

A GENERALIZED SXP RULE PROVED BY BIJECTIONS AND INVOLUTIONS

MARK WILDON

ABSTRACT. This paper proves a combinatorial rule expressing the product $s_\tau(s_{\lambda/\mu} \circ p_r)$ of a Schur function and the plethysm of a skew Schur function with a power sum symmetric function as an integral linear combination of Schur functions. This generalizes the SXP rule for the plethysm $s_\lambda \circ p_r$. Each step in the proof uses either an explicit bijection or a sign-reversing involution. The proof is inspired by an earlier proof of the SXP rule due to Remmel and Shimozono, *A simple proof of the Littlewood–Richardson rule and applications*, *Discrete Mathematics* **193** (1998) 257–266. The connections with two later combinatorial rules for special cases of this plethysm are discussed. Two open problems are raised. The paper is intended to be readable by non-experts.

1. INTRODUCTION

Let $f \circ g$ denote the plethysm of the symmetric functions f and g . While it remains a hard problem to express an arbitrary plethysm as an integral linear combination of Schur functions, many results are known in special cases. In particular, the SXP rule, first proved in [9, page 351] and later, in a different way, in [2, pages 135–140], gives a surprisingly simple formula for the plethysm $s_\lambda \circ p_r$ where s_λ is the Schur function for the partition λ of $n \in \mathbb{N}$ and p_r is the power sum symmetric function for $r \in \mathbb{N}$. It states that

$$s_\lambda \circ p_r = \sum_{\nu} \operatorname{sgn}_r(\nu^*) c_{\nu}^{\lambda} s_{\nu^*} \quad (1)$$

where the sum is over all r -multipartitions $\nu = (\nu(0), \dots, \nu(r-1))$ of n , ν^* is the partition with empty r -core and r -quotient ν , $\operatorname{sgn}_r(\nu^*) \in \{+1, -1\}$ is as defined in §2 below, and $c_{\nu}^{\lambda} = c_{(\nu(0), \dots, \nu(r-1))}^{\lambda}$ is a generalized Littlewood–Richardson coefficient, as defined at the end of §3 below.

In this note we prove a generalization of the SXP rule. The following definition is required: say that the pair of r -multipartitions (ν, τ) , denoted ν/τ , is a *skew r -multipartition* of n if $\nu(i)/\tau(i)$ is a skew partition for each $i \in \{0, \dots, r-1\}$, and $n = \sum_{i=0}^{r-1} (|\nu(i)| - |\tau(i)|)$.

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Theorem 1.1. *Let $r \in \mathbb{N}$, let τ be a partition with r -quotient $\boldsymbol{\tau}$, and let λ/μ be a skew partition of n . Then*

$$s_\tau(s_{\lambda/\mu} \circ p_r) = \sum_{\boldsymbol{\nu}} \operatorname{sgn}_r((\boldsymbol{\nu}/\boldsymbol{\tau}, \boldsymbol{\tau})^*) c_{\boldsymbol{\nu}/\boldsymbol{\tau}:\mu}^\lambda s_{(\boldsymbol{\nu}/\boldsymbol{\tau}, \boldsymbol{\tau})^*}$$

where the sum is over all r -multipartitions $\boldsymbol{\nu}$ such that $\boldsymbol{\nu}/\boldsymbol{\tau}$ is a skew r -multipartition of n , $(\boldsymbol{\nu}/\boldsymbol{\tau}, \boldsymbol{\tau})^*$ is the partition, defined formally in Definition 2.1, obtained from τ by adding r -hooks in the way specified by $\boldsymbol{\nu}/\boldsymbol{\tau}$, and $\boldsymbol{\nu}/\boldsymbol{\tau}:\mu$ is the skew $(r+1)$ -multipartition obtained from $\boldsymbol{\nu}/\boldsymbol{\tau}$ by appending μ .

Each step in the proof uses either an explicit bijection or a sign-reversing involution on suitable sets of tableaux. The critical second step uses a special case of a rule for multiplying a Schur function by the plethysm $h_\alpha \circ p_r$, where h_α is the complete symmetric function for the composition α . This rule was first proved in [3, page 29] and is stated here as Proposition 2.3. A reader familiar with the basic results on symmetric functions and willing to assume this rule should find the proof largely self-contained. In particular, we do not assume the Littlewood–Richardson rule. We show in §6.1 that two versions of the Littlewood–Richardson rule follow from Theorem 1.1 by setting $r = 1$ and taking either τ or μ to be the empty partition. The penultimate step in our proof is (12), which restates Theorem 1.1 in a form free from explicit Littlewood–Richardson coefficients. In §6.2 we discuss the connections with other combinatorial rules for plethysms of the type in Theorem 1.1, including the domino tableaux rule for $s_\tau(s_\lambda \circ p_2)$ proved in [1].

An earlier proof of both the Littlewood–Richardson rule and the SXP rule, as stated in (1), was given by Remmel and Shimozono in [17], using an involution on semistandard skew tableaux defined by Lascoux and Schützenberger in [8]. The proof given here uses the generalization of this involution to tuples of semistandard tableaux of skew shape. We include full details to make the paper self-contained, while admitting that this generalization is implicit in [8] and [17], since, as illustrated after Example 3.3, a tuple of skew tableaux may be identified (in a slightly artificial way) with a single skew tableau. The significant departure from the proof in [17] is that we replace monomial symmetric functions with complete symmetric functions. This dualization requires different ideas. It appears to offer some simplifications, as well as leading to a more general result.

The plethysm operation \circ is defined in [12, §2.3], or, with minor changes in notation, in [14, I.8], [18, A2.6]. For plethysms of the form $f \circ p_r$ the definition can be given in a simple way: write f as a formal infinite sum of monomials in the variables x_1, x_2, \dots and substitute x_i^r for each x_i to obtain $f \circ p_r$. For example, $s_{(2)} \circ p_2 = x_1^4 + x_2^4 + x_3^4 + \dots + x_1^2 x_2^2 + x_1^2 x_3^2 + x_2^2 x_3^2 + \dots = s_{(4)} - s_{(3,1)} + s_{(2,2)}$. By [12, page 167, P1], $f \circ p_r = p_r \circ f$; several of the formulae we use are stated in the literature in this equivalent form.

Outline. The necessary background results on quotients of skew partitions and ribbon tableaux are given in §2 below, where we also recall the plethystic Murnaghan–Nakayama rule and the Jacobi–Trudi formula. In §3 we give a generalization of the Lascoux–Schützenberger involution and define the generalized Littlewood–Richardson coefficients appearing in Theorem 1.1. The proof of Theorem 1.1 is then given in §4. An example is given in §5. Further examples and connections with other combinatorial rules are given in §6. In particular we deduce the Littlewood–Richardson rule as stated in [6, Definition 16.1] and, originally, in [10, Theorem III]. In the appendix we prove a ‘shape-content’ involution that implies the version of the Littlewood–Richardson rule proved in [19]. We also prove a technical result motivating Conjecture 6.7.

2. PREREQUISITES ON r -QUOTIENTS, RIBBONS AND TABLEAUX

We assume the reader is familiar with partitions, skew partitions and border strips, as defined in [18, Chapter 7]. Fix $r \in \mathbb{N}$ throughout this section. We represent partitions using an r -runner abacus, as defined in [5, page 78], on which the number of beads is always a multiple of r ; the r -quotient of a partition is then unambiguously defined by [5, 2.7.29]. (See §6.2 for a remark on this convention.) The further unnumbered definitions below are taken from [3, page 28], [4, §3] and [17, §3], and are included to make this note self-contained.

Signs and quotients of skew partitions. Let $n \in \mathbb{N}_0$ and let ν/τ be a skew partition of rn . We say that ν/τ is r -decomposable if there exist partitions

$$\tau = \sigma^{(0)} \subset \sigma^{(1)} \subset \dots \subset \sigma^{(n)} = \nu$$

such that $\sigma^{(j)}/\sigma^{(j-1)}$ is a border strip of size r (also called an r -border strip) for each $j \in \{1, \dots, n\}$. In this case we define the r -sign of ν/τ by

$$\operatorname{sgn}_r(\nu/\tau) = \prod_{j=1}^n (-1)^{\operatorname{ht}(\sigma^{(j)}/\sigma^{(j-1)})}.$$

(Here $\operatorname{ht}(\sigma^{(j)}/\sigma^{(j-1)})$ is the height of the border strip $\sigma^{(j)}/\sigma^{(j-1)}$, defined to be one less than the number of rows of $\sigma^{(j)}$ that it meets.) By [5, 2.7.26] or [20, Proposition 3], this definition is independent of the choice of the $\sigma^{(j)}$. If ν/τ is not r -decomposable, we set $\operatorname{sgn}_r(\nu/\tau) = 0$.

