,j)}]_q}$$

where $h_{(i,j)}$ is the hook length for the box (i, j) in the Young diagram of λ, $[c]_q = 1 + q + \cdots + q^{c-1}$ is the usual quantum integer and $b(\lambda)$ is a suitably defined power. For instance, to prove Hermite reciprocity using Stanley's Hook Content Formula, we write

$$\begin{aligned} s_n(1, q, \ldots, q^m) &= \frac{[m + 1]_q \ldots [m + n]_q}{[1]_q \ldots [m]_q} = \binom{m+n}{m}_q \\ &= \binom{m+n}{n}_q = \frac{[n + 1]_q \ldots [m + n]_q}{[1]_q \ldots [n]_q} = s_m(1, q, \ldots, q^n). \end{aligned}$$

It is a simple exercise to give a similar proof of the Wronksian isomorphism $\bigwedge^\ell \mathrm{Sym}^{\ell+m-1} E \cong \mathrm{Sym}^\ell \mathrm{Sym}^m E$. In joint work with Paget [15] we used the three step machine above and Stanley's Hook Content Formula to give many more necessary and sufficient conditions for isomorphisms of representations of $\mathrm{SL}_2(\mathbf{C})$. The following simultaneous generalization of Hermite reciprocity and the Wronksian isomorphism is a typical example.

Theorem (Theorem 1.6 in [15]). *Let λ be a partition with at most ℓ parts. There is an isomorphism $\nabla^\lambda \mathrm{Sym}^\ell E \cong \mathrm{Sym}^a \mathrm{Sym}^b E$ of representations of $\mathrm{SL}_2(\mathbf{C})$ if and only if λ is obtained by adding columns of length $\ell + 1$ to one of the partitions $(a), (1^a), (b), (1^b), (a^b), (b^a)$, and ℓ is respectively b, $a+b-1$, a, $a + b - 1$, b, a.*

2.2. **Modular plethysms.** Replacing $\mathbf{C}$ with a field of prime characteristic reveals some further interesting features. To give some flavour of this, take the setting of Example 1.1 and consider the left action of $\mathrm{GL}_2(\mathbf{C})$, first on $\mathrm{Sym}^2 E$ defined as usual as a quotient of $E \otimes E$, and then on $\mathrm{Sym}_2 E$ defined as a subspace of $E \otimes E$.

$$\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \xmapsto{\mathrm{Sym}^2} \begin{array}{c} \begin{array}{ccc} e_1^2 & e_2^2 & e_1e_2 \end{array} \\ \begin{pmatrix} \alpha^2 & \beta^2 & \alpha\beta \\ \gamma^2 & \delta^2 & \gamma\delta \\ 2\alpha\gamma & 2\beta\delta & \alpha\delta - \beta\gamma \end{pmatrix} \end{array} \qquad \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \xmapsto{\mathrm{Sym}_2} \begin{array}{c} \begin{array}{ccc} e_1\otimes e_1 & e_2\otimes e_2 & e_1\otimes e_2+e_2\otimes e_1 \end{array} \\ \begin{pmatrix} \alpha^2 & \beta^2 & 2\alpha\beta \\ \gamma^2 & \delta^2 & 2\gamma\delta \\ \alpha\gamma & \beta\delta & \alpha\delta - \beta\gamma \end{pmatrix} \end{array}$$

These matrices show that, in characteristic 2, $\mathrm{Sym}^2 E$ has a two-dimensional simple submodule with quotient the determinant representation, whereas $\mathrm{Sym}_2 E$ has the determinant in its socle, with the two-dimensional composition factor at its top. In fact $\mathrm{Sym}_2 E$ is isomorphic to the contravariant dual of $\mathrm{Sym}^2 E$, in the sense of [7, §2.7], and also, one can show, to $\bigwedge^2 \mathrm{Sym}^2 E$. (As partial motivation for this, recall for a k-dimensional representation V of a finite group G we have $\bigwedge^{k-1} V \cong V^\star$.) This suggests the correct modular generalization of the Wronskian isomorphism.

Theorem (Theorem 1.4 in [13])**.** *Let F be a field and let E be the natural representation of $\mathrm{SL}_2(F)$. For m, $n \in \mathbf{N}$ there is an isomorphism of $\mathrm{SL}_2(F)$-representations*

$$\mathrm{Sym}_m \mathrm{Sym}^n E \cong \bigwedge^m \mathrm{Sym}^{m+n-1} E.$$

I like to think of this as a categorification the binomial identity $\left(\!\binom{n+1}{m}\!\right) = \binom{m+n}{m}$, where $\left(\!\binom{a}{b}\!\right)$ is the number of b-multisubsets of a set of size a.

As a corollary we obtain a modular version of Hermite reciprocity, namely $\mathrm{Sym}_m \mathrm{Sym}^n E \cong \mathrm{Sym}^n \mathrm{Sym}_m E$, holding over any field. Our proof gives an explicit isomorphism. See [1, Remark 3.2] for a different proof of this result. In [13] we also give an infinite family of examples of plethysms in which an isomorphism holds working over $\mathbf{C}$, but not over a field of given prime characteristic p, even after considering all possible dualities. This demonstrates that the existence of such 'modular plethystic isomorphisms' is far from obvious.

3. Highest weight vectors and plethysms

In my joint paper [4] with de Boeck and Paget we used highest weight vectors to prove some new results on plethysms. We also generalized and gave unified proofs for several older results. An important preliminary is to define a canonical module isomorphic to $\nabla^\nu\big(\nabla^\mu(E)\big)$ with a basis indexed by the plethystic semistandard tableaux seen in Example 1.1.

3.1. **Plethystic semistandard tableaux.** Throughout let $d \in \mathbf{N}$ and let ν and μ be partitions.

Definition 3.1. A *plethystic semistandard tableau* of *shape* μ^ν is a semistandard ν-tableau whose entries are semistandard μ-tableaux. We denote by

$\mathrm{PSSYT}_{\leq d}(\nu, \mu)$ the set of plethystic semistandard tableaux of *outer shape* ν whose μ-tableaux have entries from $\{1, \ldots, d\}$.

