TWO STABILITY THEOREMS ON PLETHYSMS OF SCHUR FUNCTIONS

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Abstract. The plethysm product of Schur functions corresponds to composition of polynomial representations of infinite general linear groups. Finding the plethysm coefficients $\langle s_\nu \circ s_\mu, s_\lambda \rangle$ that express an arbitrary plethysm $s_\nu \circ s_\mu$ as a sum $\sum_\lambda \langle s_\nu \circ s_\mu, s_\lambda \rangle s_\lambda$ of Schur functions is a fundamental open problem in algebraic combinatorics. We prove two stability theorems for plethysm coefficients under the operations of adding and/or joining an arbitrary partition to either μ or ν. In both theorems μ may be replaced with an arbitrary skew partition. As special cases we obtain all stability results on plethysms of Schur functions in the literature to date. The proofs are entirely combinatorial using plethystic semistandard tableaux with positive and negative entries.

Contents

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1. Introduction

1.1. **Background.** Determining the decomposition of an arbitrary plethysm product $s_\nu \circ s_\mu$ into Schur functions was identified by Richard Stanley in [22] as a central open problem in algebraic combinatorics. It is equivalent to decomposing a polynomial representation of $\mathrm{GL}_n(\mathbb{C})$ defined by a composition of Schur functors into a direct sum of Schur functors, and to decomposing a representation of a symmetric group induced from an arbitrary irreducible representation of a wreath product subgroup into a direct sum of irreducible representations. The *plethysm coefficients* are the multiplicities in these decompositions. We refer the reader the introduction to [18] for a full account of these connections.

1.2. **Stability.** In this paper we prove two theorems showing that certain sequences of plethysm coefficients are ultimately constant, with *explicit* bounds for when stability occurs. We also give practical sufficient conditions for the stable value to be zero. These theorems include as special cases all stability results on plethysm products in the current literature, sometimes with new bounds when none were proved originally. For instance a special case of Theorem 1.1, first proved in [5, p354], is that $\langle s_\nu \circ s_{\mu+M\kappa}, s_{\lambda+|\nu|M\kappa}\rangle$ is ultimately constant for large M, while a special case of Theorem 1.2 is a key motivating result, recently proved in [13] without an explicit bound, that if d is even then $\big\langle s_{\nu+(M)} \circ s_{(m)}, s_{\lambda+M(m-d)\sqcup(d^M)}\big\rangle$ is ultimately constant for large M. Our proofs are entirely combinatorial, using the plethystic semistandard signed tableaux defined in Definition 3.10 below.

1.3. **Main results.** In both our main theorems, ν is a partition, $\mu/\mu_\star$ is a skew partition and λ is a partition of $|\nu||\mu/\mu_\star|$. We define $\mu/\mu_\star \oplus (\gamma,\delta) = \big((\mu+\delta)\sqcup\gamma'\big)/\mu^\star$ and $\mu/\mu_\star \oplus M(\gamma,\delta) = \mu/\mu_\star \oplus (M\gamma, M\delta)$, where $\sqcup$ is the join of partitions, defined formally before (3.1). The order $\trianglelefteq$ on pairs of partitions is defined in Definition 4.1 by reading the pair as a composition and then applying the dominance order. Example 11.10 in §2 motivates the conjugation seen when $|\kappa^-|$ is odd.

Theorem 1.1 (Signed inner stability)**.** *Let κ^- and κ^+ be partitions. If $|\kappa^-|$ is even then set $\nu^{(M)} = \nu$ for all M; if $|\kappa^-|$ is odd then set $\nu^{(M)} = \nu$ if M is even and $\nu^{(M)} = \nu'$ if M is odd. Then*

$$\big\langle s_{\nu^{(M)}} \circ s_{\mu/\mu_\star \oplus M(\kappa^-,\kappa^+)}, s_{\lambda \oplus nM(\kappa^-,\kappa^+)}\big\rangle$$

is constant for M at least the explicit bound in Theorem 11.15. Moreover if η^- and η^+ are partitions with $\ell(\eta^-) \le \ell(\kappa^-)$ and $(\eta^-,\eta^+) \ntrianglelefteq (\kappa^-,\kappa^+)$ then

$$\big\langle s_{\nu^{(M)}} \circ s_{\mu/\mu_\star \oplus M(\kappa^-,\kappa^+)}, s_{\lambda \oplus nM(\eta^-,\eta^+)}\big\rangle$$

is zero for M greater than the explicit bound in Proposition 11.2.