If ν/τ is r -decomposable then it is possible to obtain an abacus for ν by starting with an abacus for τ and making n single-step downward bead moves, that is, moves of a bead in a position β to position $\beta + r$. It follows that if $(\nu(0), \dots, \nu(r-1))$ is the r -quotient of ν and $(\tau(0), \dots, \tau(r-1))$ is the r -quotient of τ then $\nu(i)/\tau(i)$ is a skew partition for each i . We define the r -quotient of ν/τ , denoted $\boldsymbol{\nu}/\boldsymbol{\tau}$, to be the skew r -multipartition $(\nu(0)/\tau(0), \dots, \nu(r-1)/\tau(r-1))$. Conversely, the pair $(\boldsymbol{\nu}/\boldsymbol{\tau}, \tau)$ determines ν .

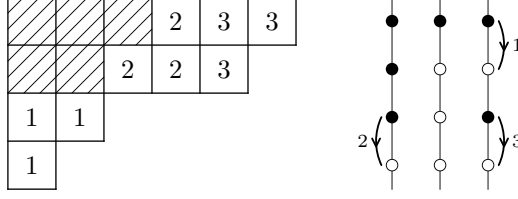


FIGURE 1. The skew partition $(6, 5, 2, 1)/(3, 2)$ is a horizontal 3-ribbon strip of size 9, with $\sigma^{(1)} = (3, 2, 2, 1)$ and $\sigma^{(2)} = (4, 4, 2, 1)$. The border strip $\sigma^{(i)}/\sigma^{(i-1)}$ is marked i ; the row-numbers are 3, 1 and 1, in increasing order of i . The corresponding bead moves on an abacus representing $(3, 2)$ are shown; note these satisfy the condition in Lemma 2.2(ii). The 3-quotient of $(6, 5, 2, 1)/(3, 2)$ is $((1), \emptyset, (2, 1)/(1))$, and so $((1), \emptyset, (2, 1)/(1), (3, 2))^* = (6, 5, 2, 1)$.

Definition 2.1. Let τ be a partition with r -quotient $\boldsymbol{\tau} = (\tau(0), \dots, \tau(r-1))$. Let $\boldsymbol{\nu}/\boldsymbol{\tau}$ be a skew r -multipartition of n . We define $(\boldsymbol{\nu}/\boldsymbol{\tau}, \boldsymbol{\tau})^*$ to be the unique partition ν such that ν/τ is a skew partition of rn with r -quotient $\boldsymbol{\nu}/\boldsymbol{\tau}$.

Working with abaci with 6 beads, we have $((1), \emptyset, (2, 1)/(1), (3, 2))^* = (6, 5, 2, 1)$ as shown in Figure 1 above, $((1), \emptyset, (2, 1)/(1), (3))^* = (6, 2, 2, 2)$ and $((1), (2), (1)/(1), (3, 2))^* = (4, 4, 4, 1, 1)$. Here we use the convention that a skew partition ν/\emptyset is written simply as ν .

Ribbons. Let ν/σ be a border strip in the partition ν . If row a is the least numbered row of ν meeting ν/σ then we say that ν/σ has *row number* a and write $R(\nu/\sigma) = a$. Let $r \in \mathbb{N}$ and $q \in \mathbb{N}_0$. A skew partition ν/τ of rq is a *horizontal r -ribbon strip* if there exist partitions

$$\tau = \sigma^{(0)} \subset \sigma^{(1)} \subset \dots \subset \sigma^{(q)} = \nu \quad (2)$$

such that $\sigma^{(j)}/\sigma^{(j-1)}$ is an r -border strip for each $j \in \{1, \dots, q\}$ and

$$R(\sigma^{(1)}/\sigma^{(0)}) \geq \dots \geq R(\sigma^{(q)}/\sigma^{(q-1)}). \quad (3)$$

For examples see Figure 1 above and Figure 3 in §5.

The following lemma, which is used implicitly in [3], is needed in the proof of Theorem 1.1. Informally, (iii) says that the border strips forming a horizontal r -ribbon strip are uniquely determined by its shape. Note also that (iv) explains the sense in which horizontal r -ribbon strips are ‘horizontal’.

Lemma 2.2. *Let $q \in \mathbb{N}_0$ and let ν/τ be a skew partition of rq . The following are equivalent:*

- (i) ν/τ is a horizontal r -ribbon strip;
- (ii) if A is an abacus representing τ then, for each $i \in \{0, 1, \dots, r-1\}$, there exists $c \in \mathbb{N}_0$ and unique positions β_1, \dots, β_c and $\gamma_1, \dots, \gamma_c$ on runner i

of A with

$$\beta_1 < \gamma_1 < \dots < \beta_c < \gamma_c$$

such that moving the bead in position β_j down to the space in position γ_j , for each $j \in \{1, \dots, c\}$ and $i \in \{0, 1, \dots, r-1\}$, gives an abacus representing ν ;

(iii) there exist unique partitions $\sigma^{(0)}, \dots, \sigma^{(a)}$ satisfying (2) and (3);

(iv) each skew partition $\nu(i)/\tau(i)$ in the r -quotient of ν/τ has at most one box in each column of its Young diagram.

Proof. Let A be an abacus representing τ . If β is a position in A containing a bead then the row-number of the r -border strip corresponding to a single-step downward move of this bead is one more than the number of beads in the positions $\{\beta + r + j : j \in \mathbb{N}\}$ of A . Thus a sequence of single-step downward bead moves, moving beads in positions β_1, \dots, β_c in that order, adds r -border strips in decreasing order of their row number, as required by (3), if and only if $\beta_1 \leq \dots \leq \beta_c$. It follows that (i) and (ii) are equivalent. It is easily seen that (ii) is equivalent to (iii) and (iv). \square

Ribbon tableaux. Let $n \in \mathbb{N}_0$. Let ν/τ be a skew partition of rn and let α be a composition of n with exactly ℓ parts. An r -ribbon tableau of shape ν/τ and weight α is a sequence of partitions

$$\tau = \rho^{(0)} \subset \rho^{(1)} \subset \dots \subset \rho^{(\ell)} = \nu \tag{4}$$

such that $\rho^{(j)}/\rho^{(j-1)}$ is a horizontal r -ribbon strip of size $r\alpha_j$ for each $j \in \{1, \dots, \ell\}$. We say that $\rho^{(j)}/\rho^{(j-1)}$ has label j . We denote the set of all r -ribbon tableaux of shape ν/τ and weight α by $r\text{-RT}(\nu/\tau, \alpha)$. For an example see §5 below.

A plethystic Murnaghan–Nakayama rule. In the second step of the proof of Theorem 1.1 we need the following combinatorial rule. Recall that h_α denotes the complete symmetric function for the composition α .

Proposition 2.3. *Let $n \in \mathbb{N}_0$. If α is a composition of n and τ is a partition then*

$$s_\tau(h_\alpha \circ p_r) = \sum_{\nu} |r\text{-RT}(\nu/\tau, \alpha)| \operatorname{sgn}_r(\nu/\tau) s_\nu$$

where the sum is over all partitions ν such that ν/τ is a skew partition of rn .

This rule was first proved in [3, page 29], using Muir’s rule [15]. For an involutive proof of Muir’s rule see [11, Theorem 6.1]. The special case when $\tau = \emptyset$ and α has a single part is proved in [14, I.8.7]. In this case the result also follows from Chen’s algorithm, as presented in [2, page 130]. The special case when α has a single part was proved by the author in [20] using a sign-reversing involution. The general case then follows easily by induction, using that $h_{(\alpha_1, \dots, \alpha_\ell)} \circ p_r = (h_{\alpha_1} \circ p_r) \dots (h_{\alpha_\ell} \circ p_r)$.

The Jacobi–Trudi formula. Let $\ell \in \mathbb{N}$. The symmetric group Sym_ℓ acts on \mathbb{Z}^ℓ by place permutation. Given $\alpha \in \mathbb{Z}^\ell$ and $g \in \text{Sym}_\ell$, we define $g \cdot \alpha = g(\alpha + \rho) - \rho$ where $\rho = (\ell - 1, \dots, 1, 0)$. For later use we note that if $k \in \{1, \dots, \ell - 1\}$ then

$$(k, k + 1) \cdot \alpha = (\alpha_1, \dots, \alpha_{k+1} - 1, \alpha_k + 1, \dots, \alpha_\ell) \quad (5)$$

where the entries in the middle are in positions k and $k + 1$.

The Jacobi–Trudi formula states that if λ is a partition with exactly ℓ parts and λ/μ is a skew partition then

$$s_{\lambda/\mu} = \sum_{g \in \text{Sym}_\ell} \text{sgn}(g) h_{g \cdot \lambda - \mu},$$

where if α has a strictly negative entry then we set $h_\alpha = 0$. A proof of the formula is given in [18, page 342] by a beautiful involution on certain tuples of paths in \mathbb{Z}^2 .