Example 1.1 shows the six elements of $\mathrm{PSSYT}\big((2),(2)\big)$. Note that to interpret 'semistandard' at the level of the outer ν-tableau one needs a total order on the set of semistandard μ-tableaux. For $s, t \in \mathrm{SSYT}_{\leq d}(\mu)$, we set $t < u$ if and only if the greatest entry appearing in a different column of t to u appears further to the right in u. (It might be absent from t.) The precise choice of order is irrelevant, for the same reason that one can pick any order on $\{1, \ldots, d\}$ to define 'semistandard' in the usual integer case.

We refer the reader to [4, §2] for the general construction of $\nabla^\nu\big(\nabla^\mu(E)\big)$ using plethystic semistandard tableaux. Here we shall consider the important special case when $\nu = (1^n)$. In this case $\nabla^{(1^n)}\big(\nabla^\mu(E)\big)$ may be identified with $\bigwedge^n \nabla^\mu(E)$. Let $F(t) \in \nabla^\mu(E)$ be the basis vector defined in [4, §2] canonically labelled by $t \in \mathrm{SSYT}_{\leq d}(\mu)$. For example, when $\mu = (2,2)$ we have

$$F\left(\begin{array}{|c|c|}\hline a & c \\ \hline b & d \\ \hline\end{array}\right) = e_a e_c \otimes e_b e_d - e_b e_c \otimes e_a e_d - e_a e_d \otimes e_b e_c + e_b e_d \otimes e_a e_c.$$

A basis for $\bigwedge^n \nabla^\mu(E)$ is all $F(t_1) \wedge \cdots \wedge F(t_n)$ where $t_1, \ldots, t_n \in \mathrm{SSYT}_{\leq d}(\mu)$ and $t_1 < t_2 < \ldots < t_n$. Since an element of $\mathrm{PSSYT}\big((1^n), \mu\big)$ is uniquely determined by the set of entries in the single column of the outer tableau, we see that this basis is in bijection with the set $\mathrm{PSSYT}\big((1^n), \mu\big)$.

3.2. **Maximal tableau families.** A first step towards understanding an arbitrary plethysm $s_\nu \circ s_\mu$ is to understand its maximal and minimal constituents in the dominance order.

Example 3.2. When $\mu = (m)$ we may identify $\nabla^{(m)}(E)$ with $\mathrm{Sym}^m E$. We shall use this to show that $s_{(1^4)} \circ s_{(3)}$ has maximal constituents $s_{(9,1,1,1)}$, $s_{(8,3,1)}$ and $s_{(6,6)}$. To show these appear as summands, it is equivalent to show that if $\dim E = 4$ then $\bigwedge^4 \mathrm{Sym}^3 E$ contains the irreducible representations $\nabla^{(9,1,1,1)}(E)$, $\nabla^{(8,3,1)}(E)$ and $\nabla^{(6,6)}(E)$. The following are eigenvectors for the subgroup of 4×4 diagonal matrices having the weights $(9,1,1,1)$, $(8,3,1,0)$ and $(6,6,0,0)$:

$$e_1^3 \wedge e_1^2 e_2 \wedge e_1^2 e_3 \wedge e_1^2 e^4,$$
$$e_1^3 \wedge e_1^2 e_2 \wedge e_1 e_2^2 \wedge e_1^2 e_3,$$
$$e_1^3 \wedge e_1^2 e_2 \wedge e_1 e_2^2 \wedge e_2^3.$$

$$\begin{array}{|c|}\hline \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline\end{array} \\ \hline \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline\end{array} \\ \hline \begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline\end{array} \\ \hline \begin{array}{|c|c|c|}\hline 1 & 1 & 4 \\ \hline\end{array} \\ \hline\end{array}$$

Note that the first is

$$F\big(\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline\end{array}\big) \wedge F\big(\begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline\end{array}\big) \wedge F\big(\begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline\end{array}\big) \wedge F\big(114\big)$$

which (after identifying $\nabla^{(1^3)}$ with $\bigwedge^3$) is the basic vector of $\nabla^{(1^3)}\big(\nabla^{(3)}(E)\big)$ labelled by the plethystic semistandard tableaux shown in the margin. Similarly the other two weight vectors are elements of the canonical basis of

$\nabla^{(1^3)}\big(\nabla^{(3)}(E)\big)$ labelled by elements of $\mathrm{PSSYT}_{\leq 4}\big((1^4),(2,2)\big)$. A convenient way to see that these weight vectors are indeed highest weight uses the Lie algebra action of $\mathsf{sl}_d(\mathbb{C}$ on $\bigwedge^4 \mathrm{Sym}^3 E$. For $i \in \{2,3,4\}$ define $X^{(i)} \in \mathsf{sl}_4(\mathbb{C})$ by $X^{(i)}e_i = e_{i-1}$ and $X^{(i)}e_j = 0$ if $j \neq i$. A weight vector is highest weight if and only if it is killed by each of the $X^{(i)}$. For instance,

$$\begin{aligned} X^{(2)} \cdot e_1^3 \wedge e_1^2 e_2 \wedge e_1 e_2^2 \wedge e_2^3 &= e_1^3 \wedge e_1^3 \wedge e_1 e_2^2 \wedge e_2^3 \\ &\quad + e_1^3 \wedge e_1^2 e_2 \wedge 2e_1^2 e_2 \wedge e_2^3 + e_1^3 \wedge e_1^2 e_2 \wedge e_1 e_2^2 \wedge 3e_1 e_2^2 = 0. \end{aligned}$$

We leave it to the reader to continue by similar methods to see that no more dominant partition can be the weight of a weight vector, and that no other maximal constituents appear.

To generalize from this example, we say that a set of n distinct μ-tableaux is a *tableau family* of *shape* μ^n. We define its *weight* $\mathrm{wt}(\mathcal{T})$ to be the sum of the weights of its μ-tableau elements and its *type* $\mathrm{type}(\mathcal{T})$ to be the conjugate of its weight, assuming that this is a partition. A μ-tableau family is *maximal* if its weight is maximal in the dominance order, over all μ-tableau families of its shape. The following is a special case of [4, Theorem 1.5].