Our second main theorem requires the strongly maximal signed weights defined in Definition 4.10 and first exemplified in Example 4.12. To orient the reader, we remark that, by Lemma 4.17, $(\varnothing,\mu)$ and $(\mu',\varnothing)$ are strongly maximal signed weights of shape μ and size 1; their signs are 1 and $(-1)^{|\mu|}$, respectively. The strongly maximal signed weight relevant to the stability of $\langle s_{\nu+(M)} \circ s_{(m)}, s_{\lambda+M(m-d)\sqcup(d^M)}\rangle$ for even d is $\bigl((1^d),(m-d)\bigr)$. This signed weight has shape (m), size 1 and sign $(-1)^d$: see Example 4.18(i).

Theorem 1.2 (Signed outer stability)**.** *Let $R \in \mathbb{N}$. Let (κ^-,κ^+) be a strongly maximal signed weight of shape $\mu/\mu_\star$ and size R. Set $\nu^{(M)} = \nu + (M^R)$ if (κ^-,κ^+) has sign $+1$ and $\nu^{(M)} = \nu \sqcup (R^M)$ if (κ^-,κ^+) has sign -1. Then*

$$\bigl\langle s_{\nu^{(M)}} \circ s_{\mu/\mu_\star}, s_{\lambda \oplus M(\kappa^-,\kappa^+)}\bigr\rangle$$

is constant for M at least the explicit bound in Theorem 14.7. Moreover if η^- and η^+ are partitions with $\ell(\eta^-) \le \ell(\kappa^-)$ and $(\kappa^-,\kappa^+) \lhd (\eta^-,\eta^+)$ then

$$\bigl\langle s_{\nu^{(M)}} \circ s_{\mu/\mu_\star}, s_{\lambda \oplus M(\eta^-,\eta^+)}\bigr\rangle$$

is zero for M greater than the explicit bound in Proposition 14.1.

The full versions of both theorems give practical sufficient conditions for the constant multiplicity in their first parts to be zero. For instance, as we explain after Example 11.16, the final part of Theorem 11.15 implies that $\langle s_\nu \circ s_{\mu \oplus M(\kappa^-,\kappa^+)}, s_{\lambda \oplus nM(\kappa^-,\kappa^+)}\rangle$ is zero for M sufficiently large, unless $(\lambda^-,\lambda^+) \unlhd n(\mu^-,\mu^+)$. Here (λ^-,λ^+) and (μ^-,μ^+) are the $\ell(\kappa^-)$-decompositions of λ and μ, as defined in Definition 6.1, and $\unlhd$ is the signed dominance order in Definition 4.1.

In Corollaries 12.1, 12.2 and 15.9, we give the corollaries of our two main theorems for the special cases where $\kappa^- = \varnothing$ and $\mu_\star = \varnothing$, showing how the explicit bounds and conditions in Theorems 11.15 and 14.7 simplify. Corollary 15.1 is the case $R = 1$ of Theorem 1.2 and is also of significant interest in its own right.

1.4. **Strongly maximal signed weights.** An important motivation for strongly maximal signed weights is that if $\mu/\mu_\star$ is a skew partition and κ is the lexicographically maximal partition labelling a Schur function summand of $s_{(1^R)} \circ s_{\mu/\mu_\star}$ then $(\varnothing,\kappa)$ is a strongly maximal signed weight of a $\mu/\mu_\star$-tableau family of size R. We plan to prove this result in a separate paper on signed maximal constituents of plethysms. Many further examples of strongly maximal signed weights, with full proofs, are given in §4.4. In particular we mention Lemma 4.20 which was motivated by (9) in [4] by Briand, Orellana and Rosas, as we discuss in §1.7.

1.5. **Skew partitions.** It is worth noting that the results on plethysms $s_\nu \circ s_{\mu/\mu^\star}$ where $\mu/\mu^\star$ is a skew partition with $\mu^\star \neq \varnothing$ are entirely novel to this paper: it is a feature of our method that this extension from partitions to skew partitions is mostly routine. See Examples 4.21 and 15.2 for examples exploiting this generality. Remark 5.1 explains why the further extension replacing ν with a skew partition is a straightforward corollary of our main theorems.

