Some answers to vacation questions.

If a question is pure bookwork then I have left you to look up the proof in notes or a book. Except for question 10 (which had a misprint I thought I should clear up) I haven’t given answers to the optional questions; ask me if you want any help with them.

1. True or false: (give brief proofs or counterexamples as appropriate)
   (i) A convergent sequence is bounded.
   True. Let \((a_n)\) be a convergent sequence. Take \(\epsilon = 1\) in the definition of convergence to obtain
   \[
   \exists a \in \mathbb{R} \exists N \in \mathbb{N} \forall n \geq N \ |a_n - a| < 1.
   \]
   From this it follows that \(|a_n| \leq |a| + 1\) if \(n \geq N\). To get an overall upper bound just take \(A = \max(a_1, a_2, \ldots, a_{N-1}, |a| + 1)\).
   
   (ii) A bounded sequence is convergent.
   False. For example \(a_n = (-1)^n\).
   
   (iii) If the sequence \((a_n)\) does not tend to infinity then there is a constant \(K\) such that \(|a_n| < K\) for all \(n \geq 1\).
   False. Remember we say that the sequence \((a_n)\) tends to infinity if given any \(K \in \mathbb{R}\) there exists an \(N \in \mathbb{N}\) such that \(|a_n| > K\) for all \(n \geq N\). By this definition the sequence
   
   \[
   a_n = \begin{cases} 
   0 & \text{if } n \text{ is even} \\
   n & \text{if } n \text{ is odd}
   \end{cases}
   \]
   
   does not tend to infinity. But \(|a_{2m}| = 2m\) so the sequence is not bounded.
   
   (iv) If \(X\) and \(Y\) are non-empty sets of real numbers and \(x < y\) for all \(x \in X\) and \(y \in Y\) then \(\sup X\) and \(\inf Y\) exist and \(\sup X \leq \inf Y\).
   True. Take any \(y \in Y\). By hypothesis \(x \leq y\) for all \(x \in X\) so \(X\) is bounded above. By the completeness property of \(\mathbb{R}\), \(X\) has a supremum, \(\sup X\). A similar argument shows that \(Y\) has an infimum. Now \(\sup X < y\) for all \(y \in Y\) so \(\sup X \leq \inf Y\). Here we used the (often helpful) result that \(A \leq y\) for all \(y \in Y\) implies \(A \leq \inf Y\).
   
   Further exercise: Can the conclusion be strengthened to \(\sup X < \inf Y\)?
   
   (v) A subsequence of a convergent sequence is convergent, and has the same limit as the original sequence.
   True. Bookwork.
   
   (vi) If \(a_n > 0\) for all \(n\) and \(a_n \to 0\) as \(n \to \infty\) then \(1/a_n \to \infty\) as \(n \to \infty\).
   True. We want to show that for all \(K > 0\) there exists \(N \in \mathbb{N}\) such that \(|1/a_n| > K\) for all \(n \in N\). All we have to play with is the hypothesis that \(a_n \to 0\) as \(n \to \infty\), i.e.
   
   \[
   \forall \epsilon > 0 \exists N \in \mathbb{N} \forall n \geq N \ |a_n| < \epsilon.
   \]
   So if we put \(\epsilon = 1/K\) we will obtain a suitable \(N\).

2. (i) Let \((a_n)\) be a bounded monotone sequence of real numbers. Prove that \((a_n)\) is convergent.
   
   Bookwork. By replacing \(a_n\) with \(-a_n\) if necessary we may assume that \((a_n)\) is increasing. Now show that \(a_n\) converges to \(\sup \{a_n : n \in \mathbb{N}\}\) by using the approximation property.
(ii) Let $b > 1$ be a fixed real number. We define a sequence $(a_n)$ inductively by taking

$$a_0 = b, \quad a_{n+1} = \frac{a_n}{2} + \frac{b}{2a_n} \quad \text{for } n \geq 0.$$ 

Prove that $(a_n)$ converges. Show that if $\beta = \lim_{n \to \infty} a_n$ then $\beta > 0$ and $\beta^2 = b$.

Straightforward inductive arguments show that (a) $a_n$ is positive, (b) $a_n^2 \geq b$ and (c) $a_{n+1} \leq a_n$ for all $n$. So the sequence $(a_n)$ is decreasing and bounded below. By part (a) it must converge to some $\beta > 0$. As $\lim_{n \to \infty} a_n = \lim_{n \to \infty} a_{n+1}$, the limit $\beta$ must satisfy $\beta = \beta/2 + b/2\beta$. It follows that $\beta^2 = b$.

(iii) $(\star)$ Show that if $a_n \geq 1$ and $|a_n - \beta| < \epsilon$ then $|a_{n+1} - \beta| < \epsilon^2/2$. Deduce that if for our initial guess $a_0$ we pick the natural number whose square is nearest to $b$ then $|a_n - \beta| < 1/2^{2n-1}$. Can $a_n = \beta$ for any $n$?

The first part is just a calculation:

$$|a_{n+1} - \beta| = \left| \frac{a_n}{2} + \frac{b}{2a_n} - \beta \right|$$

$$= \left( a_n - \beta \right)^2 / 2a_n$$

$$< \epsilon^2 / 2.$$

Now $|a_0 - \beta| < 1$, as otherwise there would be a natural number nearer $\beta$ than $a_0$, and its square would be nearer $b$ than $a_0^2$. The error estimate given in the question now follows by induction on $n$.

3. (i) State the Bolzano-Weierstrass Theorem concerning sequences of real numbers.

