Some analysis questions to motivate vacation revision.

The harder questions marked ($\star$), ($\star\star$), or in one extreme case, ($\star\star\star$) are included for interest only, and should be regarded as optional.

1. (i) Let X be an open subset of $\mathbb{R}^n$. What does it mean to say that X is *connected*? Show that X is connected if and only if every continuous function $X \to \{0, 1\}$ is constant.

(ii) Suppose now that X and Y are non-empty open connected subsets of $\mathbb{R}^n$. Use the previous part to show that $X \cup Y$ is connected if and only if $X \cap Y \neq \emptyset$.

(iii) What changes if we no longer assume that the sets X and Y are open?

2. Let A be a subset of $\mathbb{C}$. A well-known theorem says that if $f : \mathbb{C} \to \mathbb{C}$ is continuous and A is compact then $f(A)$ is a compact subset of $\mathbb{C}$. Prove conversely that if $f(A)$ is compact for every continuous map $f : A \to \mathbb{C}$ then A is compact. [*Hint: use the Heine–Borel theorem.*]

3. (i) What does it mean to say that a complex function f is *holomorphic* on an open set $U \subseteq \mathbb{C}$? Writing $f(x+iy) = u(x, y) + iv(x, y)$, show that u and v satisfy the Cauchy-Riemann equations.

(ii) Suppose that f is a holomorphic function which takes only real values. Show that f is constant.

(iii) ($\star$) Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function. When is $f(\overline{z}) - \overline{f(z)}$ holomorphic?

4. (i) State Cauchy's Integral Formula. Use it to prove that if f is holomorphic in $B_a(R) = \{z \in \mathbb{C} : |z - a| < R\}$ (where $R > 0$) then there exist *unique* constants c_n such that

$$f(z) = \sum_{n=0}^{\infty} c_n (z - a)^n \quad \text{if } z \in B_a(R).$$

(ii) Deduce that if $f(a) = 0$ and f is not identically zero in $B_a(R)$ then there is some $r > 0$ such that f does not vanish anywhere in $B_a(r) \setminus \{a\}$.

(iii) Find all holomorphic functions $f : B_0(1) \to \mathbb{C}$ such that $f(1/n) = n^2 f(1/n)^3$ for all $n \in \mathbb{N}$.

5. (i) Let U be a non-empty connected open subset of $\mathbb{C} \setminus \{0\}$. Suppose that $L_1 : U \to \mathbb{C}$ and $L_2 : U \to \mathbb{C}$ are holomorphic functions satisfying $\exp L_1(z) = z$ and $\exp L_2(z) = z$ for all $z \in U$. Show that $L_1(z) - L_2(z)$ is a constant multiple of $2\pi i$.

(ii) Prove that it is not possible to define a holomorphic function $L : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ such that $\exp L(z) = z$ for all $z \neq 0$.

6. [Based on A2 2000 Q4.] State Laurent's theorem. Explain, with examples, what is meant by each of the following terms applied to a complex valued function: (i) isolated singularity; (ii) removable singularity; (iii) zero of order m; (iv) pole of order m; (v) essential singularity.

Locate and classify the singularities of the functions given by the following expressions:

$$\frac{\sin \pi z}{(z^2 + 1)^2 (z - 1)} \quad \text{and} \quad \frac{1}{z(1 + \exp(1/z))}.$$

Calculate the residues of the first function at its poles.

7. State Cauchy's Residue Theorem. For each of the following integrals explain how you would evaluate it by using contour integration. You might mention

(a) the function and contour you would use;
(b) an explanation of how the contour integral relates to the original integral;
(c) which residues you would need to calculate.

$$\text{(i)} \int_{-\infty}^{\infty} \frac{x \sin x}{x^2+1}\,\mathrm{d}x, \quad \text{(ii)} \int_0^{\infty} \frac{x}{\sinh x}\,\mathrm{d}x, \quad \text{(iii)} \int_0^{2\pi} \sin^{2n}\theta\,\mathrm{d}\theta,$$
$$\text{(iv)} \int_0^{\infty} \frac{\sin nx}{x}\,\mathrm{d}x, \quad \text{(v)}(\star) \int_0^{\infty} \frac{\log x}{1+x^2}\,\mathrm{d}x.$$

Carry out the necessary calculations in at least one case. The answers are: $\frac{\pi}{\mathrm{e}}$, $\frac{1}{4}\pi^2$, $\frac{2\pi}{4^n}\binom{2n}{n}$, $\pi/2$ (for any $n \in \mathbb{N}$), 0.

Further questions for enthusiasts only.

8. Show that there is no continuous injective mapping $\mathbb{R}^2 \to \mathbb{R}$.

9. Let f be holomorphic on $\mathbb{C}$. Prove that

(i) f is a polynomial of degree at most k if and only if there is a positive constant M such that $|f(z)| \le M(1+|z|)^k$ for every $z \in \mathbb{C}$,

(ii) $(\star)$ f is a non-constant polynomial if and only if $|f(z)| \to \infty$ as $|z| \to \infty$. [*Hint: first look up the Casorati–Weierstrass theorem.*]

10. Some questions connecting countability with analysis.

(i) $(\star)$ Can the closed interval $[0,1]$ be expressed as a countably infinite union of disjoint closed intervals? (Note that if $x \in \mathbb{R}$ then $[x,x] = \{x\}$ is a closed interval.)

(ii) $(\star)$ Is there an uncountable family of subsets of a countable set with the property that any 2 sets in this family have finite intersection? [*Hint: one solution uses the result that given any $r \in \mathbb{R}$ there is a sequence of rational numbers that converges to r.*]

(iii) $(\star\star)$ It is possible to draw uncountably many Y shapes in the plane, with no two intersecting? (You may assume, if you like, that in any particular Y, the 3 'arms' are of the same non-zero length.)

11. Suppose that $f : \mathbb{C} \to \mathbb{C}$ is holomorphic, and that $f(z) = f(z+1)$ for all $z \in \mathbb{C}$. Show that there is a holomorphic function $g : \mathbb{C} \to \mathbb{C}$ such that $g(\exp(2\pi i z)) = f(z)$ for all $z \in \mathbb{C}$. By applying Laurent's theorem to g, show that f may be written as an everywhere convergent Fourier series

$$f(z) = \sum_{n \in \mathbb{Z}} c_n \exp(2\pi i n z)$$

for some explicitly determined coefficients c_n.

12. Show that if $f : \mathbb{C} \to \mathbb{C}$ is a holomorphic function satisfying the functional equation $f(z) = f(-z)$ then there exists a holomorphic function $g : \mathbb{C} \to \mathbb{C}$ such that $f(z) = g(z^2)$ for all $z \in \mathbb{C}$. $(\star)$ Is the analogue of this result for differentiable real valued functions true? $(\star\star\star)$ Is the analogue of this result for infinitely-differentiable real valued functions true?