A SHORT PROOF OF A PLETHYSTIC
MURNAGHAN–NAKAYAMA RULE

1. Introduction

The purpose of this note is to give a short proof of a plethystic general-
ization of the Murnaghan–Nakayama rule, first stated in [1]. The key step
in the proof uses a sign-reversing involution on sequences of bead moves
on James’ abacus (see [3, page 78]), inspired by the arguments in [4]. The
only prerequisites are the Murnaghan–Nakayama rule and basic facts about
plethysms of symmetric functions.

Let \( s_{\lambda/\nu} \) denote the Schur function corresponding to the skew parti-
tion \( \lambda/\nu \) and let \( p_r \) denote the power-sum symmetric function of degree
\( r \in \mathbb{N} \). Let \( \text{sgn}(\lambda/\nu) = (-1)^{\ell} \) if \( \lambda/\nu \) is a border-strip of height \( \ell \in \mathbb{N}_0 \),
and let \( \text{sgn}(\lambda/\nu) = 0 \) otherwise. The Murnaghan–Nakayama rule (see, for
instance, [6, Theorem 7.17.1]) states that if \( \nu \) is a partition and \( r \in \mathbb{N} \) then
\[
(1) \quad s_{\nu}p_r = \sum_{\lambda/\nu \vdash r} \text{sgn}(\lambda/\nu) s_{\lambda}.
\]

To generalize (1) we need some further definitions. Let \( \lambda/\nu \) be a skew
partition and let \( c \) be minimal such that \( \lambda_c > \nu_c \). We say that an \( r \)-border-
strip \( \lambda/\mu \) is the final \( r \)-border-strip in \( \lambda/\nu \) if \( \mu/\nu \) is a skew partition and
\( \lambda_c > \mu_c \). Thus \( \lambda/\mu \) has a (necessarily unique) final \( r \)-border-strip if and only
if the \( r \) boxes at the top-right of the rim of the Young diagram of \( \lambda/\nu \) can
be removed to leave a skew partition. We say that \( \lambda/\nu \) is \( r \)-decomposable if
there exist partitions \( \mu^{(0)}, \mu^{(1)}, \ldots, \mu^{(m)} \) such that
\[
\lambda = \mu^{(0)} \supset \mu^{(1)} \supset \ldots \supset \mu^{(m)} = \nu
\]
and \( \mu^{(i)}/\mu^{(i+1)} \) is the final \( r \)-border-strip in \( \mu^{(i)}/\nu \) for each \( i \). In this case
we define \( \text{sgn}_r(\lambda/\nu) = \text{sgn}(\mu^{(0)}/\mu^{(1)}) \ldots \text{sgn}(\mu^{(m-1)}/\mu^{(m)}) \). If \( \lambda/\nu \) is not \( r \)-
decomposable we define \( \text{sgn}_r(\lambda/\mu) = 0 \). Let \( f \circ g \) denote the plethysm of
symmetric functions \( f \) and \( g \), as defined in [5, I.8] or [6, Appendix 2]. Finally
let \( h_m = s_m \) denote the complete symmetric function of degree \( m \in \mathbb{N}_0 \).

We shall prove that if \( \nu \) is a partition and \( r, m \in \mathbb{N} \) then
\[
(2) \quad s_{\nu}(p_r \circ h_m) = \sum_{\lambda/\nu \vdash rm} \text{sgn}_r(\lambda/\nu) s_{\lambda}.
\]
Taking \( m = 1 \) recovers (1). The formula for \( s_{\mu}(p_r \circ h_{m_1} \ldots h_{m_d}) \) given in
[1, page 29] follows by repeated applications of (2). This formula is proved
in [1] using Muir’s rule. Similarly (2) implies combinatorial formulae for

Date: August 15, 2014.
2010 Mathematics Subject Classification. 05E05, secondary: 05E10, 20C30.
s_\mu(p_{r_1} \ldots p_{r_c} \circ h_m), and, more generally, for s_\mu(p_{r_1} \ldots p_{r_c} \circ h_{m_1} \ldots h_{m_d}). An alternative proof of (2) using the character theory of the symmetric group was given in [2, Proposition 4.3]. The special case \nu = \emptyset of (2) follows from [5, I.8, Example 8].