3. A GENERALIZED LASCoux–SCHÜTZENBERGER INVOLUTION

We begin by presenting the coplactic maps in [13, §5.5]. For further background see [8]. Let w be a word with entries in \mathbb{N} and let $k \in \mathbb{N}$. Following the exposition in [17], we replace each k in w with a right-parenthesis ‘)’ and each $k + 1$ with a left-parenthesis ‘(’. An entry k or $k + 1$ is *k-paired* if its parenthesis has a pair, according to the usual rules of bracketing, and otherwise *k-unpaired*. Equivalently, reading w from left to right, an entry k is *k-unpaired* if and only if it sets a new record for the excess of ks over $(k + 1)s$; dually, reading from right to left, an entry $k + 1$ is *k-unpaired* if and only if it sets a new record for the excess of $(k + 1)s$ over ks . We may omit the ‘k-’ if it will be clear from the context.

For example, if $w = 3422243312311$ then the 2-unpaired entries are shown in bold and the corresponding parenthesised word is $(4))4((1)(11$

Lemma 3.1. *Let w be a word with entries in \mathbb{N} . Let $k \in \mathbb{N}$. The subword of w formed from its k -unpaired entries is $k^c(k + 1)^d$ for some $c, d \in \mathbb{N}_0$. Changing this subword to $k^{c'}(k + 1)^{d'}$, where $c', d' \in \mathbb{N}_0$ and $c' + d' = c + d$, while keeping all other positions the same, gives a new word which has k -unpaired entries in exactly the same positions as w . \square*

Proof. It is clear that any k to the right of the rightmost unpaired $k + 1$ in w is paired. Dually, any $k + 1$ to the left of the leftmost unpaired k in w is paired. Hence the subword of w formed from its unpaired entries has the claimed form. When $d \geq 1$, changing the unpaired subword from $k^c(k + 1)^d$ to $k^{c+1}(k + 1)^{d-1}$ replaces the first unpaired $k + 1$, in position i say, with a k ; since every $k + 1$ to the left of position i is paired, the new k is unpaired. The dual result holds when $c \geq 1$; together these imply the lemma. \square

Definition 3.2. Let w be a word with entries from \mathbb{N} . Suppose that the k -unpaired subword of w is $k^c(k+1)^d$. If $d > 0$, let $E_k(w)$ be defined by changing the subword to $k^{c+1}(k+1)^{d-1}$, and if $c > 0$, let $F_k(w)$ be defined by changing the subword to $k^{c-1}(k+1)^{d+1}$. Let $S_k(w)$ be defined by changing the subword to $k^d(k+1)^c$.

We now extend these maps to tuples of skew tableaux. Let $\text{cont}(t)$ denote the content of a skew tableau t , and let $w(t)$ denote its word, obtained by reading the rows of t from left to right, starting at the highest numbered row. Let $m \in \mathbb{N}$ and let $\sigma/\tau = (\sigma(1)/\tau(1), \dots, \sigma(m)/\tau(m))$ be a skew m -multipartition of $n \in \mathbb{N}$. Let $\ell \in \mathbb{N}$ and let $\alpha \in \mathbb{Z}^\ell$. Let $\mathbf{SSYT}(\sigma/\tau, \alpha)$ denote the set of all m -tuples $(t(1), \dots, t(m))$ of semistandard skew tableaux such that $t(i)$ has shape $\sigma(i)/\tau(i)$ for each $i \in \{1, \dots, m\}$ and

$$\text{cont}(t(1)) + \dots + \text{cont}(t(m)) = \alpha. \quad (6)$$

Thus if α fails to be a composition because it has a negative entry then $\mathbf{SSYT}(\sigma/\tau, \alpha) = \emptyset$. We call the elements of $\mathbf{SSYT}(\sigma/\tau, \alpha)$ *semistandard skew m -multitableaux* of shape σ/τ , or *m -multitableaux* for short. The word of an m -multitableau $(t(1), \dots, t(m)) \in \mathbf{SSYT}(\sigma/\tau, \alpha)$ is the concatenation $w(t(1)) \dots w(t(m))$. For $k \in \mathbb{N}$ we say that an entry of an m -multitableau \mathbf{t} is *k -paired* if the corresponding entry of $w(\mathbf{t})$ is k -paired. Note that, for fixed σ/τ , a word w of length n and content α uniquely determines an m -multitableau (not necessarily semistandard) of shape σ/τ satisfying (6); we denote this multitableau by $\mathbf{T}(w)$. (The skew m -multipartition σ/τ will always be clear from the context.) Abusing notation slightly, we set $E_k(\mathbf{t}) = \mathbf{T}(E_k(w(\mathbf{t})))$, $F_k(\mathbf{t}) = \mathbf{T}(F_k(w(\mathbf{t})))$ (in each case, when either is defined) and $S_k(\mathbf{t}) = \mathbf{T}(S_k(w(\mathbf{t})))$.

Example 3.3. Consider the semistandard skew 3-multitableau

$$\mathbf{t} = \left(\begin{array}{|c|c|c|} \hline 2 & \mathbf{2} & \mathbf{2} \\ \hline 3 & 4 & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 2 & \mathbf{3} \\ \hline \mathbf{3} & 3 \\ \hline 4 & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 1 \\ \hline \end{array} \right).$$

The shape of \mathbf{t} is $((3, 2), (3, 2, 1)/(1), (2))$ and the 2-unpaired entries are shown in bold. By Definition 3.2, $E_2(\mathbf{t})$ is obtained from \mathbf{t} by changing the leftmost unpaired 3 to a 2, and $F_2(\mathbf{t})$ is obtained from \mathbf{t} by changing the rightmost unpaired 2 to a 3. It follows that

$$S_2 E_2(\mathbf{t}) = \left(\begin{array}{|c|c|c|} \hline 2 & \mathbf{2} & \mathbf{3} \\ \hline 3 & 4 & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 2 & \mathbf{3} \\ \hline \mathbf{3} & 3 \\ \hline 4 & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 1 \\ \hline \end{array} \right).$$

As mentioned in the introduction, one may identify a skew m -multitableau with a single skew tableau of larger shape. For example, the semistandard

skew 3-multitableau \mathbf{t} above corresponds to

$$\begin{array}{ccccccc} & & & & & & 1 & 1 \\ & & & & & & 2 & 3 \\ & & & & & & 3 & 3 \\ & & & & & & 4 & \\ & & & & & & 2 & 2 & 2 \\ & & & & & & 3 & 4 & \end{array} .$$

This identification may be used to reduce the next two results to Proposition 4 and the argument in §3 of [17]. We avoid it in this paper, since it has an artificial flavour, and loses combinatorial data: for instance, the skew tableau above may also be identified with two different semistandard skew 2-multitableaux.

Lemma 3.4. *Let $m \in \mathbb{N}$, let σ/τ be a skew m -multipartition of $n \in \mathbb{N}_0$ and let α be a composition with exactly ℓ parts. Fix $k \in \{1, \dots, \ell - 1\}$. Let $\mathbf{SSYT}_k(\sigma/\tau, \alpha)$ and $\mathbf{SSYT}_{k+1}(\sigma/\tau, \alpha)$ be the sets of m -multitableaux in $\mathbf{SSYT}(\sigma/\tau, \alpha)$ that have a k -unpaired k or a k -unpaired $k + 1$, respectively. Let*

$$\varepsilon(k) = (0, \dots, 1, -1, \dots, 0) \in \mathbb{Z}^\ell,$$

where the two non-zero entries are in positions k and $k + 1$. The maps

$$E_k : \mathbf{SSYT}_{k+1}(\sigma/\tau, \alpha) \rightarrow \mathbf{SSYT}_k(\sigma/\tau, \alpha + \varepsilon(k))$$

$$F_k : \mathbf{SSYT}_k(\sigma/\tau, \alpha) \rightarrow \mathbf{SSYT}_{k+1}(\sigma/\tau, \alpha - \varepsilon(k))$$

$$S_k : \mathbf{SSYT}_k(\sigma/\tau, \alpha) \rightarrow \mathbf{SSYT}_{k+1}(\sigma/\tau, (k, k + 1)\alpha)$$

are bijections and $S_k E_k : \mathbf{SSYT}_{k+1}(\sigma/\tau, \alpha) \rightarrow \mathbf{SSYT}_{k+1}(\sigma/\tau, (k, k + 1) \cdot \alpha)$ is an involution.

Proof. Let $\mathbf{t} = (t(1), \dots, t(m)) \in \mathbf{SSYT}(\sigma/\tau, \alpha)$. The main work comes in showing that $E_k(\mathbf{t})$, $F_k(\mathbf{t})$ are semistandard (when defined). Suppose that $E_k(\mathbf{t}) = (t'(1), \dots, t'(m))$ and that the first unpaired $k + 1$ in $w(\mathbf{t})$ corresponds to the entry in row a and column b of tableau $t(j)$. Thus $t'(j)$ is obtained from $t(j)$ by changing this entry to an unpaired k and $t'(i) = t(i)$ if $i \neq j$.