1.6. **A stronger conjecture.** Theorem 1.2 was motivated by Proposition 5.3 in [12], which in turn was motivated by a conjecture of Bessenrodt, Bowman and Paget [1, Conjecture 1.2] that the plethysm coefficients $\langle s_{\nu \sqcup (1^M)} \circ s_{(2)}, s_{\lambda \oplus M((1),(1))}\rangle$ are non-decreasing with M. A proof of this conjecture appears to require fundamentally different methods to those used in this paper: we believe it is true and that a proof will be of wide interest. More generally, we make the following conjecture, which includes the BPP-conjecture as a special case.

Conjecture 1.3. *The sequences of plethysm coefficients in Theorems 1.1 and 1.2 are non-decreasing with respect to M.*

1.7. **Earlier work.** We believe the two main theorems in this paper imply all the stability results on Schur functions published in the literature. These include the stable version of Foulkes Conjecture. Here we survey [3] by Bowman and Paget, [4] by Briand, Orellana and Rosas, [5] by Brion, [6] by Carré and Thibon, [7] by Colmenarejo, [8] by de Boeck, Paget and Wildon, [13, 12] by Law and Okitani, [16] by Manivel and [23] by Weintraub. (Except in the case of one result from [6], we silently change the notation used by these authors to be consistent, as far as possible, with this paper.)

Bowman–Paget. THEOREM A OF [3]. This states that the plethysm coefficients $\langle s_{(n+N)} \circ s_{(m+M)}, s_{\lambda+(mN+nM+MN)}\rangle$ are ultimately constant. For M varying this is the special case of Theorem 1.1 for $\nu = (n+N)$, $\mu = (m)$ taking $(\kappa^-, \kappa^+) = \bigl(\varnothing, (1)\bigr)$. The bound from Corollary 12.2, applied replacing λ with $\lambda + (mN)$, is $M \geq (n+N-1)m - (\lambda_1 + mN) = (n-1)m - \lambda_1$ which improves on $M \geq |\lambda| = mn$ in [3]. For N varying this is the special case of Theorem 1.2 for $\nu = (n)$, $\mu = (m+M)$, again with the same choice of (κ^-, κ^+); by Lemma 4.17, $\bigl(\varnothing, (1)\bigr)$ is a strongly 1-maximal signed weight. The bound from Corollary 15.9 is in general worse than $N \geq |\lambda|$ in [3]. A corollary (see [3, Corollary 9.4]) is that the 'stable' version of Foulkes' Conjecture [9] holds with equality. We emphasise that the main contribution of [3] is to prove the result using Schur–Weyl duality with the partition algebra, thereby giving an explicit and *clearly positive* formula for the multiplicities. This goes significantly beyond the results obtainable by the general methods in this paper.

Briand–Orellana–Rosas. RESULT (7) IN [4]. This states that $\langle s_\nu \circ s_\mu, s_\lambda\rangle = \langle s_\nu \circ s_{\mu+(M^\ell)}, s_{\lambda+n(M^\ell)}\rangle$ provided that $\ell(\nu) \leq \ell$. This is a weaker version of Theorem 1.2 in [8] by de Boeck, Paget and Wildon, discussed below.

RESULT (9) IN [4]. This states that

$$\langle s_{\nu+M(1^R)} \circ s_\mu, s_{\lambda+M(q^\ell)}\rangle \tag{1.1}$$

is constant, where R is the number of semistandard tableaux of shape μ with entries from $\{1, \ldots, \ell\}$ and $q = R|\mu|/\ell$. By Theorem 1.2 applied with the strongly ℓ-maximal signed weight $\bigl(\varnothing, (q^\ell)\bigr)$ (see Lemma 4.20) the plethysm coefficient is ultimately constant. In fact this theorem implies the more general result where μ is replaced with an arbitrary skew partition. The relevant strongly maximal semistandard signed tableau family is, as one would expect from the statement of (9), all semistandard tableaux of shape μ with entries from $\{1, \ldots, \ell\}$. While the 'signed' generality is irrelevant here, this was an important motivating example for strongly maximal signed weights. Corollary 15.9 can be used to give explicit stability bounds for (1.1); Proposition 15.11 shows that in many cases of interest, stability is immediate.