2. Proof of Equation (2)

The proof is by induction on m. We begin with the identity

(3) \quad m h_m = \sum_{\ell=1}^{\infty} p_\ell h_{m-\ell},

which may be proved in a few lines working from the generating functions

\sum_{m=0}^{\infty} h_m t^m = \prod_{i=1}^{\infty} (1 - x_i t)^{-1} and \sum_{\ell=0}^{\infty} p_\ell t^\ell = 1 + \sum_{i=1}^{\infty} x_i t (1 - x_i t)^{-1}.

The map \( f \mapsto p_r \circ f \) is an endomorphism of the ring of symmetric functions (see [5, I.8.6]), so (3) implies that

\[ p_r \circ m h_m = p_r \circ \sum_{\ell=1}^{\infty} p_\ell h_{m-\ell} = \sum_{\ell=1}^{\infty} (p_r \circ p_\ell) (p_r \circ h_{m-\ell}) = \sum_{\ell=1}^{\infty} p_\ell (p_r \circ h_{m-\ell}). \]

Since \( p_r \circ m h_m = m p_r \circ h_m \), it follows that

\[ m s_\nu (p_r \circ h_m) = \sum_{\ell=1}^{\infty} s_\nu (p_r \circ h_{m-\ell}) p_\ell. \]

By (1) and induction we get

\[ m s_\nu (p_r \circ h_m) = \sum_{\ell=1}^{\infty} \sum_{\mu/\nu \vdash r (m-\ell)} \sum_{\lambda/\mu \vdash r \ell} \text{sgn}_r(\mu/\nu) \text{sgn}(\lambda/\mu) s_\lambda. \]

It is therefore sufficient to prove that if \( \lambda/\nu \) is a skew partition of \( rm \) then

(4) \quad \text{sgn}_r(\lambda/\nu) = \frac{1}{m} \sum_\mu \text{sgn}(\lambda/\mu) \text{sgn}_r(\mu/\nu)

where the sum is over all partitions \( \mu \) such that \( \lambda \supset \mu \supset \nu \) and \( |\mu/\nu| \) is a border-strip of length divisible by \( r \).

Fix an abacus display \( A \) for \( \lambda \) using an \( r \)-runner abacus, numbered so that the positions on runner \( t \) are \( \{ j \in N_0 : j \equiv t \mod r \} \) for each \( t \in \{0, 1, \ldots, r-1\} \). We may assume that one side of (4) is non-zero, and so an abacus display \( C \) for \( \nu \) can be obtained by a sequence of bead moves on the runners of \( A \). We say that runner \( t \) of \( A \) is of type

(I) if it has positions \( \beta_1 > \alpha_1 > \beta_2 > \alpha_2 > \cdots > \beta_c > \alpha_c \) such that, for each \( k \in \{1, \ldots, c\} \), position \( \beta_k \) has a bead, positions \( \beta_k - p, \ldots, \alpha_k \) have gaps, and runner \( t \) of \( C \) is obtained by moving the bead in position \( \beta_k \) to the gap in position \( \alpha_k \) for each \( k \);

(II) if it is not of type (I), but a runner of type (I) can be obtained by swapping a bead on runner \( t \) with a gap one or more positions above it;

(III) if it is neither of type (I) nor of type (II).
Runners of types (I) and (II) are illustrated in Figure 1 above.