Let $t = t(j)$, let $t' = t'(j)$ and write $u_{(a,b)}$ for the entry of a tableau u in row a and column b . If t' fails to be semistandard then $a > 1$, $(a - 1, b)$ is a box in t , and $t'_{(a-1,b)} = k$. Hence $t_{(a-1,b)} = k$. This k is to the right of the unpaired $k + 1$ in $w(t)$, so by Lemma 3.1 it is paired, necessarily with a $k + 1$ in row a and some column $b' > b$ of t . Since

$$k = t_{(a-1,b)} \leq t_{(a-1,b')} < t_{(a,b')} = k + 1$$

we have $t_{(a-1,b')} = k$. Thus $t_{(a,e)} = k + 1$ and $t_{(a-1,e)} = k$ for every $e \in \{b, \dots, b'\}$. Since $t_{(a-1,b)}$ is paired with $t_{(a,b')}$ under the k -pairing, we see that $t_{(a-1,b+j)}$ is paired with $t_{(a,b'-j)}$ for each $j \in \{0, \dots, b' - b\}$. In particular, the $k + 1$ in position (a, b) of t is paired, a contradiction. Hence $E_k(\mathbf{t})$

is semistandard. The proof is similar for F_k in the case when \mathbf{t} has an unpaired k .

It is now routine to check that $E_k F_k$ and $F_k E_k$ are the identity maps on their respective domains, so E_k and F_k are bijective. If the unpaired subword of $w(\mathbf{t})$ is $k^c(k+1)^d$ then $S_k(\mathbf{t}) = E_k^{d-c}(\mathbf{t})$ if $d \geq c$ and $S_k(\mathbf{t}) = F_k^{c-d}(\mathbf{t})$ if $c \geq d$. Hence S_k is an involution. A similar argument shows that $S_k E_k$ is an involution. By (5) at the end of §2, the image of $S_k E_k$ is as claimed. \square

We are ready to define our key involution. Say that a semistandard skew multitableau \mathbf{t} is *lattice*d if $w(\mathbf{t})$ has no k -unpaired $(k+1)$ s, for any k . Let λ be a partition of $n \in \mathbb{N}_0$ with exactly ℓ parts, let σ/τ be a skew m -multipartition of n and let

$$\mathcal{T} = \bigcup_{g \in \text{Sym}_\ell} \text{SSYT}(\sigma/\tau, g \cdot \lambda). \quad (7)$$

Observe that if $g \neq \text{id}_{\text{Sym}_\ell}$ then $g \cdot \lambda$ is not a partition, and so no element of $\text{SSYT}(\sigma/\tau, g \cdot \lambda)$ is latticed. Therefore the set

$$\text{SSYT}(\sigma/\tau, \lambda) = \{\mathbf{t} \in \text{SSYT}(\sigma/\tau, \lambda) : \mathbf{t} \text{ is lattice} \}$$

is precisely the latticed elements of \mathcal{T} . Let $\mathbf{t} \in \mathcal{T}$. If \mathbf{t} is latticed then define $G(\mathbf{t}) = \mathbf{t}$. Otherwise consider the k -unpaired $(k+1)$ s in $w(\mathbf{t})$ for each $k \in \mathbb{N}$. If the rightmost such entry is a k -unpaired $k+1$ then define $G(\mathbf{t}) = S_k E_k(\mathbf{t})$.

For instance, in Example 3.3 we have $k = 2$ and $G(\mathbf{t}) = S_2 E_2(\mathbf{t})$.

Proposition 3.5. *Let $m \in \mathbb{N}$, let σ/τ be a skew m -multipartition of $n \in \mathbb{N}_0$, and let λ be a partition of n . Let \mathcal{T} be as defined in (7). The map $G : \mathcal{T} \rightarrow \mathcal{T}$ is an involution fixing precisely the skew m -multitableaux in $\text{SSYT}(\sigma/\tau, \lambda)$. If $\mathbf{t} \in \text{SSYT}(\sigma/\tau, g \cdot \lambda)$ and $G(\mathbf{t}) \neq \mathbf{t}$ then $G(\mathbf{t}) \in \text{SSYT}(\sigma/\tau, (k, k+1)g \cdot \lambda)$ for some $k \in \{1, \dots, \ell - 1\}$.*

Proof. This follows immediately from Lemma 3.4. \square

This is a convenient place to define our generalized Littlewood–Richardson coefficients. In §6.1 we show these specialize to the original definition.

Definition 3.6. The *Littlewood–Richardson coefficient* corresponding to a partition λ of n and a skew m -multipartition σ/τ of n is

$$c_{\sigma/\tau}^\lambda = |\text{SSYT}(\sigma/\tau, \lambda)|.$$

4. PROOF OF THEOREM 1.1

Suppose that λ has exactly ℓ parts. The outline of the proof is as follows:

$$s_\tau(s_{\lambda/\mu} \circ p_r) \tag{8}$$

$$= \sum_{g \in \text{Sym}_\ell} \text{sgn}(g) s_\tau(h_{g \cdot \lambda - \mu} \circ p_r) \tag{9}$$

$$= \sum_{g \in \text{Sym}_\ell} \text{sgn}(g) \sum_{\nu} |r\text{-RT}(\nu/\tau, g \cdot \lambda - \mu)| \text{sgn}_r(\nu/\tau) s_\nu \tag{10}$$

$$= \sum_{g \in \text{Sym}_\ell} \text{sgn}(g) \sum_{\nu} |\mathbf{SSYT}(\nu/\tau, g \cdot \lambda - \mu)| \text{sgn}_r((\nu/\tau, \tau)^*) s_{(\nu/\tau, \tau)^*} \tag{11}$$

$$= \sum_{\nu} |\mathbf{SSYTL}(\nu/\tau : \mu, \lambda)| \text{sgn}_r((\nu/\tau, \tau)^*) s_{(\nu/\tau, \tau)^*} \tag{12}$$

$$= \sum_{\nu} c_{\nu/\tau; \mu}^\lambda \text{sgn}_r((\nu/\tau, \tau)^*) s_{(\nu/\tau, \tau)^*}, \tag{13}$$

where the sum in (10) is over all partitions ν such that ν/τ is a skew partition of rn , the sums in (11) and (12) are over all r -multipartitions ν such that ν/τ is a skew r -multipartition of n , and in (12) and (13), $\nu/\tau : \mu$ is the skew $(r+1)$ -multipartition $(\nu(0)/\tau(0), \dots, \nu(r-1)/\tau(r-1), \mu)$ obtained from ν/τ by appending μ .

We now give an explicit bijection or involution establishing each step. For an illustrative example see §5 below.

Proof of (9). Apply the Jacobi–Trudi formula for skew Schur functions, as stated in §2. \square

Proof of (10). Apply Proposition 2.3 to each $s_\tau(h_{g \cdot \lambda - \mu} \circ p_r)$. \square

Proof of (11). Let T be a r -ribbon tableau of shape ν/τ and weight α as in (4), so T corresponds to the sequence of partitions

$$\tau = \rho^{(0)} \subset \rho^{(1)} \subset \dots \subset \rho^{(\ell)} = \nu$$

where $\rho^{(j)}/\rho^{(j-1)}$ is a horizontal r -ribbon strip of size $r\alpha_j$ for each $j \in \{1, \dots, \ell\}$. Let ν/τ have r -quotient $\nu/\tau = (\nu(0)/\tau(0), \dots, \nu(r-1)/\tau(r-1))$, so $(\nu/\tau, \tau)^* = \nu$. Take an abacus A representing τ with a multiple of r beads. The sequence above defines a sequence of single-step downward bead moves leading from A to an abacus B representing ν . For each bead moved on runner i put the label of the corresponding horizontal r -ribbon strip in the corresponding box of the Young diagram of $\nu(i)/\tau(i)$. By Lemma 2.2(iv), this defines a semistandard skew tableau $t(i)$ of shape $\nu(i)/\tau(i)$ for each $i \in \{0, \dots, r-1\}$. Conversely, given $(t(0), \dots, t(r-1)) \in \mathbf{SSYT}(\nu/\tau, \alpha)$, one obtains a sequence of single-step downward bead moves satisfying the condition in Lemma 2.2(ii), and hence an r -ribbon tableau of shape ν/τ and content α . Thus the map sending T to $(t(0), \dots, t(r-1))$ is a bijection from $r\text{-RT}(\nu/\tau, g \cdot \lambda - \mu)$ to $\mathbf{SSYT}(\nu/\tau, g \cdot \lambda - \mu)$, as required. \square

Proof of (12). Fix a skew r -multipartition ν/τ of n . Let

$$\mathcal{T} = \bigcup_{g \in \text{Sym}_\ell} \mathbf{SSYT}(\nu/\tau : \mu, g \cdot \lambda).$$

Let G be the involution on \mathcal{T} defined in §3. Let $u(\mu)$ be the semistandard μ -tableau having all its entries in its j -th row equal to j for each relevant j . Note that $u(\mu)$ is the unique latticed semistandard μ -tableau. Thus if

$$\mathcal{T}_\mu = \{ (t(0), \dots, t(r-1), v) \in \mathcal{T} : v = u(\mu) \} \tag{14}$$

then $\mathbf{SSYTL}(\nu/\tau : \mu, \lambda) \subseteq \mathcal{T}_\mu$. Let $\mathbf{t} \in \mathcal{T}_\mu$. The final $|\mu|$ positions of $w(\mathbf{t})$ correspond to the entries of $u(\mu)$. Every entry $k+1$ in these positions is k -paired. If an entry k in one of these positions is k -unpaired then there is no k -unpaired $k+1$ to its left, so every $k+1$ in $w(\mathbf{t})$ is k -paired. It follows that the final semistandard tableau in $G(\mathbf{t})$ is $u(\mu)$ and so G restricts to an involution on \mathcal{T}_μ . By Proposition 3.5, the fixed-point set of G , acting on either \mathcal{T} or \mathcal{T}_μ , is $\mathbf{SSYTL}(\nu/\tau : \mu, \lambda)$.