Theorem 3.3. *Let m, $n \in \mathbf{N}$ and let $\mu \in \mathrm{Par}(m)$.*

(i) *The maximal partitions labelling constituents of $s_{(1^n)} \circ s_\mu$ are the weights of the maximal tableau families of shape μ^n.*

(ii) *If m is even then the minimal partitions labelling constituents of $s_{(1^n)} \circ s_\mu$ are the types of the maximal ableau families of shape μ'^n.*

This theorem can be proved using highest weight vectors by generalizing the method seen in the example above. One difficulty, which in [4] we work around using the analogue of Garnir relations for $\nabla^\mu(E)$, is that the summands of $X^{(i)} \cdot F(t)$ are not in general labelled by semistandard tableaux. For example,

$$X^{(4)} \cdot F\left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 & 4 \\ \hline\end{array}\right) = F\left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array}\right) + F\left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 & 3 \\ \hline\end{array}\right).$$

The difficulty is compounded when we replace $\bigwedge^n$ with a general Schur functor ∇^ν, as in the full version of Theorem 1.5. We indicate a simpler combinatorial proof of Theorem 1.5 in §4. See also §7 for some work in progress which strengthens Theorem 3.3, and Theorem 4.1 for another application of tableau families.

3.3. **Stability results.** Two further theorems from [4] concern stability properties of plethysm coefficients. Again they are proved using highest weight vectors. Let $(k) \sqcup \lambda$ denote the partition obtained from λ by inserting a new part of size k.

Theorem (Theorem 1.1 from [4]). *Let $\nu \in \mathrm{Par}(n)$, let $\mu \in \mathrm{Par}(m)$ and let $\lambda \in \mathrm{Par}(mn)$. If r is at least the greatest part of μ then*

$$\big\langle s_\nu \circ s_{(r)\sqcup\mu}, s_{(nr)\sqcup\lambda}\big\rangle = \big\langle s_\nu \circ s_\mu, s_\lambda\big\rangle.$$

This theorem was proved in the special case $\mu = (1^m)$ in [3]. The main step in our proof constructs an explicit bijection between the highest weight vectors in $\nabla^\nu\big(\nabla^\mu(E)\big)$ of highest weight λ and the highest weight vectors in $\nabla^\nu\big(\nabla^{(r)\sqcup\mu}(E)\big)$ of highest weight $(nr) \sqcup \lambda$.

Theorem (Theorem 1.2 from [4]). *Let $\nu \in \mathrm{Par}(n)$, let $\mu \in \mathrm{Par}(m)$ and let $\lambda \in \mathrm{Par}(mn)$. If $r \in \mathbf{N}$ then*

$$\big\langle s_\nu \circ s_{\mu+(1^r)}, s_{\lambda+(n^r)}\big\rangle \geq \big\langle s_\nu \circ s_\mu, s_\lambda\big\rangle$$

and moreover $\big\langle s_\nu \circ s_{\mu+N(1^r)}, s_{\lambda+N(n^r)}\big\rangle$ is constant for

$$N \geq n(\mu_1 + \cdots + \mu_{r-1}) + (n-1)(\mu_r + \mu_{r+1} - (\lambda_1 + \cdots + \lambda_r).$$

This theorem was first proved by Brion in [2] using geometric methods. Our proof uses highest weight vectors to show that the condition N is tight in infinitely many cases, and gives an explicit upper bound for the stable multiplicity.

4. Some new results

This section outlines ongoing work with Paget in which we prove new stability results, and generalize some of the theorems mentioned in §3 to Schur functions labelled by skew partitions. Our methods are now entirely combinatorial, using that if $\nu/\nu^\star$ and $\mu/\mu^\star$ are skew partitions then

$$(s_{\nu/\nu^\star} \circ s_{\mu/\mu^\star})(x_1, \ldots, x_d)$$

enumerates by weight the set $\mathrm{PSSYT}_{\leq d}\big(\nu/\nu^\star, \mu/\mu^\star\big)$ of plethystic semistandard tableaux of shape $\nu/\nu^\star$ having entries from $\mathrm{SSYT}_{\leq d}(\mu/\mu^\star)$. The basic observation used in this section is that, by the duality

$$\langle h_\lambda, \mathrm{mon}_\mu\rangle = \begin{cases} 1 & \text{if } \lambda = \mu \\ 0 & \text{otherwise} \end{cases}$$

between the complete homogeneous symmetric functions h_λ and the monomial symmetric functions mon_μ, we have

$$(\dagger) \qquad \langle s_{\nu/\nu^\star} \circ s_{\mu/\mu^\star}, h_\lambda\rangle = |\mathrm{PSSYT}_{\leq d}\big(\nu/\nu^\star, \mu/\mu^\star\big)_\lambda|,$$

where the subscript λ indicates the subset of plethystic semistandard tableaux of weight λ. The special case that $(s_{(2)} \circ s_{(2)})(x_1, x_2)$ is the generating function enumerating $\mathrm{PSSYT}_{\leq 2}\big((2),(2)\big)$ was seen in Example 1.1: the reader is invited to use $h_{(2,2)} = s_{(4)} + s_{(3,1)} + s_{(2,2)}$ to confirm that $\langle s_{(2)} \circ s_{(2)}, h_{(2,2)}\rangle$ is the coefficient of the monomial $x_1^2 x_2^2$.

4.1. **Revisiting stability results.** To start, we outline how to prove the second theorem in §3.3 in this setting. Since $h_\lambda = s_\lambda + g$ where g is a sum of symmetric functions labelled by partitions μ with $\mu \lhd \lambda$, the multiplicity

$$\langle s_\nu \circ s_{\mu+N(1^r)}, s_{\lambda+N(n^r)} \rangle$$

is constant for $N \geq M$ provided, for each partition $\gamma \in \mathrm{Par}(mn)$ with $\gamma \unrhd \lambda$, the multiplicity $\langle s_\nu \circ s_{\mu+N(1^r)}, h_{\gamma+N(n^r)} \rangle$ is constant (that is, dependent only on γ) for $N \geq M$. Therefore it suffices to show that, for each $\gamma \in \mathrm{Par}(mn)$ with $\gamma \unrhd \lambda$, the size

$$\left|\mathrm{PSSYT}\bigl(\nu, \mu + N(1^r)\bigr)_{\gamma+N(n^r)}\right|$$

is an increasing function of N, constant for N at least the given bound. In turn (and being somewhat rough) this holds because when N is very large, almost every μ-tableaux entry of a plethystic semistandard tableaux $T \in \mathrm{PSSYT}\bigl(\nu, \mu+N(1^r)\bigr)$ of weight $\gamma+N(n^r)$ must have, in its top-right, N columns each beginning $1, 2, \ldots, r$. An analogous result, with a more technical lower bound on N, holds replacing μ with an arbitrary skew partition and leads to a generalization of the theorem.

