Some questions for the Summer vacation

Here are a few questions on this term's work, followed by some harder but hopefully more interesting questions on all this year's work. For further routine questions see past exam papers, old problem sheets, etc.

For some light vacation reading I recommend *What is Mathematics?* by R. Courant & H. Robbins and *The Pleasures of Counting* by T. W. Körner. The first few chapters of Hilary Pristley's deservedly popular book *Introduction to Complex Analysis* have a reminder about complex numbers, and lead into the 2nd year analysis course.

1. Let $f : [a, b] \to \mathbb{R}$ be a continuous function. We define the *integral* of f by

$$\int f = \sup \left\{ \int \phi : \phi \leq f,\ \phi \text{ a step function on } [a, b] \right\}.$$

Prove carefully from this definition that if f takes non-negative values then $\int f \geq 0$. Show also that if f takes non-negative values and $f(x) > 0$ for some $x \in [a, b]$, then $\int f > 0$.

2. Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be vectors in $\mathbb{R}^3$ and let Π be the plane through $\mathbf{0}$ with normal $\mathbf{n}$. Suppose that $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are all on the positive side of Π, that is, $\mathbf{a} \cdot \mathbf{n} \geq 0$, $\mathbf{b} \cdot \mathbf{n} \geq 0$ and $\mathbf{c} \cdot \mathbf{n} \geq 0$. Let α, β, γ be the angles made between $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

Prove that $\alpha + \beta + \gamma \leq 2\pi$, with equality if and only if $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \Pi$. [*Hint: one approach uses the spherical trigonometry developed in Geometry II.*]

3. Define a function $L : \mathbb{R}^{>0} \to \mathbb{R}$ by

$$L(x) = \int_1^x \frac{\mathrm{d}t}{t}.$$

Prove the following properties of L, quoting carefully any general results you need:

(a) $L(x)$ is differentiable with derivative $L'(x) = 1/x$,

(b) L is strictly increasing with range $(-\infty, \infty)$,

(c) $L(xy) = L(x) + L(y)$ for all $x, y \in \mathbb{R}^{>0}$.

One could now define the real exponential function as the inverse function to L. Alternatively we might take exp as already known; in this case it is natural to want to prove:

(d) $\exp L(x) = x$ for all $x \in \mathbb{R}^{>0}$, [*Hint: try differentiating.*]

(e) exp and L are inverse functions,

(The point of this question is to give an alternative definition of the logarithm function. So you should not assume from the outset that $L = \log$.)

4. Suppose that $f : (-1, 1) \to \mathbb{R}$ is differentiable and that $f(0) = 0$. Show that there exists a continuous function $g : (-1, 1) \to \mathbb{R}$ such that $f(x) = xg(x)$. Under what circumstances is g differentiable?

5. (a) Show that if f and g are differentiable functions then the product function fg is differentiable, with derivative $(fg)' = f'g + g'f$. Hence derive the formula for integration by parts.

(b) Use induction and integration by parts to show that if $f : \mathbb{R} \to \mathbb{R}$ is n-times differentiable and $f^{(n)}$ is continuous, then

$$\begin{aligned} f(x) = f(0) + f'(0)x + \ldots + \frac{f^{(n-1)}(0)}{(n-1)!}x^{n-1} \\ + \frac{1}{(n-1)!}\int_0^x f^{(n)}(t)(x-t)^{n-1}\mathrm{d}t. \end{aligned}$$

(c) By applying the Second Integral Mean Value Theorem, show that if $x > 0$ then

$$f(x) = f(0) + f'(0)x + \ldots + \frac{f^{(n-1)}(0)}{(n-1)!}x^{n-1} + \frac{f^{(n)}(\zeta)}{n!}x^n$$

for some $\zeta \in (0, x)$. (This recovers the more usual remainder term in Taylor's Theorem.)

(d) Evaluate $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n2^n}$.

Further questions for enthusiasts

6. Let (a_r) be a sequence of real numbers. By analogy with the theory of infinite series, give a definition for the infinite product $\prod_{r=1}^{\infty} a_r$. (Of course, just as for infinite series, an infinite product may fail to converge.)