The part of the sum in (11) corresponding to the skew r -multipartition ν/τ is

$$\sum_{g \in \text{Sym}_\ell} \sum_{\mathbf{t} \in \mathbf{SSYT}(\nu/\tau, g \cdot \lambda - \mu)} \text{sgn}(g) \text{sgn}_r((\nu/\tau, \tau)^*) s_{(\nu/\tau, \tau)^*}.$$

The set of r -multitableaux \mathbf{t} in this sum is $\mathcal{S} = \bigcup_{g \in \text{Sym}_\ell} \mathbf{SSYT}(\nu/\tau, g \cdot \lambda - \mu)$. There is an obvious bijection $A : \mathcal{S} \rightarrow \mathcal{T}_\mu$ given by appending $u(\mu)$ to a skew r -multitableau in \mathcal{S} . By the remarks above, $A^{-1}GA$ is an involution on \mathcal{S} . Since $\text{sgn}(g) = -\text{sgn}((k, k+1)g)$, it follows from Proposition 3.5 that the contributions to (11) from r -multitableaux $\mathbf{t} \in \mathcal{S}$ such that $A(\mathbf{t}) \notin \mathbf{SSYTL}(\nu/\tau : \mu, \lambda)$ cancel in pairs, leaving exactly the r -multitableaux \mathbf{t} such that $A(\mathbf{t}) \in \mathbf{SSYTL}(\nu/\tau : \mu, \lambda)$. This proves (12). \square

Proof of (13). This is true by our definition of the Littlewood–Richardson coefficient $c_{\nu/\tau : \mu}^\lambda$. \square

5. EXAMPLE

We illustrate (11) and (12) in the proof of Theorem 1.1. Let $r = 3$, let $\lambda = (3, 3)$, $\mu = \emptyset$ and $\tau = (3, 2)$. Take $\nu = (6, 5, 5, 5, 2)$. From the abaci shown in Figure 2 below, we see that $\nu/\tau = ((1), (2), (2, 2))/(1)$. We have

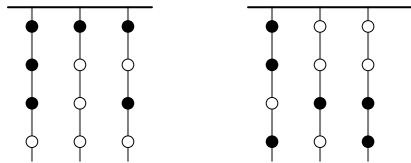


FIGURE 2. Abaci for $(3, 2)$ and $(6, 5, 5, 5, 2)$.

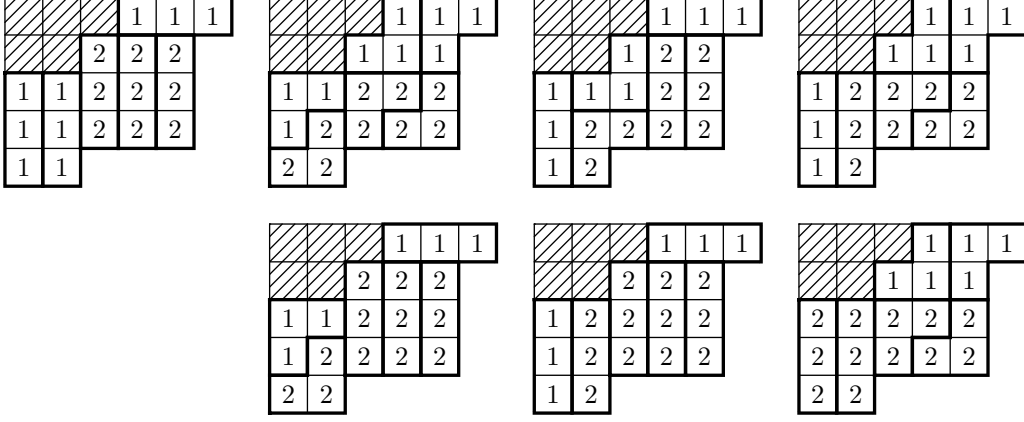


FIGURE 3. The four 3-ribbon tableau tableaux in $3\text{-RT}((6, 5, 5, 5, 2)/(3, 2), (3, 3))$ and the corresponding 3-ribbon tableaux in $3\text{-RT}((6, 5, 5, 5, 2)/(3, 2), (2, 4))$ under the G involution. The first tableaux in the top line is latticed, and so is fixed by G . In each case the ribbon with label i is marked i , and its unique partition into 3-border strips, as in (2), is shown by heavy lines.

$(1, 2) \cdot \lambda = (2, 4)$. The horizontal 3-ribbon tableaux of shape $(6, 5, 5, 5, 2)$ and weights $(3, 3)$ and $(2, 4)$ are shown in Figure 3 above. Applying the bijection

$$3\text{-RT}((6, 5, 5, 5, 2)/(3, 2), (3, 3)) \rightarrow \mathbf{SSYT}(((1), (2), (2, 2)/(1)), (3, 3))$$

in the proof of (11) we obtain the 3-multitableaux

$$\begin{aligned} \mathbf{t}_1 &= \left(\boxed{2}, \boxed{1 \mid 2}, \boxed{1 \mid \begin{array}{c} 1 \\ 2 \end{array}} \right), & \mathbf{t}_2 &= \left(\boxed{1}, \boxed{2 \mid 2}, \boxed{1 \mid \begin{array}{c} 1 \\ 2 \end{array}} \right), \\ \mathbf{t}_3 &= \left(\boxed{2}, \boxed{1 \mid 1}, \boxed{2 \mid \begin{array}{c} 1 \\ 2 \end{array}} \right), & \mathbf{t}_4 &= \left(\boxed{1}, \boxed{1 \mid 2}, \boxed{2 \mid \begin{array}{c} 1 \\ 2 \end{array}} \right) \end{aligned} \quad (15)$$

in the order corresponding to the top line in Figure 3. Here \mathbf{t}_1 is latticed and $\mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4$ are not. Applying the involution G in §3 to $\mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4$, as in the proof of (12), we obtain the 3-multitableaux

$$\left(\boxed{2}, \boxed{2 \mid 2}, \boxed{1 \mid \begin{array}{c} 1 \\ 2 \end{array}} \right), \left(\boxed{2}, \boxed{1 \mid 2}, \boxed{2 \mid \begin{array}{c} 1 \\ 2 \end{array}} \right), \left(\boxed{1}, \boxed{2 \mid 2}, \boxed{2 \mid \begin{array}{c} 1 \\ 2 \end{array}} \right)$$

in the order corresponding to the bottom line in Figure 3. As expected, these are the images of the three horizontal 3-ribbon tableaux of shape $(6, 5, 5, 5, 2)$ and weight $(2, 4)$ under the bijection

$$3\text{-RT}((6, 5, 5, 5, 2)/(3, 2), (2, 4)) \rightarrow \mathbf{SSYT}(((1), (2), (2, 1)/(1)), (2, 4)).$$

Therefore all but one of the seven summands in (11) is cancelled by G . Since $\text{sgn}_3((6, 5, 5, 5, 2)/(3, 2)) = 1$, we have $\langle s_{(3,2)}(s_{(3,3)} \circ p_3), s_{(6,5,5,5,2)} \rangle = 1$.

