An introduction to modular plethysms

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(joint work with Eoghan McDowell, Rowena Paget)

As motivation I began my talk with the observation that the vector spaces $\mathrm{Sym}^2 \mathbb{C}^{d-1}$ and $\bigwedge^2 \mathbb{C}^d$ both have the same dimension, namely $\binom{d}{2}$. An appealing explanation is that if E is the natural 2-dimensional representation of $\mathrm{SL}_2(\mathbb{C})$ then $\mathrm{Sym}^2 \mathrm{Sym}^{d-1} E \cong \bigwedge^2 \mathrm{Sym}^d E$ as representations of $\mathrm{SL}_2(\mathbb{C})$. This is generalized by the Wronskian isomorphism

$$\mathrm{Sym}^r \mathrm{Sym}^\ell E \cong \bigwedge^r \mathrm{Sym}^{l+r-1} E,$$

categorifying the counting identity that the number of r-multisubsets of $\{1, \ldots, \ell+1\}$ is the number of r-subsets of $\{1, \ldots, \ell+r\}$. It is natural to ask if the Wronskian isomorphism holds over fields other than $\mathbb{C}$. The answer is 'yes', provided that a suitable duality is introduced, replacing a Sym^r with its dual functor Sym_r; this corresponds to taking invariants rather than coinvariants in a tensor power.

Theorem (McDowell–W, Theorem 1.4 in [4]). *Let F be a field and let E be the natural representation of $\mathrm{SL}_2(F)$. For r, $\ell \in \mathbb{N}$ there is an isomorphism of $\mathrm{SL}_2(F)$-representations*

$$\mathrm{Sym}_r \mathrm{Sym}^\ell E \cong \bigwedge^r \mathrm{Sym}^{r+\ell-1} E.$$

Isomorphisms of $\mathrm{SL}_2(\mathbb{C})$-modules such as $\bigwedge^r \mathrm{Sym}^{r+\ell-1} E$ were studied systematically in my joint paper [5] with Rowena Paget. An essential result was the following equivalent characterisations:

(i) $\nabla^\lambda \mathrm{Sym}^\ell E \cong_{\mathrm{SL}_2(\mathbb{C})} \nabla^\mu \mathrm{Sym}^m E$;
(ii) $(s_\lambda \circ s_{(\ell)})(q, q^{-1}) = (s_\mu \circ s_{(m)})(q, q^{-1})$;
(iii) $s_\lambda(q^\ell, q^{\ell-2}, \ldots, q^{-\ell}) = s_\mu(q^m, q^{m-2}, \ldots, q^{-m})$;
(iv) $s_\lambda(1, q, \ldots, q^\ell) = s_\mu(1, q, \ldots, q^m)$ up to a power of q

Here ∇^λ is the Schur functor for the partition λ and $s_\lambda \circ s_\ell$ is the *plethysm product* of the two Schur functions, defined by substituting the monomials in s_ℓ for the variables in s_λ. In this case, the chosen variables are q and q^{-1}, and since $s_\ell(x, y) = x^\ell + x^{\ell-1}y + \cdots + y^\ell$, the monomials are $q^\ell, q^{\ell-2}, \ldots, q^{-\ell}$. Because of this connection with symmetric functions, we refer to isomorphisms, such as those in the theorem stated above, as *modular plethysms*. For further background and some motivation for why composition of Schur functors corresponds to the plethysm product on Schur functions, see [3].

Example. *Hermite reciprocity is the isomorphism*

$$\mathrm{Sym}^r \mathrm{Sym}^\ell E \cong \mathrm{Sym}^\ell \mathrm{Sym}^r E.$$

By the equivalence of (i) *and* (iv), *taking $\lambda = (r)$ and $m = r$, it is equivalent to prove that $s_{(r)}(1, q, \ldots, q^\ell) = s_{(\ell)}(1, q, \ldots, q^r)$. Remembering that $s_{(n)}$ is the complete symmetric function, this follows by interpreting the left-hand side as the* *generating function enumerating partitions whose Young diagram is contained in a box with r rows and ℓ columns, and the right-hand side similarly, using the box with ℓ rows and r columns.*

In this example we saw a combinatorial proof of an algebraic isomorphism. To continue in this theme, a very useful result is Stanley's Hook Content Formula [6, Theorem 7.21.2], which states that there is a power $q^{b(\lambda)}$ such that

$$s_\lambda(1, q, \ldots, q^m) = q^{b(\lambda)} \frac{\prod_{(i,j)\in[\lambda]}[j-i+m+1]_q}{\prod_{(i,j)\in[\lambda]}[h_{(i,j)}]_q}$$

where $[r]_q$ is the quantum integer $(q^r-1)/(q-1)$ and $h_{(i,j)}$ is the hook length of the box (i,j) of the Young diagram $[\lambda]$. Note that $j-i$ is the content of the box $(i,j)\in[\lambda]$, so the quantum integers in the numerator are the contents of $[\lambda]$, shifted by $\ell+1$. Using this result Paget and I proved the following simultaneous generalization of Hermite reciprocity and the Wronksian isomorphism.