Prove that $\displaystyle\prod_{r=1}^{\infty} \frac{2^r - 1}{2^r}$ converges to a real number between $1/10$ and $1/2$.

Show that $\displaystyle\prod \frac{1}{1 - p^{-1}}$ diverges, where the product is taken over all primes p. [*Hint: expand each term as a geometric series and then multiply them together.*] Deduce that there are infinitely many primes.

7. Suppose that $f : (-1, 1) \to \mathbb{R}$ has the property that for all $x \in (-1, 1)$, the limit $\lim_{t\to x} f(t)$ exists. Define $g : (-1, 1) \to \mathbb{R}$ by $g(x) = \lim_{t\to x} f(t)$.

(a) Need f be continuous?

(b) Can f have countable many points of discontinuity?

(c) ($\star\star$) Can f have uncountably many points of discontinuity?

(d) Show that g is continuous.

8. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Find a necessary and sufficient condition on f for the following statement to hold: for any sequence (a_n) of real numbers, $a_n \to 0$ as $n \to \infty$ if and only if $f(a_n) \to 0$ as $n \to \infty$.

9. Let M be the set of Möbius transformations of the extended complex plane, $\mathbb{C} \cup \{\infty\}$.

(a) Show that M is a group under composition of maps.

(b) Let G be the general linear group of complex 2×2 matrices, so

$$G = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{C}, ad - bc \neq 0 \right\}.$$

Show that the map $\phi : G \to M$ defined by letting $\phi \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be the Möbius transformation $z \mapsto \dfrac{az+b}{cz+d}$ is a surjective group homomorphism. What is its kernel?

(c) Let $A \in G$. Show that $\begin{pmatrix} \lambda \\ \mu \end{pmatrix}$ is an eigenvector of A if and only if $\dfrac{\lambda}{\mu}$ is a fixed point of $\phi(A)$.

(d) Hence, or otherwise, show that any non-identity Mobius transformation has exactly 1 or exactly 2 fixed points. Give examples to show both possibilities may occur.

10. Let $n \geq 1$ and let A be the $n \times n$ matrix such that $A_{ij} = 1$ if $i \neq j$ and $A_{ij} = 0$ if $i = j$; thus

$$A = \begin{pmatrix} 0 & 1 & \dots & 1 & 1 \\ 1 & 0 & \dots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \dots & 0 & 1 \\ 1 & 1 & \dots & 1 & 0 \end{pmatrix}.$$

(a) Show that $n-1$ and -1 are eigenvalues of A and find bases of the associated eigenspaces.

(b) Find an invertible $n \times n$ matrix P such that

$$P^{-1}AP = \begin{pmatrix} n-1 & & & \\ & -1 & & \\ & & \ddots & \\ & & & -1 \end{pmatrix}.$$

(c) One says that a permutation of the numbers $\{1, 2, \dots, n\}$ is a *derangement* if it has no fixed points. So for permutations of $\{1, 2, 3, 4\}$, $(12)(34)$ and (1234) are derangements, but (123) is not, as $4(123) = 4$.

Let e_n be the number of derangements of $\{1, 2, \dots, n\}$ that are even permutations and let o_n be the number of derangements of $\{1, 2, \dots, n\}$ that are odd permutations. By evaluating the determinant of A in 2 different ways prove that

$$e_n - o_n = (-1)^{n-1}(n-1).$$

(d) ($\star$) For $n \geq 1$ let $d_n = e_n + o_n$ be the total number of derangements in the symmetric group S_n. Show that $d_n = (n-1)(d_{n-1} + d_{n-2})$ and hence that

$$d_n = n! \left(1 - \frac{1}{2} + \dots + \frac{(-1)^n}{n} \right).$$

(There are many other proofs of this result. For example, you might try looking at `www.cut-the-knot.org/arithmetic/combinatorics/InclusionExclusion.shtml`.)

11. ($\star$) The leader of the pirates (see past vacation work *passim*) has acquired a necklace on which hang 168 beads; of these, 84 are white and 84 are black. Can it be cut, and the ends resewn, so that two necklaces each having 42 beads of either colour are obtained? (See right for one solution for an 8 bead necklace.) It is, of course, strictly against the pirates' moral code to remove the beads and rethread them in a different order.