Brion. THEOREM [5, §2.1]. This states that $\langle s_\nu \circ s_{\mu+M\kappa}, s_{\lambda+nM\kappa}\rangle$ is ultimately constant. There is a bound implicitly defined using the root system of type A. This is the special case of Theorem 1.1 taking $(\kappa^-, \kappa^+) = (\varnothing, \kappa)$. The bound from Theorem 11.15 is the same.

THEOREM [5, §3.1]. This states that $\langle s_{\nu+(n)} \circ s_\mu, s_{\lambda+n\mu}\rangle$ is ultimately constant with an explicit bound. This is the special case of Theorem 1.2 taking $(\kappa^-, \kappa^+) = (\varnothing, \mu)$; by Lemma 4.17 this is a strongly $\ell(\mu)$-maximal signed weight. Brion's bound improves on the bound from Theorem 14.7 or Corollary 15.9 by using orthogonality in the type A root system.

Carré–Thibon. We first note that Jp in [6] is, in our notation $(p) \sqcup J$, where J is a partition. If J has first part a and $p \geq a$ then $(p) \sqcup J = \big(J \sqcup (a)\big) + (p-a)$, and so, by taking p sufficiently large, we can interpret Jp as an addition of $(p-a)$ to the partition $J \sqcup (a)$.

THEOREM 4.1 IN [6]. The special case (see the remark after the proof in [6]) relevant to plethysm coefficients is equivalent, by the previous notational remark, to the theorem in §2.1 of Brion [5], discussed above.

THEOREM 4.2 IN [6]. It follows very similarly that the special case relevant to plethysm coefficients is that $\langle s_{\nu+(M)} \circ s_\mu, s_{\lambda+(M|\mu|)}\rangle$ is ultimately constant. When $\mu = (m)$ this is a special case of the theorem in §3.1 of Brion [5] discussed above. When $\mu \neq (m)$ we have $\mu \lhd (m)$ and so the stable multiplicity is zero by the 'moreover' part of Theorem 1.2 applied with the strongly maximal signed weight $(\varnothing, \mu)$. (By Lemma 4.17 this is a strongly $\ell(\mu)$-maximal signed weight.)

We remark that [6] precedes [5] and the method of vertex operators used in [6] is completely different to Brion's geometric arguments. It is also worth noting that $Jp = J \oplus \big((1^a), (p-a)\big)$, and so our results can also be applied without first taking p sufficiently large.

Colmenarejo. THEOREM 1.1 IN [7]. This states four stability results. The first is the special case of the second taking, in the notation of [7], $\pi = (1)$. The remaining three are:

- $\langle s_\nu \circ s_{\mu+M\kappa}, s_{\lambda+nM\kappa}\rangle$ is ultimately constant. As just seen, this is the special case of Theorem 1.1 taking $(\kappa^-, \kappa^+) = \big(\varnothing, \kappa\big)$.
- $\langle s_{\nu+(M)} \circ s_\mu, s_{\nu+M\mu}\rangle$ is ultimately constant. This is the special case of Theorem 1.2 taking $(\kappa^-, \kappa^+) = (\varnothing, \mu)$; by Lemma 4.17 this is a strongly $\ell(\mu)$-maximal signed weight.

- $\langle s_{\nu+(M)} \circ s_\mu, s_{\nu+M(|\mu|)}\rangle$ is ultimately constant. This is the same as Theorem 4.2 in Carré and Thibon [6] already discussed.

deBoeck–Paget–Wildon. THEOREM 1.1 IN [8]. This states the equality $\langle s_\nu \circ s_{(r)\sqcup\mu}, s_{(nr)\sqcup\lambda}\rangle = \langle s_\nu \circ s_\mu, s_\lambda\rangle$ provided r is at least the greatest part of μ. Applying the ω-involution (see [15, page 21] or [21, §7.6]) this becomes

$$\langle s_{\nu^\dagger} \circ s_{\mu'+(1^r)}, s_{\lambda'+(1^{nr})}\rangle = \langle s_\nu \circ s_\mu, s_\lambda\rangle,$$

provided $M \geq \ell(\mu)$, where $\nu^\dagger = \nu$ if r is even and $\nu^\dagger = \nu'$ if r is odd. Observe that when $M \geq \ell(\mu')$ we have $\mu' + (1^{M+1}) = \bigl(\mu' + (1^M)\bigr) \sqcup (1)$ and when $nM \geq \ell(\lambda')$ we have $\lambda' + (1^{n(M+1)}) = \bigl(\lambda' + (1^{nM})\bigr) \sqcup (1^M)$. The plethysm coefficient above is therefore

$$\langle s_{\nu^\dagger} \circ s_{\mu'+(1^{\ell(\mu')})\sqcup(1^M)}, s_{\lambda'+(1^{n\ell(\mu')})\sqcup(1^{nM})}\rangle. \tag{1.2}$$

