Some algebra questions to motivate vacation revision

Questions 1–5 are roughly in order of increasing difficulty. After them there are some further questions more in the style of examination questions.

1. For $\alpha \in \mathbb{R}$, let $T_\alpha : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map defined by

$$T_\alpha \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + \alpha y \\ y + \alpha z \\ z \end{pmatrix}$$

(i) Find the matrix representing T_α with respect to the standard basis of $\mathbb{R}^3$ (as both initial and final basis).

(ii) Find bases $\mathcal{E}$ and $\mathcal{F}$ for $\mathbb{R}^3$ such that the matrix representing T_α with respect to $\mathcal{E}$ as initial basis and $\mathcal{F}$ as final basis is the 3×3 identity matrix.

(iii) Prove that T_α is diagonalisable if and only if $\alpha = 0$.

2. Let V and W be a finite dimensional real vector spaces of dimensions m and n respectively and let $T : V \to W$ be a linear map.

(i) Show that there is a basis of V, $\mathcal{E} = (e_1, \ldots, e_m)$ and a basis of W, $\mathcal{F} = (f_1, \ldots, f_n)$ such that if $r = \operatorname{rank} T$,

$$Te_i = \begin{cases} f_i & : \ 1 \le i \le r \\ 0 & : \ r < i \le m \end{cases} .$$

[*Hint: Adapt the proof of the rank-nullity theorem.*]

(ii) Suppose now that $\mathcal{E}'$ is a basis of V and $\mathcal{F}'$ is a basis of W. Let A be the matrix representing T with respect to $\mathcal{E}'$ as initial basis and $\mathcal{F}'$ as final basis. Show that there exist invertible matrices P and Q such that

$$QAP^{-1} = J(r)$$

where $J(r)$ is the $n \times m$-matrix satisfying

$$J(r)_{ij} = \begin{cases} 1 & : \ i = j \text{ and } 1 \le i \le r \\ 0 & : \ \text{otherwise} \end{cases} .$$

3. (i) Suppose that U and W are vector subspaces of a vector space V. Show that there is a basis of V containing bases for $U \cap W$, U and W.

[*You may assume that if X is a vector subspace of the vector space Y then any basis of X can be extend to a basis of Y.*]

(ii) Deduce that $V = U \oplus W$ if and only if $U \cap W = 0$ and $\dim U + \dim W = \dim V$.

(iii) If U_1, U_2 and U_3 are vector subspaces of a vector space V, must there be a basis of V containing bases for each of U_1, U_2 and U_3?

4. Let a, b and c be any 3 complex numbers, and let $A = \begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix}$.

(i) Let $\omega = \exp(2\pi i/3)$. Show that A has eigenvalues $a + b + c$, $a + \omega b + \omega^2 c$ and $a + \omega^2 b + \omega c$.

(ii) Let α, β and γ be complex numbers. By diagonalising A, or otherwise, give necessary and sufficient conditions for the following system of linear equations over $\mathbb{C}$ to have a solution:

$$\begin{aligned} ax + by + cz &= \alpha \\ cx + ay + bz &= \beta \\ bx + cy + az &= \gamma. \end{aligned}$$

5. Let $n \geq 1$ and let A be the $n \times n$ matrix such that $A_{ij} = 1$ if $i \neq j$ and $A_{ij} = 0$ if $i = j$,

$$A = \begin{pmatrix} 0 & 1 & \dots & 1 & 1 \\ 1 & 0 & \dots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \dots & 0 & 1 \\ 1 & 1 & \dots & 1 & 0 \end{pmatrix}.$$

(i) Show that $n-1$ and -1 are eigenvalues of A and find bases of the associated eigenspaces.

(ii) Find an invertible $n \times n$ matrix P such that

$$P^{-1}AP = \begin{pmatrix} n-1 & & & \\ & -1 & & \\ & & \ddots & \\ & & & -1 \end{pmatrix}.$$

(iii) (This part may be regarded as optional.) One says that a permutation of the numbers $\{1, 2, \dots, n\}$ is a *derangement* if it has no fixed points. So for permutations of $\{1, 2, 3, 4\}$, $(12)(34)$ and (1234) are derangements, but (123) is not, as $4(123) = 4$.