The skew partition \( \lambda/\mu \) has a final \( r \)-border-strip if and only if there is a gap immediately above the greatest numbered bead of \( A \) that is moved in a sequence of single-step bead moves (that is, moves of a bead into a gap immediate above it) leading to \( C \). It follows that \( \lambda/\nu \) is \( r \)-decomposable if and only if all runners in \( A \) have type (I). In this case, exactly \( m \) skew partitions \( \mu/\nu \) can be obtained by removing a border-strip of length divisible by \( r \) from \( \lambda/\nu \). In the notation used in the definition of type (I), these skew partitions are obtained by choosing a runner, and then moving a bead in one of the positions \( \beta_k \) to a gap in one of the positions \( \beta_k - p, \ldots, \alpha_k \). Each such skew partition \( \mu/\nu \) is \( r \)-decomposable, and \( \text{sgn}(\lambda/\mu) \text{sgn}(\mu/\nu) = \text{sgn}(\lambda/\nu) \) for each \( \mu \). Hence (4) holds in this case.

If there is a runner in \( A \) of type (III), or two or more runners of type (II) then both sides of (4) are zero.

In the remaining case there is a unique runner of \( A \), say runner \( t \), of type (II). Since runner \( t \) is not of type (I), there are beads \( d \) and \( d^* \) on this runner, in positions \( \delta \) and \( \delta^* \) respectively, such that \( \delta > \delta^* \) and in any sequence of single-step bead moves leading from \( A \) to \( C \), bead \( d \) finishes above position \( \delta^* \). It is clear that to obtain a runner of type (I), either bead \( d \) or bead \( d^* \) must be swapped with a gap above position \( \delta^* \) on runner \( t \). Let \( P \) be the set of pairs \((e, \gamma)\) such that \( e \in \{d, d^*\} \) and the runner obtained by swapping bead \( e \) with the gap in position \( \gamma \) has type (I). Note that \((d, \gamma) \in P \) if and only if \((d^*, \gamma) \in P \).
Let \((d, \gamma) \in P\), let \(B\) be the abacus obtained by swapping bead \(d\) with the gap in position \(\gamma\) and let \(\mu\) be the partition represented by \(B\). Let \(\mathcal{I}\) be the set of pairs \((b, b')\) where \(b\) and \(b'\) are beads on \(A\) such that \(b\) begins in a higher-numbered position of \(A\) than \(b'\) and, after the bead move from \(A\) to \(B\), and a sequence of single-step bead moves from \(B\) to \(C\), \(b\) finishes in a lower-numbered position of \(C\) than \(b'\). Define \(B^*\), \(\mu^*\) and \(\mathcal{I}^*\) analogously by replacing \(d\) with \(d^*\), and (in the definition of \(\mathcal{I}^*\)) replacing \(B\) by \(B^*\).

It follows from the results in [3, page 81] that \(\text{sgn}(\lambda/\mu) \text{sgn}_r(\mu/\nu) = (-1)^{|\mathcal{I}|}\) and \(\text{sgn}(\lambda/\mu^*) \text{sgn}_r(\mu^*/\nu) = (-1)^{|\mathcal{I}^*|}\). Let \(\alpha\) be the final position of bead \(d^*\) after a sequence of single-step bead moves from \(B\) to \(C\). Let \(b\) be a bead on \(A\) such that \(b \neq d, d^*\). Suppose that \(b\) is in position \(\beta\). Considering the location of \(\beta\) in the chain of inequalities \(\delta > \delta^* > \alpha > \gamma\) shows that if \(e\) is any bead then
\[
(e, b) \in \mathcal{I} \setminus \mathcal{I}^* \iff e = d\text{ and } \alpha > \beta > \gamma,
\]
\[
(e, b) \in \mathcal{I}^* \setminus \mathcal{I} \iff e = d^*\text{ and } \alpha > \beta > \gamma.
\]
Since \((d, d^*) \in \mathcal{I} \setminus \mathcal{I}^*\) it follows that \(|\mathcal{I}| = |\mathcal{I}^*| + 1\) and so the involution \((d, \gamma) \leftrightarrow (d^*, \gamma)\) on \(P\) pairs up contributions to the right-hand side of (4) with opposite signs. Hence the right-hand side of (4) is zero. This completes the proof.

Acknowledgements

The author thanks Anton Evseev and Rowena Paget for many helpful and stimulating discussions and for comments on a draft of this paper.

References