The Bolzano-Weierstrass Theorem states that any bounded sequence of real numbers has a convergent subsequence. (There is an analogous version for sequences of complex numbers.)

(ii) What does it mean to say that a sequence is a Cauchy sequence? Prove that a sequence of real numbers is a Cauchy sequence if and only if it converges.

Bookwork.

(iii) Deduce that if the series $\sum_{r=1}^{\infty} a_r$ converges absolutely then it converges. Give an example to show that the converse of this result is false.

The first part is bookwork. The series $\sum_{r=1}^{\infty} a_r$ where $a_r = (-1)^r / \sqrt{r}$ is convergent (by the alternating series test), but not absolutely convergent.

4. [Based on Q1 1999 Mods Analysis.] Let $(a_n)$ be a sequence of real numbers. What is meant by the statement that $(a_n)$ is convergent?

Let $(a_n)$ and $(b_n)$ be sequences converging to the limits $l$ and $m$ respectively. Show that:

(i) The sequence $(a_n + b_n)$ converges to $l + m$.

(ii) The sequence $(a_nb_n)$ converges to $lm$.

(iii) If $a_n \leq b_n$ for all $n$ then $l \leq m$.

Yet more bookwork.

Give an example to show that if $a_n < b_n$ for all $n$ then it is not necessarily true that $l < m$.

One could take $a_n = 0$ for all $n$ and $b_n = 1/n$. The moral of this example is that limits don’t in general preserve ‘sharp’ inequalities, i.e. inequalities involving < or > signs.
Now suppose that $l = 0$. Define a new sequence $(c_n)$ by $c_n = \frac{1}{n} \sum_{r=1}^{n} a_r$. Show that $(c_n)$ also converges to 0.

Fix $\epsilon > 0$. Let $|a_r| < \epsilon$ for all $r \geq N$. If $n \geq N$ then

$$\left| \frac{1}{n} \sum_{r=1}^{n} a_r \right| \leq \frac{1}{n} \sum_{r=1}^{N} |a_r| + \frac{1}{n} \sum_{r=N+1}^{n} |a_r| \leq \frac{1}{n} \sum_{r=1}^{N} |a_r| + \frac{n-N}{n} \epsilon.$$ 

As $n$ tends to $\infty$ the first summand tends to 0, and the second is bounded by $\epsilon$. So \( \lim_{n \to \infty} c_n = 0 \).

5. [Based on Q4 2001 Mods Analysis] State the comparison test and derive the integral test for a series of real numbers $\sum_{r=1}^{\infty} a_r$.

It's important to include all the necessary conditions in the integral test: let $f : \mathbb{R} \to \mathbb{R}$ be a decreasing function that takes values in the positive real numbers. The integral test states that the series with $n$th term $f(n)$ converges if and only if the integral

$$\int_{1}^{\infty} f(x) \, dx$$

converges.

Prove that the series $\sum_{r=1}^{\infty} r^{-\alpha}$ converges if $\alpha > 1$ and diverges if $\alpha \leq 1$. Determine whether the following series converge or diverge:

$$\sum_{r=2}^{\infty} \frac{1}{r \log r}, \quad \sum_{r=1}^{\infty} \frac{1}{r} \sin \frac{1}{r}.$$

The first part follows from an application of the integral test with the function $f(x) = x^{-\alpha}$. For the next part apply the integral test with the function $f(x) = 1/x \log x$, which has integral

$$\int_{2}^{t} \frac{dx}{x \log x} = \log \log t - \log \log 2.$$

For the last one use the inequality $\sin x \leq x$, which is valid for all $x \geq 0$ to get

$$\sum_{r=1}^{n} \frac{1}{r} \sin \frac{1}{r} \leq \sum_{r=1}^{n} \frac{1}{r^2}.$$

The comparison test now shows that the series converges.

9. (*) Let $\alpha > 0$ be an irrational number. Show that given $N > 0$ there exist natural numbers $m$ and $n$ such that $n \leq N$ and

$$|\alpha - \frac{m}{n}| < \frac{1}{nN}.$$

[Hint: let $\{x\}$ denote the fractional part of $x \in \mathbb{R}$. Apply the pigeonhole principle to the $N+1$ numbers $0, \{\alpha\}, \{2\alpha\}, \ldots, \{N\alpha\}$. Deduce that for any $\epsilon > 0$ there exist points $(m,n) \in \mathbb{N} \times \mathbb{N}$ lying within a distance $\epsilon$ of the line $y = \alpha x$.]

The numbers $0, \{\alpha\}, \{2\alpha\}, \ldots, \{N\alpha\}$ all lie between 0 and 1. So one of the disjoint intervals $(0, 1/N), (1/N, 2/N), \ldots, ((N-1)/N, 1)$ must contain 2 of them. (We don’t need to worry about endpoints as we supposed that $\alpha$ was irrational.) If both $\{r\alpha\}$ and $\{s\alpha\}$ appear in one such interval then there is an integer $m$ such that

$$|r\alpha - s\alpha - m| < \frac{1}{N}.$$
Put $r - s = n$ and divide through to obtain

$$\left| \alpha - \frac{m}{n} \right| < \frac{1}{Nn}.$$  

Fix $\epsilon > 0$. Choose $N > 1/\epsilon$. We have shown that there exist $m, n \in \mathbb{N}$ such that $|\alpha - m/n| < \epsilon/n$. Multiplying by $n$ gives $|n\alpha - m| < \epsilon$, so the point $(m, n)$ lies within $\epsilon$ of the line $y = m\alpha$. 