4.2. **New stability results.** The maximal tableaux families from §3.2 reappear in the following new result.

Theorem 4.1. *Let $\nu \in \mathrm{Par}(m)$ and let $\mu/\mu^\star$ be a skew partition of m. Let κ be the weight of a maximal $\mu/\mu^\star$-tableaux family of size r. The multiplicity $\langle s_{\nu+N(1^r)} \circ s_{\mu/\mu^\star}, s_{\lambda+N\kappa} \rangle$ is constant for N sufficiently large.*

Two special cases of Theorem 4.1, both with $\mu^\star = \varnothing$, have already appeared in the literature.

(i) There is a unique maximal μ-tableaux family of size 1, whose single element is the unique μ-tableaux of weight μ. In this case Theorem 4.1 states that $\langle s_{\nu+N} \circ s_\mu, s_{\lambda+N\mu} \rangle$ is constant for N sufficiently large. This result was first proved by Brion in [2, Theorem 3.1].

(ii) For each $d \in \mathbf{N}$, there is a maximal μ-tableaux family consisting of *every* semistandard Young tableaux of shape μ with entries from $\{1, \ldots, d\}$. Suppose there are r such tableaux. Since these tableaux are enumerated by the symmetric function $s_\mu(x_1, \ldots, x_d)$, each element of $\{1, \ldots, d\}$ appears overall the same number rm/d times as an entry in these r tableaux. The weight of the family is therefore (q^d), where $q = rm/d$. Theorem 4.1 states that $\langle s_{\nu+N(1^r)} \circ s_\mu, s_{\lambda+N(q^d)} \rangle)$ is constant for N sufficiently large. In fact, by [5, (9)], the multiplicity is constant for all N. It remains to be seen if a more careful proof of Theorem 4.1 will give an explicit bound on when the multiplicity becomes constant strong enough to imply the result in [5].

The following example shows a case more typical of the full generality of Theorem 4.1.

Example 4.2. Applied with the maximal (2)-tableaux family

$$\left\{ \boxed{1\,|\,1},\ \boxed{1\,|\,2},\ \ldots,\ \boxed{1\,|\,n} \right\}$$

of weight $(n+1, 1^{n-1})$, a special case of Theorem 4.1 implies that, for any $\varepsilon \in \mathbb{Z}^r$, the multiplicity

$$\bigl\langle s_{N(1^r)} \circ s_{(2)}, s_{\varepsilon + N(n+1,1^{n-1})} \bigr\rangle$$

is constant for N sufficient large.

4.3. **Maximal and minimal constituents.** The following theorem generalizes Theorem 3.3 above and Theorem 1.5 in [4].

Theorem 4.3 (Maximals for skew partitions)**.** *Let $\mu/\mu^\star \in \mathrm{SPar}(m)$ and let $\nu/\nu^\star \in \mathrm{SPar}(n)$. The maximal partitions λ in the dominance order such that s_λ is a constituent of $s_\nu \circ s_\mu$ are precisely the maximal weights of the plethystic semistandard tableaux of shape $(\mu/\mu^\star)^{\nu/\nu^\star}$.*

Note that when $\mu^\star = \varnothing$ and $\nu/\nu^\star = (1^n)$ the relevant plethystic semistandard tableaux are uniquely determined by the μ-tableau family of entries in their single column, and this theorem reduces to Theorem 3.3(i). The main idea needed in the proof was seen earlier in §4.1.

5. The Law–Okitani Theorem

Let $\kappa \sqcup (1^j)$ denote the partition obtained from κ by inserting j new parts of size 1. In [9], Law and Okitani prove the following result.

Proposition (Proposition 5.3 of [9])**.** *The sequence of multiplicities*

$$\bigl\langle s_{\nu \sqcup (1^r)} \circ s_{(2)}, s_{\lambda + (r) \sqcup (1^r)} \bigr\rangle$$

is constant for r sufficiently large.

At the July meeting in Hannover in memory of Prof. Christine Bessenrodt, Law announced a generalization of this proposition in which (2) is replaced with a general (m) and $\lambda + (r) \sqcup (1^r)$ with $\lambda + (r(m-1)) \sqcup (1^r)$. I find it striking that the partition $(r, 1^r)$ is the weight of the maximal (1^2)-tableaux family of size r

$$\left\{ \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array},\ \begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline \end{array},\ \ldots,\ \begin{array}{|c|}\hline 1 \\ \hline r^+ \\ \hline \end{array} \right\}$$

where r^+ denotes $r+1$. Also, generalizing Example 4.2, $\bigl(r(m-1)+1, 1^{r-1}\bigr)$ is the weight of the following maximal (m)-tableaux family of size r

$$\left\{ \boxed{1\,|\,1\,|\,\cdots\,|\,1}\quad \boxed{1\,|\,1\,|\,\cdots\,|\,2}\quad \cdots\quad \boxed{1\,|\,1\,|\,\cdots\,|\,r} \right\}$$

The following question therefore seems worth pursuing.

Question 5.1. *Is there a further generalization of Proposition 5.3 of* [9] *in which the hypotheses include an arbitrary maximal tableaux family, and λ grows in a way defined by the weight of this family?*

It appears not to be possible to prove Proposition 5.3 in [9] using only the combinatorial methods of §4.1. It may however be possible to give an alternative proof using highest weight vectors as in §3 and the methods from [4]. Such a proof might suggest how to answer Question 5.1 in the affirmative.

6. Problems on modular plethysms

We end with three questions concerning categorifications of certain symmetric function identities.

6.1. **Newell's theorem.** To motivate our first question we recall a theorem of Newell (used several times in [9]) that if $\lambda \in \mathrm{Par}(mn)$ then

$$\langle s_{(n)} \circ s_{(m+1)}, s_{(1^n)+\lambda} \rangle = \langle s_{(1^n)} \circ s_m, s_\lambda \rangle.$$

This is stated as Theorem 4.4 in [9] and proved using deflations (in the sense of [6]). Another reference is [16]. The proof below uses the permutation module $H^{(a^b)}$ of S_{ab} acting on set partitions of $\{1,\ldots,ab\}$ into b sets each of size a.