We now find $\langle s_{(3,2)}(s_{(4,3)/(1)} \circ p_3), s_{(6,5,5,5,2)} \rangle$ using the full generality of Theorem 1.1. Following the proof of (12), we append $\boxed{1}$ to each of

the four 3-multitableaux in $\mathbf{SSYT}(((1), (2), (2, 1)/(1)), (3, 3))$ before applying G . This gives three latticed 4-multitableaux,

$$\left(\begin{array}{|c|} \hline 2 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array}, \begin{array}{|c|} \hline 1 \\ \hline \end{array}, \begin{array}{|c|} \hline 1 \\ \hline \end{array} \right), \left(\begin{array}{|c|} \hline 1 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 2 & 2 \\ \hline \end{array}, \begin{array}{|c|} \hline 1 \\ \hline \end{array}, \begin{array}{|c|} \hline 1 \\ \hline \end{array} \right), \left(\begin{array}{|c|} \hline 2 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 1 \\ \hline \end{array}, \begin{array}{|c|} \hline 2 \\ \hline \end{array}, \begin{array}{|c|} \hline 1 \\ \hline \end{array} \right),$$

all fixed by G , and one unlatticed 4-multitableau, obtained by appending $\begin{array}{|c|} \hline 1 \\ \hline \end{array}$ to \mathbf{t}_4 ; its image under G is given by

$$\left(\begin{array}{|c|} \hline 1 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array}, \begin{array}{|c|} \hline 2 \\ \hline \end{array}, \begin{array}{|c|} \hline 1 \\ \hline \end{array} \right) \xleftarrow{G} \left(\begin{array}{|c|} \hline 2 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 2 & 2 \\ \hline \end{array}, \begin{array}{|c|} \hline 2 \\ \hline \end{array}, \begin{array}{|c|} \hline 1 \\ \hline \end{array} \right).$$

There are now five summands in (11), of which two are cancelled by G , and so $\langle s_{(3,2)}(s_{(4,3)/(1)} \circ p_3), s_{(6,5,5,5,2)} \rangle = 3$. Alternatively, we can get the same result by using (a very special case of) the Littlewood–Richardson rule to write $s_{(4,3)/(1)} = s_{(4,2)} + s_{(3,3)}$. From above we have $\langle s_{(3,2)}(s_{(3,3)} \circ p_3), s_{(6,5,5,5,2)} \rangle = 1$ and since $h_{(2,4)} = h_{(4,2)}$, we have

$$\langle s_{(3,2)}(h_{(4,2)} \circ p_3), s_{(6,5,5,5,2)} \rangle = |\mathbf{3-RT}((6, 5, 5, 5, 2), (2, 4))| = 3.$$

Since $|\mathbf{3-RT}((6, 5, 5, 5, 2), (1, 5))| = 1$, we get $\langle s_{(3,2)}(s_{(4,3)/(1)} \circ p_3), s_{(6,5,5,5,2)} \rangle = (4 - 3) + (3 - 1) = 3$, as before. This extra cancellation suggests that the general form of the SXP rule in Theorem 1.1 may have some computational advantages.

6. CONNECTIONS WITH OTHER COMBINATORIAL RULES

6.1. Non-plethystic rules. Let $\mathbf{SSYT}(\nu/\tau, \lambda)$ be the set of semistandard skew tableaux of shape ν/τ and content λ . We say that a skew tableau t is *latticed* if the corresponding skew 1-multitableau (t) is latticed. Let $\mathbf{SSYTL}(\nu/\tau, \lambda)$ be the set of latticed semistandard tableaux of shape ν/τ and content λ .

Let λ/μ be a skew partition of $n \in \mathbb{N}_0$. Setting $r = 1$ in Theorem 1.1 we obtain

$$s_\tau s_{\lambda/\mu} = \sum_\nu c_{(\nu/\tau, \mu)}^\lambda s_\nu \tag{16}$$

where the sum is over all partitions ν such that ν/τ is a skew partition of n . (For the remainder of this subsection we usually rely on the context to make such summations clear.) Specialising (16) further by setting $\mu = \emptyset$ we get

$$s_\tau s_\lambda = \sum_\nu c_{(\nu/\tau)}^\lambda s_\nu. \tag{17}$$

By definition $c_{(\nu/\tau)}^\lambda = |\mathbf{SSYTL}(\nu/\tau, \lambda)|$. Thus (17) is the original Littlewood–Richardson rule, as proved in [10, Theorem III].

Specialising (16) in a different way by setting $\tau = \emptyset$, and then changing notation for consistency with (17), we get

$$s_{\nu/\tau} = \sum_\lambda c_{(\lambda, \tau)}^\nu s_\lambda. \tag{18}$$

By (17) and (18), we have

$$\begin{aligned} \langle s_\tau s_\lambda, s_\nu \rangle &= c_{(\nu/\tau)}^\lambda = |\text{SSYTL}(\nu/\tau, \lambda)| \\ &= |\mathbf{SSYTL}((\lambda, \tau), \nu)| = c_{(\lambda, \tau)}^\nu = \langle s_\lambda, s_{\nu/\tau} \rangle \end{aligned} \quad (19)$$

where the middle equality follows from Proposition 7.1 in the appendix. This gives a combinatorial proof of the fundamental adjointness relation for Schur functions. By (16) and this relation we have $\langle s_{\lambda/\mu}, s_{\nu/\tau} \rangle = c_{(\nu/\tau, \mu)}^\lambda$. If \mathbf{t} is a latticed skew 2-multitableau of shape $(\nu/\tau, \mu)$ then, as seen in (14), $\mathbf{t} = (t, u(\mu))$ for some ν/τ -tableau t . Thus

$$\langle s_{\nu/\tau}, s_{\lambda/\mu} \rangle = c_{(\nu/\tau, \mu)}^\lambda = |\{t \in \text{SSYT}(\nu/\tau, \lambda - \mu) : (t, u(\mu)) \text{ is latticed}\}|. \quad (20)$$

This is equivalent to the skew-skew Littlewood–Richardson rule proved in [17, §4]. The non-obvious equalities $|\text{SSYTL}(\nu/\tau, \lambda)| = |\text{SSYTL}(\nu/\lambda, \tau)|$ and $|\mathbf{SSYTL}((\lambda, \tau), \nu)| = |\mathbf{SSYTL}((\tau, \lambda), \nu)|$ are also corollaries of (19).

As a final exercise, we show that our definition of generalized Littlewood–Richardson coefficients is consistent with the algebraic generalisation of (16) to arbitrary products of Schur functions.

Lemma 6.1. *Let $m \in \mathbb{N}$. If ν/τ is a skew m -multipartition of $n \in \mathbb{N}_0$ and λ is a partition of n then*

$$s_{\nu(1)/\tau(1)} \cdots s_{\nu(m)/\tau(m)} = \sum_{\lambda} c_{\nu/\tau}^\lambda s_\lambda$$

where the sum is over all partitions λ of n .

Proof. By induction, the fundamental adjointness relation and (20) we have

$$\begin{aligned} &\langle s_{\nu(1)/\tau(1)} s_{\nu(2)/\tau(2)} \cdots s_{\nu(m)/\tau(m)}, s_\lambda \rangle \\ &= \left\langle \sum_{\gamma} s_{\nu(1)/\tau(1)} c_{((\nu(2)/\tau(2), \dots, \nu(m)/\tau(m))}^\gamma s_\gamma, s_\lambda \right\rangle \\ &= \sum_{\gamma} \langle s_{\nu(1)/\tau(1)}, s_{\lambda/\gamma} \rangle c_{((\nu(2)/\tau(2), \dots, \nu(m)/\tau(m))}^\gamma \\ &= \sum_{\gamma} c_{(\nu(1)/\tau(1), \gamma)}^\lambda c_{((\nu(2)/\tau(2), \dots, \nu(m)/\tau(m))}^\gamma \end{aligned}$$

where the sums are over all partitions γ of $n - (|\nu(1)| - |\tau(1)|)$. The right-hand side counts the number of pairs of semistandard skew multitableaux $((t, u(\gamma)), \mathbf{t})$ such that $t \in \text{SSYTL}(\nu(1)/\tau(1), \lambda - \gamma)$ and

$$\mathbf{t} \in \mathbf{SSYTL}((\nu(2)/\tau(2), \dots, \nu(m)/\tau(m)), \gamma).$$

Such pairs are in bijection with $\mathbf{SSYTL}((\nu(1)/\tau(1), \dots, \nu(m)/\tau(m)), \lambda)$ by the map sending $((t, u(\gamma)), \mathbf{t})$ to the concatenation $(t : \mathbf{t})$. The lemma follows. \square

6.2. Plethystic rules. By Theorem 1.1 and the fundamental adjointness relation, we have $\langle s_\lambda \circ p_r, s_{\nu/\tau} \rangle = \text{sgn}_r(\nu/\tau) c_{\nu/\tau}^\lambda$. Hence, by Lemma 6.1,

$$\langle s_\lambda \circ p_r, s_{\nu/\tau} \rangle = \begin{cases} \langle s_\lambda, s_{\nu(0)/\tau(0)} \cdots s_{\nu(r-1)/\tau(r-1)} \rangle & \text{if } \nu/\tau \text{ is } r\text{-decomposable,} \\ 0 & \text{otherwise.} \end{cases} \quad (21)$$

This adjointness relation was first proved in [7]: for a more recent proof see [3, after (39)]. It is perhaps a little surprising that (21) implies that the absolute value of the coefficient of $s_{(\nu/\tau, \tau)^*}$ in $s_\tau(s_\lambda \circ p_r)$, namely $c_{\nu/\tau}^\lambda = |\mathbf{SSYTL}(\nu/\tau, \lambda)|$, is the same for all $r!$ permutations of the r -quotient ν/τ .