Let e_n be the number of derangements of $\{1, 2, \dots, n\}$ that are even permutations and let o_n be the number of derangements of $\{1, 2, \dots, n\}$ that are odd permutations. By evaluating the determinant of A in 2 different ways prove that

$$e_n - o_n = (-1)^{n-1}(n-1).$$

(You might first check this holds for small n, e.g. $n = 2$, $n = 3$, $\dots$)

Exam style questions

1. **(a)** Let V and W be finite dimensional vector spaces over the real numbers and let $T : V \to W$ be a linear transformation. Define the *kernel*, $\ker T$ and the *image*, $\operatorname{im} T$.

Prove that $\ker T$ is a subspace of V and $\operatorname{im} T$ is a subspace of W.

Prove that T is one-to-one if and only if $\ker T = \{0_V\}$.

State the *rank-nullity formula*

Suppose that $\dim(V) = \dim(W)$. Prove that T maps V onto W if and only T is one-to-one.

(b) Let $T : V \to V$ be a linear transformation of the finite dimensional real vector space V. Show that $\operatorname{rank} T = \operatorname{rank} T^2$ if and only if $V = \operatorname{im} T \oplus \ker T$.

2. **(a)** Let V be a finite dimensional real vector space and let $T : V \to V$ be a linear map. Explain carefully what is meant be an *eigenvalue* of T and by an associated *eigenvector* of T.

Show that if $\lambda_1, \ldots, \lambda_r$ are distinct eigenvalues of T and $v_1, \ldots, v_r$ are associated eigenvectors then $v_1, \ldots, v_r$ are linearly independent.

(b) Let V be the set of all differentiable functions on $\mathbb{R}$. (You may assume that V is a real vector space). Let $n \geq 1$ and let U be the subspace of V spanned by the functions

$$\{\sin mx, \cos mx : m = 1 \ldots n\}.$$

Show that differentiation defines a linear transformation from U onto itself.

Prove that if for some $n \geq 1$

$$a_1 \sin x + a_2 \sin 2x + \ldots + a_n \sin nx = 0 \quad \forall x \in \mathbb{R}$$

then $a_1 = \ldots = a_n = 0$.

3. **(a)** Let S be a finite subset of a vector space V. Explain what is meant by

(i) the *span* of S,

(ii) S is *linearly independent*,

(iii) S is a *basis* of V.

Let $n \geq 1$ and let V be the vector space of all polynomials of degree at most n. Show that if $\alpha \in \mathbb{R}$ then

$$\{f \in V : f(\alpha) = 0\}$$

is a subspace of V, and determine its dimension.

(b) Now suppose that $n = 4$. Find, with proof, a basis of V which contains bases for each of

$$U = \left\{ f : \frac{\mathrm{d}^3 f}{\mathrm{d}x^3} = 0 \right\} \quad \text{and} \quad W = \{f \in V : f(1) = f(2) = 0\}.$$

4. **(a)** Let π be a permutation of the set $\{1, 2, \ldots, n\}$. What is the cycle decomposition of π? Illustrate your answer by giving the cycle decomposition of

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 5 & 3 & 2 & 1 & 8 & 9 & 7 & 4 & 6 \end{pmatrix}.$$

The *conjugate by* π of the permutation θ is defined to be the permutation $\pi^{-1}\theta\pi$. Let θ be the 3-cycle (abc). Show that $\pi^{-1}\theta\pi$ is the 3-cycle $(a\pi\ b\pi\ c\pi)$.

We say that θ and π *commute* if $\theta\pi = \pi\theta$. Show that θ and π commute if and only if the conjugate by π of θ is θ.

(b) Now let $n = 6$ and let α be the permutation

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 5 & 4 & 6 \end{pmatrix}.$$

Express α as a product of disjoint cycles and find all permutations that commute with it. Show that each such permutation is a power of α.

Let

$$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 5 & 6 & 4 \end{pmatrix}.$$

Is every permutation which commutes with β a power of β?