Proof of Newell's Theorem. Observe that, by Pieri's rule, $s_\lambda s_{(1^n)} = s_{\lambda+(1^n)} + g$ where g is a sum of Schur functions labelled by partitions with strictly more than n parts. Since no such Schur functions appear in $s_{(n)} \circ s_{(m+1)}$, for example because this plethysm is contained in $(s_{(m+1)})^n$, we have

$$(\ddagger) \qquad \langle s_{(n)} \circ s_{(m+1)}, s_{(1^n)+\lambda} \rangle = \langle s_{(n)} \circ s_{(m+1)}, s_\lambda s_{(1^n)} \rangle.$$

We now interpret the right-hand side in the symmetric group and apply Frobenius reciprocity to get

$$\begin{aligned}\big\langle s_{(n)} \circ s_{(m+1)}, s_\lambda s_{(1^n)} \big\rangle &= \dim \mathrm{Hom}_{FS_{n(m+1)}} \big(H^{(m+1)^n}, S^\lambda \otimes \mathrm{sgn}_{S_n} \big\uparrow_{S_{mn}\times S_n}^{S_{(m+1)n}} \big) \\ &= \dim \mathrm{Hom}_{FS_{nm}\times S_n} \big(H^{(m+1)^n} \big\downarrow_{S_{mn}\times S_n}, S^\lambda \otimes \mathrm{sgn}_{S_n} \big)\end{aligned}$$

where S^λ is the Specht module labelled by λ. The maximal submodule M of $H^{(m+1)^n} \downarrow_{S_{mn}\times S_n}$ on which $S_{\{mn+1,\ldots,mn+n\}}$ acts as the sign representation is spanned by the anti-symmetrized set partitions

$$\{A_1, \ldots, A_n\} \sum_{\sigma \in S_{\{mn+1,\ldots,mn+n\}}} \sigma \, \mathrm{sgn}(\sigma).$$

Note that this quantity is non-zero if and only if $\{mn+1, \ldots, mn+n\}$ is a transversal set of representatives for $A_1, \ldots, A_n$. We may therefore assume that A_i contains $mn+i$. Observing that if $\tau \in S_{mn}$ swaps two of the sets $A_i \backslash \{mn+i\}$ then it acts with a sign on the antisymmetrised sum, we see that $M \cong \big(\mathrm{Inf}_{S_n}^{S_m \wr S_n} \mathrm{sgn}_{S_n}\big)\big\uparrow^{S_m \wr S_n}$. This is the symmetric group module corresponding to the plethysm $s_{(1^n)} \circ s_m$, and so it follows that the right hand side of (‡) is $\langle s_{(1^n)} \circ s_m, s_\lambda \rangle$, as required. $\square$

In the proof we implicitly constructed an adjoint functor to the functor $U \mapsto (U \otimes \mathrm{sgn}_{S_n})\big\uparrow_{S_r \times S_n}^{S_{r+n}}$ from mod$-S_r$ to mod$-S_{r+n}$. Since we used that M is the maximal direct summand on which sgn_{S_n} acts isotypically, it is defined only in characteristic zero. More generally, one can define such an adjoint for any partition κ, using the deflation/restriction functors in [6].

Question 6.1. *Let F be a field of prime characteristic. When is there a left- or right- adjoint to the functor L_κ : mod-$FS_n \to$ mod-FS_{n+r} defined by $L_\kappa(V) = (V \otimes S^\kappa)\big\uparrow_{S_n \times S_r}^{S_{n+r}}$?*

6.2. **Modular adjoints to multiplication by a Schur function.** We now give the general linear group version of the previous problem. Fix an infinite field F and let mod$-\mathrm{GL}_d(F)_s$ denote the category of polynomial representations of $\mathrm{GL}_d(F)$ of polynomial degree s. For any partitions $\nu \in \mathrm{Par}(n), \kappa \in \mathrm{Par}(r)$ and $\lambda \in \mathrm{Par}(n+r)$ we have

$$\langle s_\lambda s_\kappa, s_\nu \rangle = \langle s_\lambda, s_{\nu/\kappa} \rangle.$$

Working over $\mathbf{C}$, the special case for partitions with at most d parts is equivalent to

$$\dim \mathrm{Hom}_{\mathrm{GL}_d(\mathbf{C})}\big(\nabla^\lambda E \otimes \nabla^\kappa E, \nabla^\nu E\big) = \dim \mathrm{Hom}_{\mathrm{GL}_d(\mathbf{C})}\big(\nabla^\lambda E, \nabla^{\nu/\kappa} E\big).$$

Thus the functor $-\otimes \nabla^\kappa(E) : \mathrm{mod}-\mathrm{GL}_d(F)_n \to \mathrm{mod}-\mathrm{GL}_d(\mathbf{C})_{s+n}$ is adjoint to the functor $\mathrm{mod}-\mathrm{GL}_d(\mathbf{C})_{s+n} \to \mathrm{mod}-\mathrm{GL}_d(\mathbf{C})_s$ defined by $\nabla^\nu(E) \mapsto \nabla^{\nu/\kappa}(E)$. (It is routine to generalise the construction in §2 of [4] of $\nabla^\lambda(E)$ to skew partitions.)

Question 6.2. *Let F be an infinite field of prime characteristic. When does the functor $-\otimes \nabla^\kappa(E) : \mathrm{mod}-\mathrm{GL}_d(F)_n \to \mathrm{mod}-\mathrm{GL}_d(F)_{s+n}$ have a left- or right-adjoint?*

In §6.1 we saw the special case when κ is (1^n) of the analogous result for the symmetric group.

It may seem natural to try to apply the tensor-hom adjunction, but since $\nabla^\kappa(E)$ only has the structure of a representation of $\mathrm{GL}_d(\mathbf{C})$, rather than a bimodule structure, the functor $\mathrm{Hom}_{\mathrm{GL}_d(F)}(\nabla^\kappa, -)$ maps into vector spaces rather than the required $\mathrm{mod}-\mathrm{GL}_d(\mathbf{C})_n$. Questions 6.1 and 6.2 are clearly related, but given the subtle nature of the inverse Schur functor (see [7, §6.2] and [12]) it is not obvious that they will have the same answer.