Theorem (Paget–W, Theorem 1.6 in [5])**.** *Let λ be a partition with at most ℓ parts. There is an isomorphism $\nabla^\lambda \operatorname{Sym}^\ell E \cong \operatorname{Sym}^a \operatorname{Sym}^b E$ of representations of $\mathrm{SL}_2(\mathbb{C})$ if and only if λ is obtained by adding columns of length $\ell+1$ to one of the partitions $(a), (1^a), (b), (1^b), (a^b), (b^a)$, and ℓ is respectively b, $a+b-1$, a, $a+b-1$, b, a.*

A related result is the converse of a theorem of King. For a fixed s, let $\lambda^{\bullet d}$ denote the complement of the partition λ having largest part at most s in a $d\times s$ box. For example if $s=5$ then $(4,3,3,1)^{\bullet 4} = (4,2,2,1)$.

Theorem (King 1985 [if], Paget–W 2019 [only if])**.** *Let λ have at most d parts. Then*

$$\nabla^\lambda \operatorname{Sym}^\ell E \cong \nabla^{\lambda^{\bullet d}} \operatorname{Sym}^\ell E$$

if and only if $\lambda = \lambda^{\bullet d}$ or $\ell = d-1$.

Using Stanley's Hook Content Formula one can obtain an attractive combinatorial interpretation of this theorem. To illustrate it by example, again take $s=5$ and $\lambda=(4,3,3,1)$. By the theorem, $\nabla^\lambda \operatorname{Sym}^3 E \cong \nabla^{\lambda^{\bullet 4}} \operatorname{Sym}^3 E$. The two tableaux below show the hook lengths of $[\lambda]$ and $[\lambda^{\bullet 4}]$ in ordinary type numbers, and the shifted contents in bold. Please ignore the subscripts for the moment.

$\mathbf{4}_0$	$\mathbf{5}_1$	$\mathbf{6}_2$	$\mathbf{7}_3$	1_0
$\mathbf{3}_0$	$\mathbf{4}_1$	$\mathbf{5}_2$	1_0	3_1
$\mathbf{2}_0$	$\mathbf{3}_1$	$\mathbf{4}_2$	2_0	4_1
$\mathbf{1}_0$	1_0	2_1	5_2	7_3

7_3	5_2	4_1	1_0	$\mathbf{1}_0$
5_2	3_1	2_0	$\mathbf{3}_1$	$\mathbf{2}_0$
4_2	2_1	1_0	$\mathbf{4}_1$	$\mathbf{3}_0$
1_0	$\mathbf{7}_3$	$\mathbf{6}_2$	$\mathbf{5}_1$	$\mathbf{4}_0$

Thus the left-hand tableau has the quantum integers appearing in

$$\prod_{(i,j)\in[\lambda]} [i-j+4]_q \prod_{(i,j)\in[\lambda^{\bullet 4}]} [h_{(i,j)}(\lambda)]_q$$

and the right-hand tableaux has the quantum integers appearing in the analogous product swapping λ and $\lambda^{\bullet 4}$. By Stanley's formula, the two products are equal. Hence, by a unique factorization result for quantum integers, proved as Lemma 3.2 in [5], the multisets of entries in the two tableaux are equal. I gave a combinatorial proof of this fact in [7]. After seeing this paper, Prof. Christine Bessenrodt [2] observed that by [1] a stronger combinatorial result holds, in which the hooks and shifted contents are paired with their corresponding arm lengths, as shown in the tableaux above as subscripts.

Problem. *Give an algebraic proof of Bessenrodt's observation using Jack symmetric functions.*

Returning to the original algebraic theorem, it is natural to ask when its isomorphism holds over other fields. The following result gives, we believe, the more precise possible answer.

Theorem (McDowell–W, Theorem 1.2 in [4])**.** *Let G be a group. Let V be a d-dimensional representation of G over an arbitrary field. Let $s \in \mathbb{N}$, and let λ be a partition with $\ell(\lambda) \leq d$ and first part at most s. There is an explicit isomorphism*

$$\nabla^{\lambda} V \cong \nabla^{\lambda^{\bullet d}} V^{\star} \otimes (\det V)^{\otimes s}.$$

In the final part of my talk I emphasised that the existence of such modular plethysms is far from obvious, and there are many cases where an isomorphism known to hold for $\mathrm{SL}_2(\mathbb{C})$ does not generalize to arbitrary fields.

Theorem (McDowell–W 2020, Theorem 1.6 in [4])**.** *Let F be an infinite field of prime characteristic p. There exist infinitely many pairs (a, b) such that, provided e is sufficiently large, the eight representations of $\mathrm{SL}_2(F)$ obtained from $\nabla^{(a+1,1^b)} \operatorname{Sym}^{p^e+b} E$ by*

- *Replacing ∇ with Δ (duality)*
- *Replacing $(a+1, 1^b)$ with $(b+1, 1^a)$ and $p^e + b$ with $p^e + a$ (King conjugation);*
- *Replacing $\operatorname{Sym}^{\ell} E$ with $\operatorname{Sym}_{\ell} E$ (another duality);*

are all non-isomorphic.

Even establishing the non-existence of an isomorphism is not easy, because the existence of an isomorphism over $\mathrm{SL}_2(\mathbb{C})$ means that many of the standard techniques, for example, considering the image of representations in the Grothendieck ring, are inapplicable. I recommend the further study of these modular plethysms.

References

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[5] Rowena Paget and Mark Wildon, *Plethysms of symmetric functions and representations of* $\mathrm{SL}_2(\mathbf{C})$, arXiv:1907.07616 (July 2019), 51 pages.
[6] Richard P. Stanley, *Enumerative combinatorics. Vol. 2*, Cambridge Studies in Advanced Mathematics, vol. 62, Cambridge University Press, Cambridge, 1999, With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin.
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