Further exercises for a5 algebra

Questions 1 and 2 are there if you would like some revision of normal sub-

1. (Based on algebra moderations 2000 Q9). What are the elements and what is the definition of multiplication in the quotient $G/N$ of a group $G$ by a normal subgroup $N$? State carefully (without proof) the isomorphism theorem for groups.

Let $G$ be the set of all $2 \times 2$ real matrices of the form $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ with $a \neq 0$, and $N$ the set of matrices in $G$ with $a = 1$.

(a) Prove that $G$ is a group under matrix multiplication.

(b) Define $\phi : G \to \mathbb{R}^*$ by $\phi \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = a$. (Here $\mathbb{R}^*$ is the multiplicative group of non-zero real numbers.) Prove that $\phi$ is a group homomorphism.

(c) Deduce that $N$ is a normal subgroup of $G$ and that $G/N$ is isomorphic to $\mathbb{R}^*$. Describe explicitly the elements of $G/N$.

(d) Are there any other normal subgroups of $G$?

2. Throught this question $G$ is a finite group.

(a) Explain what it means to say that $G$ is a simple group.

(b) Prove Cayley’s theorem, that if $G$ has order $n$, and $\rho : G \to S_n$ is the map sending each $g \in G$ to the permutation $\rho_g$ it induces on $G$, then $G \cong \text{im} \rho$.

(c) Suppose that the order of $G$ is even. Show that $G$ has an element of order 2. [Hint: one approach is to consider a partitioning of $G$ into subsets of the form $\{g, g^{-1}\}$.]

(d) Suppose that $G$ has order $2m$ where $m$ is odd. Prove that if $t \in G$ has order 2 then $\rho_t$ is an odd permutation of the elements of $G$. Hence prove that $G$ has a normal subgroup of order $m$.

3. In this question we consider various groups of transformations of the plane, $\mathbb{R}^2$. Let $O_2(\mathbb{R})$ be the group of all distance-preserving linear maps from $\mathbb{R}^2$ to itself. Let $SO_2(\mathbb{R}) = \{T \in O_2(\mathbb{R}) : \det T = 1\}$. Let $T_a : \mathbb{R} \to \mathbb{R}$ be translation by $a \in \mathbb{R}^2$, i.e. $T_a(x) = a + x$. Let $T = \{T_a : a \in \mathbb{R}^2\}$ be the group of all translations. Let $E_2(\mathbb{R})$ be the group of isometries of the plane generated by $O_2(\mathbb{R})$ and $T_2(\mathbb{R})$.

(a) Show that if $x \in O_2(\mathbb{R})$ is represented by the matrix $A$ with respect to the standard basis of $\mathbb{R}^2$ then $A^{tr}A = AA^{tr} = I$.

(b) Show that $SO_2(\mathbb{R})$ is a normal subgroup of $O_2(\mathbb{R})$. Describe geometrically the cosets of $SO_2(\mathbb{R})$ in $O_2(\mathbb{R})$.

(c) Prove that $T$ is a normal subgroup of $E_2(\mathbb{R})$. Hence show that $E_2(\mathbb{R}) = \{T_aS : S \in O_2(\mathbb{R}), a \in \mathbb{R}^2\}$.

Use this to give a rigorous proof that $\text{Stab}_{E_2(\mathbb{R})}(0) = O_2(\mathbb{R})$.

4. Let $G$ be a group of order $p^a$ for some prime $p$. By considering the orbits in the action of $G$ on itself by conjugacy, show that the centre of $G$ has order at least $p$. [The centre of a group $G$ is $\{g \in G : xg = gx \forall x \in G\}$].

5. (Based on a3 algebra 2000 Q2). Let $G$ be a non-trivial finite group of rotations of $\mathbb{R}^3$. Let $P$ be the set of points of the unit sphere that are fixed by some non-identity element of $G$.

(a) Show that if $a \in P$ and $x \in G$ then the image of $a$ under $x$ lies in $P$.

(b) Prove that $|P|$ is an even integer with $2 \leq |P| \leq 2(|G| - 1)$.

(c) Let $k$ be the number of orbits of $G$ on $P$. Prove that $(k - 2)|G| = |P| - 2$. [Hint: use Burnside’s lemma.] Deduce that $2 \leq k \leq 3$.

(d) Show that if $|G|$ is odd then $|P| = k = 2$ and all non-identity elements of $G$ have the same axis.

(e) Give an example where $k = 3$. 