6.3. **A modular ω involution.** The ω involution is defined on symmetric functions by $s_\lambda \mapsto s_{\lambda'}$. Working with representations over $\mathbf{C}$ it lifts to the endofunctor of $\mathrm{mod} - GL_d(\mathbf{C})$ defined by $\nabla^\lambda(E) \mapsto \nabla^{\lambda'}(E)$.

Question 6.3. *Let F be an infinite field of prime characteristic. When is there a modular lift of the ω involution to an endofunctor on* $\text{mod}{-}\mathrm{GL}_d(\mathbf{C})$*?*

The analogous question for the symmetric group has a positive answer: the functor $\Omega : \text{mod}{-}S_n \to \text{mod}{-}S_n$ defined by $\Omega(V) = V \otimes \text{sgn}$ satisfies $\Omega(S^\lambda) = (S^{\lambda'})^\star$ by [8, Theorem 8.15]. The unexpected duality arises because in prime characteristic we have $S^\lambda \otimes \text{sgn} \cong (S^{\lambda'})^\star$. This makes me think that the answer to Question 6.3 may be negative in general.

7. Appendix: Comparing maximal and minimal constituents

This section is on work in progress with Paget.

7.1. Motivation.

The motivating example for the conjecture in this section is the plethysm $s_{(1^n)} \circ s_{(2)}$: we repeat some of the proof of Corollary 8.6 in [14] below. There are many other routes to this result: see for instance [11, 20, I.8, Exercise 6(d)] or [17, Lemma 7].

$$\begin{array}{|c|}\hline \boxed{1\,|\,1} \\ \hline \boxed{1\,|\,2} \\ \hline \boxed{1\,|\,3} \\ \hline \boxed{2\,|\,2} \\ \hline \boxed{2\,|\,3} \\ \hline \end{array}$$

Example 7.1. We shall prove that

$$s_{(1^5)} \circ s_{(2)} = s_{(6,1,1,1,1)} + s_{(5,3,1,1)} + s_{(4,4,2)}.$$

The marginal plethystic semistandard tableaux corresponds to the multiset family $\big\{\{1,1\},\{1,2\},\{1,3\}\big\} \cup \big\{\{2,2\},\{2,3\}\big\}$ of weight $(4,1,1)+(0,3,1) = (4,4,2)$. The tableau below indicates the contribution from $(0,3,1)$ to the weight $(4,4,2)$ in bold:

$$\begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 1 \\ \hline 2 & \mathbf{2} & \mathbf{2} & \mathbf{2} \\ \hline 3 & \mathbf{3} \\ \cline{1-2} \end{array}\,.$$

Note that the shape of the hook uniquely determines the entries in the two multisets $\{2,2\}$ and $\{2,3\}$. Using Theorem 3.3(i), it follows that $s_{(4,4,2)}$ is a maximal constituent of $s_{(1^5)} \circ s_{(2)}$. Similarly, using Theorem 3.3(ii), the minimal constituents of $s_{(1^n)} \circ s_{(2)}$ are labelled by the types of the maximal families of n distinct 2-sets. The maximal set family $\big\{\{1,2\},\{1,3\},\{1,4\}\big\} \cup \big\{\{2,3\},\{2,4\}\big\}$ has type $(4,1,1)+(0,3,1) = (4,4,2)$. The tableau below indicates the contribution from $(0,3,1)$ to the type $(4,4,2)$ in bold:

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 1 & \mathbf{2} & \mathbf{3} & \mathbf{4} \\ \hline 1 & \mathbf{2} \\ \cline{1-2} \end{array}$$

Therefore $s_{(4,4,2)}$ is both a maximal and a minimal constituent of $s_{(1^5)} \circ s_{(2)}$. We leave it to the reader to verify by similar arguments that $s_{(6,1,1,1,1)}$ and $s_{5,3,1,1)}$ are also simultaneous maximal and minimal constituents. Since any constituent is dominates by a maximal and dominates a minimal, there are no further summands in $s_{(1^5)} \circ s_{(2)}$.

To generalize from this example, given a partition $(\alpha_1, \ldots, \alpha_s)$ with distinct parts, let $2[\alpha]$ denote the partition whose first s parts are $\alpha_1+1, \alpha_2+2, \ldots, \alpha_s+s$ and whose main diagonal hook lengths are $2\alpha_1, 2\alpha_2, \ldots, 2\alpha_s$. Thus $2[(3,2)] = (4,4,2)$.

Proposition 7.2. *For $n \in \mathbf{N}$ we have*

$$s_{(1^n)} \circ s_{(2)} = \sum_{\alpha} s_{2[\alpha]}$$

where the sum is over all partitions α of n with distinct parts.

Proof. By Theorem 3.3(i), the maximal constituents of $s_{(1^n)} \circ s_{(2)}$ are labelled by the maximal weights of families of n distinct 2-multisets. A maximal multiset family $\mathcal{T}$ of shape $(2)^n$ can be decomposed uniquely as $\mathcal{T}_1 \cup \ldots \cup \mathcal{T}_r$ where each $\mathcal{T}_i$ has the form $\bigl\{\{i,i\}, \ldots, \{i, i+b_i\}\bigr\}$. The multisets in $\mathcal{T}_i$ contribute the $2(b_i+1)$ boxes

$$(i,i), (i,i+1), \ldots (i, i+b_i), (i+1, i), (i+b_i-1, i) \tag{$\star\star$}$$

to the Young diagram of the partition $\mathrm{wt}(\mathcal{T})$. Therefore the maximal weights of the families of n distinct 2-multisets are precisely the partitions of the form $2[\alpha]$ for α a partition of n with distinct parts. Similarly a maximal multiset families of shape $(1^2)^n$ can be decomposed uniquely as $\mathcal{T}_1' \cup \ldots \cup \mathcal{T}_r'$ where each $\mathcal{T}_i'$ has the form $\{i, i+1\}, \ldots, \{i, i+1+b_i\}$. The sets in $\mathcal{T}_i'$ contribute exactly the same boxes as in $(\star\star)$ to the Young diagram of the partition $\mathrm{type}(\mathcal{T})$. Therefore, by Theorem 3.3(ii), the maximal constituents of $s_{(1^n)} \circ s_{(2)}$ are precisely the minimal constituents, every constituent is both maximal and minimal, and the decomposition is as claimed. $\square$

7.2. **Conjugate tableau families.** In the course of the proof of Proposition 7.2 we constructed a bijection between the maximal tableaux families of shape $(2)^n$ and the maximal tableaux families of shape $(1^2)^n$. We generalize this bijection as follows.