Note that we obtain only a numerical equality: even cyclic permutations of skew r -multitableaux, do not, in general preserve the lattice property. For example, changing the abaci in Figure 2 in §5 so that 7 beads are used to represent $(3, 2)$ and $(6, 5, 5, 5, 2)$ induces a rightward cyclic shift of the skew tableaux forming the skew 3-multitableaux $\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4$. After one or two such shifts, the unique latticed skew 3-multitableaux are the shifts of \mathbf{t}_3 and \mathbf{t}_2 , respectively; \mathbf{t}_4 remains unlatticed after any number of shifts. The identification of \mathbf{t}_1 as the unique skew 3-multitableau contributing to the coefficient of $s_{(6,5,5,5,2)}$ in $s_{(3,2)}(s_{(3,3)} \circ p_3)$ is therefore canonical, but not entirely natural.

The author is aware of two combinatorial rules in the literature for special cases of the product $s_\tau(s_\lambda \circ p_r)$ that avoid this undesirable feature of the SXP rule. To state the first, which is due to Carré and Leclerc, we need a definition from [1]. Let T be an r -ribbon tableau of shape ν/τ and weight λ . Represent T , as in Figure 3, by a tableau of shape ν/τ in which the boxes of the α_j disjoint r -border strips forming the horizontal r -ribbon in T labelled j all contain j . The *column word* of T is the word of length n obtained by reading the columns of this tableau from bottom to top, starting at the the leftmost (lowest numbered) column, and recording the label of each r -border strip when it is first seen, in its leftmost column.

Theorem 6.2 ([1, Corollary 4.3]). *Let $r \in \mathbb{N}$ and let $n \in \mathbb{N}_0$. Let ν/τ be a skew partition of rn and let λ be a partition of n . Up to the sign $\text{sgn}_2(\nu/\tau)$, the multiplicity $\langle s_\tau(s_\lambda \circ p_2), s_\nu \rangle$ is equal to the number of 2-ribbon tableaux T of shape ν/τ and weight λ whose column word is latticed.*

For example, there are two 2-ribbon tableaux of shape $(5, 5, 2, 2)/(3, 1)$ and content $(3, 1, 1)$ having a latticed column word (see Figure 4 overleaf), and so $\langle s_{(3,1)}(s_{(3,1,1)} \circ p_2), s_{(5,5,2,2)} \rangle = 2$. These correspond to the skew 2-multitableaux of shape $((3, 1)/(2), (2, 1))$

$$\left(\begin{array}{|c|} \hline 1 \\ \hline \end{array} \quad \begin{array}{|c|} \hline 1 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \end{array} \right), \quad \left(\begin{array}{|c|} \hline 1 \\ \hline \end{array} \quad \begin{array}{|c|} \hline 1 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & \end{array} \right),$$

respectively. Only the second is latticed in the multitableau sense.

			1	1
	1	1	2	2
1	3			
1	3			

			1	1
	1	1	1	1
2	2			
3	3			

FIGURE 4. The two 2-ribbon tableaux of shape $(5, 5, 2, 2)/(3, 1)$ and content $(3, 1, 1)$ whose column words, namely 13121 and 32111, are latticed.

In Theorem 6.3 of [4], Evseev, Paget and the author applied character theoretic arguments to the case $\lambda = (a, 1^b)$, considering arbitrary $r \in \mathbb{N}$. To state this result in our setting, we introduce the following definition.

Definition 6.3. The *row-number tableau* of an r -ribbon tableau T is the row-standard tableau $\text{RNT}(T)$ defined by putting an entry i in its row a for each r -border strip of row number a in the r -ribbon strip of T labelled i .

If T has weight λ then the content of $\text{RNT}(T)$ is λ . The shape of $\text{RNT}(T)$ is in general a composition, possibly with some zero parts. The row-number tableaux of the four 3-ribbon tableaux in $3\text{-RT}((6, 5, 5, 5, 2)/(3, 2), (3, 3))$, shown in the top line of Figure 3 in §5, are

1		1	1		1	2	2		1	1
2	2	2			1	2	2		1	2
1	1				1				1	2
					2				1	2

The definition of latticed extends to row-number tableaux in the obvious way. The second row-number tableau above, with word 212211, is the only one that is latticed.

Corollary 6.4 (see [4, Theorem 6.3]). *Let $r \in \mathbb{N}$, let $a \in \mathbb{N}$ and let $b \in \mathbb{N}_0$. Let ν/τ be a skew partition of $r(a + b)$. Then $\langle s_\tau(s_{(a, 1^b)} \circ p_r), s_\nu \rangle$ is equal, up to the sign $\text{sgn}_r(\nu/\tau)$, to the number of r -ribbon tableaux of shape ν/τ and weight $(a, 1^b)$ whose row-number tableau is latticed. The column word of such an r -ribbon tableau is $(b + 1)b \dots 21 \dots 1$, where the number of 1s is a .*

Proof. By Theorem 6.3 in [4], up to the sign $\text{sgn}_r(\nu/\tau)$, the multiplicity $\langle s_\tau(s_{(a, 1^b)} \circ p_r), s_\nu \rangle$ is the number of $(a, 1^b)$ -like border-strip r -diagrams of shape ν/τ , as defined in [4, Definition 6.2]. (The required translation from character theory to symmetric functions is outlined in [4, §7].) To relate these objects to r -ribbon tableaux, we define a skew partition ρ/τ to be a *vertical r -ribbon strip* if ρ'/τ' is a horizontal r -ribbon strip.

Let T be an r -ribbon tableau of shape ν/τ and weight $(a, 1^b)$. There is a unique partition ρ such that ρ/τ is the horizontal a -ribbon strip in T and ν/ρ is a vertical b -ribbon strip, formed from the border strips labelled 2,

$\dots, b + 1$. Suppose $\text{RNT}(T)$ is latticed. Then the row numbers of these border strips are increasing. Moreover, the rightmost border strip in either of the ribbons ρ/τ and ν/ρ lies in the Young diagram of ρ/τ , and the skew partition formed from this border strip and ν/ρ is a vertical $(b + 1)$ -ribbon strip. Therefore T corresponds to an $(a, 1^b)$ -like border-strip r -diagram of shape ν/τ , and the column word of T is as claimed. Conversely, each such r -ribbon tableau arises in this way. \square

The second claim in Corollary 6.4 implies that if T is an r -ribbon tableau of weight $(a, 1^b)$ whose row-number tableau $\text{RNT}(T)$ is latticed, then the word of $\text{RNT}(T)$ agrees with the column word of T . Hence the combinatorial rules for $\langle s_\tau(s_{(a,1^b)} \circ p_2), s_\nu \rangle$ obtained by taking $\lambda = (a, 1^b)$ in Corollary 4.3 of [1] or $r = 2$ in Corollary 6.4 count the same sets of r -ribbon tableaux. For example, in Figure 4 we have $a = 3$ and $b = 2$; the first 2-ribbon tableau has a horizontal 2-ribbon strip of shape $(5, 3, 1, 1)/(3, 1)$, a vertical 2-ribbon strip of shape $(5, 5, 2, 2)/(5, 3, 1, 1)$, and the augmented vertical 2-ribbon strip has shape $(5, 5, 2, 2)/(3, 3, 1, 1)$.

For general weights we have the following result.

Proposition 6.5. *Let $r \in \mathbb{N}$. Let T be an r -ribbon tableau. If the column word of T is latticed then the row-number tableau of T is latticed.*

The proof is given in the appendix. The converse of Proposition 6.5 is false. For example $\langle s_{(2,2)} \circ p_3, s_{(3,3,3,3)} \rangle = 1$. The two 3-ribbon tableaux in $r\text{-RT}((3, 3, 3, 3), (2, 2))$ are shown below. Both have a latticed row-number tableau, with word 2211. The column words are 2112 and 2121 respectively; only the second is latticed.

1	1	1
1	1	2
1	2	2
2	2	2

1	1	1
1	1	1
2	2	2
2	2	2

In both Theorem 6.2 and Corollary 6.4 there is a lattice condition that refers directly to certain sets of r -ribbon tableaux, without making use of r -quotients. In the following problem, which the author believes is open under its intended interpretation (except when $r = 2$ or $\lambda = (a, 1^b)$ for some $a \in \mathbb{N}$ and $b \in \mathbb{N}_0$) we say that such conditions are *global*.