Definition 7.3. Let $\mu \in \mathrm{Par}(m)$. Given a μ-tableau t, we define the *conjugate tableaux* t' by $(i,j)t' = (j,i)t - j + i$. Given a μ-tableau family $\mathcal{T}$ the *conjugate tableau family* $\mathcal{T}'$ is defined by conjugating each μ-tableau element of $\mathcal{T}$.

It is easily seen that the map $t \mapsto t'$ defines a self-inverse bijection $\mathrm{SSYT}_{\le t+\ell(\mu)}(\mu) \to \mathrm{SSYT}_{\le t+\ell(\mu')}\, \mu'$. The map $\mathcal{T} \mapsto \mathcal{T}'$ does not in general preserve maximality. It does however preserve the weaker *closed* property from [4], stated below in Definition 7.8, and this suffices for our application.

Example 7.4. The closed families of shape $(3,1)^3$ and of shape $(2,1,1)^3$ are shown in the table below.

Closed family	Weight	Closed conjugate family	Type
$\left\{\ytableaushort{111,2},\ \ytableaushort{111,3},\ \ytableaushort{111,4}\right\}$	$(9,1,1,1)$	$\left\{\ytableaushort{11,2,3},\ \ytableaushort{12,2,3},\ \ytableaushort{13,2,3}\right\}$	$(3,3,3,3)$
$\left\{\ytableaushort{111,2},\ \ytableaushort{111,3},\ \ytableaushort{112,2}\right\}$	$(8,3,1)$	$\left\{\ytableaushort{11,2,3},\ \ytableaushort{12,2,3},\ \ytableaushort{11,2,4}\right\}$	$(4,3,2,2,1)$
$\left\{\ytableaushort{111,2},\ \ytableaushort{112,2},\ \ytableaushort{113,2}\right\}$	$(7,4,1)$	$\left\{\ytableaushort{11,2,3},\ \ytableaushort{11,2,4},\ \ytableaushort{11,2,5}\right\}$	$(5,2,2,1^3)$
$\left\{\ytableaushort{111,2},\ \ytableaushort{112,2},\ \ytableaushort{122,2}\right\}$	$(6,6)$	$\left\{\ytableaushort{11,2,3},\ \ytableaushort{11,2,4},\ \ytableaushort{11,3,4}\right\}$	$(4,4,1^4)$

The family of shape $(3,1)^4$ and weight $(7,4,1)$ is closed but not maximal. But all four closed families of shape $(2,1,1)^4$ have maximal weight, and minimal type. Observe that in each row, the weight of the family $\mathcal{T}$ strictly dominates the weight of the conjugate family $\mathcal{T}'$.

Conjecture 7.5. *Let $\mu \in \mathrm{Par}(m)$ with $m \geq 2$. Let $\mathcal{T}$ be a closed tableau family of shape μ^n. Then $\mathrm{wt}(\mathcal{T})$ strictly dominates $\mathrm{type}(\mathcal{T}')$ with equality in and only if $m = 2$.*

We outline a plan to prove this conjecture below. While examples may suggest the result holds, with room to spare, some care is certainly needed. For instance, the conjecture does not hold when $m = 1$, since then the unique closed tableau family is $\left\{\boxed{1},\ \boxed{2},\ \boxed{n}\right\}$ of weight (1^n) and type (n). Since this tableau family is its own conjugate, in this case, the type dominates the weight.

Corollary 7.6 (Conditional on Conjecture 7.5)**.** *Let $m \in \mathbf{N}$ be even and let $n \in \mathbf{N}$. Let $\mu \in \mathrm{Par}(m)$ and let $\nu \in \mathrm{Par}(n)$. Suppose that λ labels a maximal constituent of $s_{(1^n)} \circ s_\mu$. Let $\mathcal{T}$ be the corresponding closed tableau family of shape μ^n. Let $\lambda^\star$ be the type of the closed tableau family $\mathcal{T}'$. Then $\langle s_{(1^n)} \circ s_\mu, s^\star_\lambda \rangle \geq 1$ and $\lambda \trianglerighteq \lambda^\star$ with equality if and only if $m = 2$.*

Proof. This follows from Conjecture 7.5 and Theorem 3.3. $\square$

We remark that Corollary 7.6 can be generalized to an arbitrary partition ν, still with m even, by consider the tableau families corresponding to each column in a plethystic semistandard tableau of shape μ^ν. We believe that there are no new cases where equality holds.

Problem 7.7. *Is there a way to compare maximal and minimal constituents of $s_{(1^n)} \circ s_\mu$, or more generally, $s_\nu \circ s_\mu$, when the size of the partition μ is odd?*

7.3. **Plan to prove Conjecture 7.5.**

Preliminaries. Given a finite multisubset S of $\mathbf{N}$ with entries in $\{1,\ldots,d\}$, define $v(S) \in \mathbf{N}^d$ so that $v(S)_i$ is the multiplicity of i in S. For example, $v\big(\{3,4,4,6\}\big) = (0,0,1,2,0,1,0,\ldots)$. We define the *dominance order* on finite multisubsets of $\mathbf{N}$ by setting $S \trianglerighteq T$ if and only if $v(S) \trianglerighteq v(T)$. Thus $\{3,4,4,6\} \trianglelefteq \{3,4,5,6\}$ but $\{3,4,4,6\} \not\trianglerighteq \{3,5,5,5\}$. Given μ-tableaux t and u, we set $t \succeq u$ if u can be obtained from t by successively decrementing entries of t.

Definition 7.8. A tableau family is *closed* if it is a downset for $\succeq$.

We remark that closed tableau families have weights of partition shape and so have well-defined types.

Reduction to downsets problem. Let $\mathcal{T}$ be a closed tableau family of shape μ^n. Order its elements $t^{(1)},\ldots,t^{(n)}$ so that each initial segment

$$\mathcal{T}^{<\ell} = \big\{t^{(1)},\ldots,t^{^{(\ell)}}\big\}$$

is closed. Let $\omega^{^{(\ell)}} = \mathrm{wt}(\mathcal{T}^{<\ell})$. Observe that each $\mathcal{T}'^{<\ell}$ is closed. Let $\tau^{^{(\ell)}} = \mathrm{type}(\mathcal{T}'^{\leq \ell})$. It suffices to prove that $\omega^{(\ell)} \trianglerighteq \tau^{(\ell)}$ for each ℓ. This is true when $\ell = 1$ since $t^{(1)}$ is the unique least μ-tableau, having μ_i entries of i in its row i, and this tableau is its own conjugate. For the inductive step, we may suppose that

- the Young diagram $[\omega^{(\ell)}]$ is obtained from the Young diagram $[\omega(\ell-1)]$ by adding boxes in rows $x_1 \leq \ldots \leq x_m$, corresponding to entries in $t^{^{(\ell)}}$ of $x_1,\ldots,x_m$;
- x_α is in box (a_α, b_α) of $t^{^{(\ell)}}$.

Then the tableau $t'^{(\ell)}$ has an entry $y_\alpha = x_\alpha - a_\alpha + b_\alpha$ in its box (b_α, a_α) for each α. Moreover $[\tau^{(\ell)}]$ is obtained from $[\tau^{(\ell-1)}]$ by adding boxes in columns $y_1,\ldots,y_m$. Order the x_α so that $x_1 \leq \ldots \leq x_m$ and break ties where $x_\gamma = x_\delta$ so that $y_1 \leq \ldots \leq y_m$. Let z_α be the row of the box added to $[\tau^{(\ell-1)}]$ for y_α. Since $y_1 \leq \ldots \leq y_m$, we have $z_1 \geq \ldots \geq z_m$.

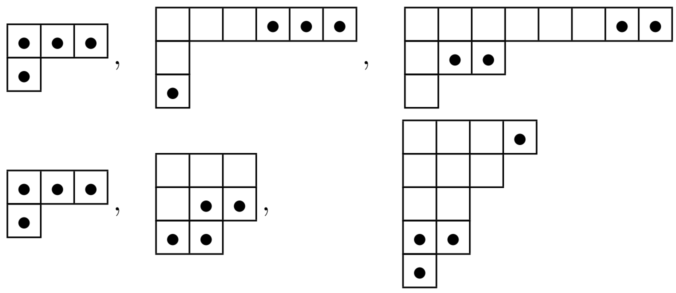
Suppose that y appears exactly once in $t'^{(\ell)}$. Then y appears exactly once in the multiset $\{y_1,\ldots,y_m\}$, as y_γ say, and, by the definition of type, we have $z_\gamma = \mathrm{wt}\big(\mathcal{T}^{<\ell}\big)_y + 1$. In general, y appears as y_γ, $\ldots$, y_δ where $\gamma \leq \delta$. (Thus $y_{\gamma-1} \neq y = y_\gamma = \ldots = y_\delta \neq y_{\delta+1}$.) We now have $z_\gamma > \ldots > z_\delta$ where $z_\gamma = \mathrm{wt}\big(\mathcal{T}^{<\ell}\big)_y + \mathrm{wt}(t)_y$ and $z_\delta = \mathrm{wt}\big(\mathcal{T}^{<\ell}\big)_y + 1$.

Therefore, by the characterisation of the dominance order in terms of single box shifts, it suffices to prove that

$$(\dagger\dagger) \qquad \{x_1,\ldots,x_m\} \;\trianglerighteq\; \bigcup_y \big\{\mathrm{wt}\big(\mathcal{T}^{<\ell}\big)_y + 1, \ldots, \mathrm{wt}\big(\mathcal{T}^{<\ell}\big)_y + \mathrm{wt}(t)_y\big\}$$

where the union is over the distinct elements y of t'. Note that the left-hand side is the multiset of entries of t.

Example 7.9. Take the maximal tableau family of shape $(3,1)^3$ and weight $(8,3,1)$ seen in Example 7.4. The chain $\omega^{(1)}, \omega^{(2)}, \omega^{(3)}$ and the chain $\tau^{(1)}, \tau^{(2)}$, $\tau^{(3)}$ are shown below with the added boxes in each stage marked by $\bullet$:



For instance in the final step we add boxes in columns $1, 1, 2, 4$ of $[\tau^{(2)}]$, corresponding to the entries in the marginal $(2,1,1)$-tableau t' . The corresponding rows are $5, 4, 4, 1$, respectively, obtained from the multiplicities of 1, 2 and 4 read from

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}$$

$$\mathrm{wt}\left\{ \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array} \right\} = (5,3,2,0)$$

by adding the multiplicities in t'.

Special case. Observe that $1 \in \{z_m, \ldots, z_1\}$ if and only if $z_m = 1$, and so if and only if ${t'}^{(\ell)}$ has an entry not seen in any previous $t'^{(j)}$ for $j < \ell$. Since $\mathcal{T}^{<\ell}$ is closed, and $m \geq 2$, there is at least one entry of 1 in $t^{(\ell)}$. Hence $x_1 = 1$ and the first case we need of the dominance condition in (††) holds.

Reduction. Our required condition (††) follows from the lemma below taking $\mathcal{T} = \mathcal{T}^{(\ell)}$.

Lemma 7.10. *Let $\mu \in \mathrm{Par}(m)$ with $m \geq 2$. Let $\mathcal{T}$ be a closed tableau family of shape μ^ℓ and let t be a maximal element of $\mathcal{T}$, so $\mathcal{T}\backslash\{t\}$ is again closed. Then*

$$\{x : x \in t\} \trianglerighteq \big\{\mathrm{wt}(\mathcal{T}')_y + 1, \ldots, \mathrm{wt}(\mathcal{T}')_y + \mathrm{wt}(t)_y : y \in t'\big\}$$

with equality if and only if $m = 2$.

It is not too hard to prove the lemma when μ has either just one row (and so $\mathcal{T}$ is a multiset family) or just one column (and so $\mathcal{T}$ is a set family). These are in some sense the 'tightest' cases, but it will need some care to extend the arguments to the general case.

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