Problem 6.6. *Find a combinatorial rule, simultaneously generalizing Theorem 6.2 and Corollary 6.4, that expresses $\langle s_\tau(s_\lambda \circ p_r), s_\nu \rangle$ as the product of $\text{sgn}_r(\nu/\tau)$ and the size of a set of r -ribbon tableaux of shape ν/τ satisfying a global lattice condition.*

The obvious generalizations of Theorem 6.2 and Corollary 6.4 fail even to give correct upper and lower bounds on the multiplicity in Problem 6.6. Counterexamples are shown in the table below. The second column gives the

number of r -ribbon tableaux of the relevant shape and weight and the final two columns count those r -ribbon tableaux whose column word is latticed (CWL), and whose row-number tableau is latticed (RNTL), respectively.

plethysm $\langle s_\tau(s_\lambda \circ p_r), s_\nu \rangle$	$ r\text{-RT}(\nu/\tau, \lambda) $	CWL	RNTL
$\langle s_{(3,3)} \circ p_3, s_{(6,6,6)} \rangle = 1$	6	0	2
$\langle s_{(2,2,2)} \circ p_4, s_{(7,4,4,4,4,1)} \rangle = -1$	9	0	0
$\langle s_{(1)}(s_{(3,3)} \circ p_3), s_{(6,6,6,1)} \rangle = 1$	6	0	0
$\langle s_{(1)}(s_{(2,2)} \circ p_4), s_{(5,4,4,4)} \rangle = 1$	2	2	2

Despite this, there are some signs that row-number tableaux are a useful object in more general settings than Corollary 6.4. In particular, the following conjecture holds when $r \leq 4$ and $n \leq 10$ and when $r \leq 6$ and $n \leq 6$. (Haskell [16] source code to verify this claim is available from the author.) When $r = 2$ it holds by Theorem 6.2 and Proposition 6.5, replacing (a, b) with a general partition λ ; by row 2 of the table above, this more general conjecture is false when $r = 4$. By Corollary 6.4, the conjecture holds, with equality, when $b = 1$.

Conjecture 6.7. *Let $r \in \mathbb{N}$, let $n \in \mathbb{N}_0$, let ν be a partition of rn and let (a, b) be a partition of n . The number of r -ribbon tableaux T of shape ν and weight (a, b) such that the row-number tableau $\text{RNT}(T)$ is latticed is an upper bound for the absolute value of $\langle s_{(a,b)} \circ p_r, s_\nu \rangle$.*

APPENDIX: THE SHAPE-CONTENT INVOLUTION AND PROOF OF PROPOSITION 6.5

In the proof of (19) we used the following proposition.

Proposition 7.1. *If λ, μ and ν are partitions then*

$$|\{t \in \text{SSYT}(\nu, \lambda - \mu) : (t, u(\mu)) \text{ is latticed}\}| = |\text{SSYTL}(\lambda/\mu, \nu)|.$$

This proposition follows immediately from Lemma 7.2(iii) below, by setting $\alpha = \nu$ and $\beta = \emptyset$. The ‘shape-content involution’ given in this lemma is surely well known to experts, but the author has not found it in the literature in this generality. The lemma may also be used to show that the final corollary in Stembridge’s involutive proof of the Littlewood–Richardson rule [19] is equivalent to (17); this is left as a ‘not-too-difficult exercise’ in [19].

Let λ/μ and α/β be skew partitions of the same size. Let $\text{RSYT}(\lambda/\mu, \alpha/\beta)$ be the set of all row-standard λ/μ tableaux t such that $\beta + \text{cont}(t) = \alpha$. Let

$$\text{RSYTL}(\lambda/\mu, \alpha/\beta) = \{t \in \text{RSYT}(\lambda/\mu, \alpha/\beta) : (t, u(\beta)) \text{ is latticed}\},$$

$$\text{SSYTL}(\lambda/\mu, \alpha/\beta) = \text{RSYTL}(\lambda/\mu, \alpha/\beta) \cap \text{SSYT}(\lambda/\mu, \alpha/\beta).$$

Given $t \in \text{RSYT}(\lambda/\mu, \alpha/\beta)$, let $SC(t)$ be the row-standard tableau of shape α/β defined by putting a k in row a of $SC(t)$ for every a in row k of t .

Lemma 7.2 (Shape/content involution).

- (i) $SC : \text{RSYT}(\lambda/\mu, \alpha/\beta) \rightarrow \text{RSYT}(\alpha/\beta, \lambda/\mu)$ is an involution.
- (ii) SC restricts to a involution $\text{SSYT}(\lambda/\mu, \alpha/\beta) \rightarrow \text{RSYT}(\alpha/\beta, \lambda/\mu)$.
- (iii) SC restricts to a involution $\text{SSYT}(\lambda/\mu, \alpha/\beta) \rightarrow \text{SSYT}(\alpha/\beta, \lambda/\mu)$.

Proof. (i) is obvious. For (ii) observe that if $t \in \text{RSYT}(\lambda/\mu, \alpha/\beta)$ then $(t, u(\beta))$ is not latticed if and only if there exists $k \in \mathbb{N}$ and an entry $k+1$ in row a of t and position i of $w(t)$ such that

$$|\{j : w(t)_j = k+1, j \geq i\}| + \beta_{k+1} = |\{j : w(t)_j = k : j > i\}| + \beta_k + 1.$$

Let b be the common value. The first $b-1-\beta_k$ entries in row k of $SC(t)$ are at most $a-1$, and the next entry is the number of a row a' with $a' \geq a$. The entry below is the $(b-\beta_{k+1})$ -th entry in row $k+1$ of $SC(t+1)$, namely a . Therefore $SC(t)$ is not semistandard. The converse may be proved by reversing this argument. It follows from (ii) that SC restricts to involutions $\text{SSYT}(\lambda/\mu, \alpha/\beta) \rightarrow \text{RSYT}(\alpha/\beta, \lambda/\mu)$ and $\text{RSYT}(\lambda/\mu, \alpha/\beta) \rightarrow \text{SSYT}(\alpha/\beta, \lambda/\mu)$; taking the common domain and codomain of these involutions we get (iii). \square

We end with the proof of Proposition 6.5. One final definition will be useful. Let D be a subset of the boxes of a Young diagram of a partition ν . If column b is the least numbered column of ν meeting D , then we say that D has *column number* b , and write $C(D) = b$. (Thus if D is a border strip in ν then D has column number b if and only if the conjugate border strip D' in ν' has row number b .) For an example see Figure 5 overleaf.

Proof of Proposition 6.5. Suppose that the labels of the r -ribbons in T are $\{1, \dots, \ell\}$. Fix $k < \ell$. Let D_1, \dots, D_q be the subsets of the Young diagram of ν/τ that form the r -border strips lying in the r -ribbon strips of T labelled k and $k+1$, written in the order corresponding to the column word of T . Thus

$$C(D_1) \leq \dots \leq C(D_q)$$

and if $C(D_j) = C(D_{j+1})$ then $R(D_j) > R(D_{j+1})$. Let $N(D_j) \in \{k, k+1\}$ be the label of D_j . Let

$$w = N(D_1)N(D_2) \dots N(D_q)$$

be the subword of the column word of T formed from the entries k and $k+1$. By hypothesis, w has no k -unpaired $k+1$.

Let v be the subword of the word of the row-number tableau $\text{RNT}(T)$ formed from the entries k and $k+1$. We may obtain v by reading the rows of T from left to right, starting at the highest numbered row, and writing down the label $N(D_j)$ of D_j on the final occasion when we see a box of D_j . By (3), if $N(D_j) = N(D_{j+1})$ then $R(D_j) \geq R(D_{j+1})$. Moreover, if $N(D_j) = k+1$ and $N(D_{j+1}) = k$ then $R(D_j) > R(D_{j+1})$. Therefore $N(D_j)$ is written after $N(D_{j+1})$ when writing v if and only if $N(D_j) = k$, $N(D_{j+1}) = k+1$

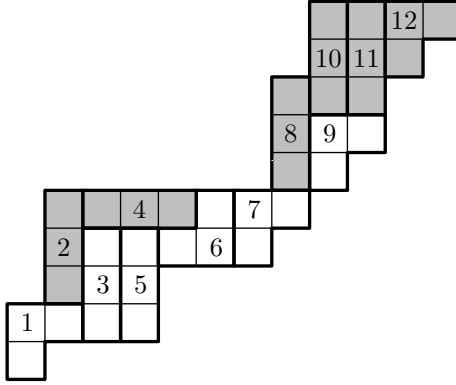


FIGURE 5. Border strips D_1, \dots, D_{12} labelled k (grey) or $k+1$ (white) forming the 3-ribbons in a 3-ribbon tableau T are shown. Numbers are as in the proof of Proposition 6.5. For example, $R(D_9) = 4$ and $C(D_9) = 9$. The subword of the column word with entries k and $k+1$ is $k^+kk^+kk^+k^+k^+kk^+kkk$, where k^+ denotes $k+1$. The inversions are 2, 4 and 8. The subword of the row word of the row-number tableau of T with entries k and k^+ is obtained by sorting the entries in positions 2, 3, 4, 5 and 8, 9 into decreasing order, giving $k^+k^+k^+kkk^+k^+k^+kkkk$.

and $R(D_j) < R(D_{j+1})$. We say that such j are *inversions*. If there are no inversions, then v and w are equal. Otherwise, let j be minimal such that j is an inversion, and let s be maximal such that $R(D_j) < R(D_{j+s})$; note that $N(D_{j+s}) = k+1$, by (3). The word v is obtained from w sorting its entries in positions $j, j+1, \dots, j+s$ into decreasing order, and then continuing inductively with the later positions. It is clear that this procedure does not create a new k -unpaired $k+1$. Hence v has no k -unpaired $k+1$. \square

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