That it is ultimately constant now follows from Theorem 1.1, taking $\kappa^- = (1)$, and $\kappa^+ = \varnothing$ and replacing μ with $\mu'+(1^{\ell(\mu')})$ and λ with $\lambda'+(1^{n\ell(\mu')})$. As we show in Example 11.16, the explicit bounds in Theorem 11.15 show that in fact the plethysm coefficient (as stated in the second displayed equation) is immediately constant provided $\ell(\lambda') \leq n\ell(\mu')$.

THEOREM 1.2 IN [8]. This states that $\langle s_\nu \circ s_{\mu+M(1^r)}, s_{\lambda+M(n^r)}\rangle$ is constant for M greater than an explicit bound. By Theorem 1.1, applied with $\kappa^- = \varnothing$ and $\kappa^+ = (1^r)$, the plethysm coefficient is ultimately constant. The bound from Theorem 11.15 is the same, as we show at the end of §11.

Law–Okitani. PROPOSITION 5.3 IN [12]. This states that $\langle s_{\nu\sqcup(1^M)} \circ s_{(2)}, s_{\lambda\oplus M((1),(1))}\rangle$ is ultimately constant. This is the special case of Theorem 1.2 taking $\mu = (2)$ and $(\kappa^-, \kappa^+) = \bigl((1),(1)\bigr)$; by Lemma 4.17 $\bigl((1),(1)\bigr)$ is a strongly 1-maximal signed weight.

THEOREM 1 IN [13]. This paper followed [12]. An equivalent statement of Theorem 1 is that when d is even

$$\langle s_{\nu+(M)} \circ s_{(m)}, s_{\lambda\oplus M((1^d),(m-d))}\rangle \tag{1.3}$$

is ultimately constant and when d is odd

$$\langle s_{\nu\sqcup(1^M)} \circ s_{(m)}, s_{\lambda\oplus M((1^d),(m-d))}\rangle \tag{1.4}$$

is ultimately constant. This result was briefly known between March 2022 and September 2022 as Wildon's Conjecture: it was an important motivation for Theorem 1.2, and is exemplified in §8.2. No bounds on M were proved in [13]. These results are unified as the special case of Theorem 1.2 taking $\mu = (m)$ and $(\kappa^-, \kappa^+) = \bigl((1^d),(m-d)\bigr)$; by Example 4.18(i) this is a strongly 1-maximal signed weight.

Manivel. MAIN RESULT AND THEOREM 4.3.1 IN [16]. This is the same result as Theorem A in [3] by Bowman and Paget, already discussed. We emphasise that the proof in [16] is by novel geometric methods.

Weintraub. THEOREM 0.1 IN [23]. This states that $\langle s_{\nu+(M)} \circ s_\mu, s_{\lambda+M(|\mu|)}\rangle$ is ultimately constant. It is the same as Theorem 4.2 in [6] and the final result of Colmenarejo [7] both discussed above; we mention that Weintraub's proof precedes both these papers and the methods used are different from either.

1.8. **Outline.** This paper is split into the five parts indicated in the table of contents. Each section is written to be read independently as far as possible.

Introduction and overview (§1–2). The introductory material ends in §2 with an overview of the proof and some small examples: we hope this will persuade the reader that while the proof is lengthy, because of many minor technical difficulties, the overall concept of finding stable bijections between certain semistandard signed tableaux and between certain plethystic semistandard signed tableaux is quite simple.

Preliminaries (§3–6). In §3 we give basic definitions. In particular we define plethystic semistandard signed tableaux in Definition 3.10. The reader should be able to skip this section and then use it as a reference. In §4 we define the signed weights needed to prove Theorem 1.1 and the strongly maximal signed weights in Theorem 1.2. In §5 we give background results on plethysms of symmetric functions. Finally in §6 we define the ℓ^--twisted dominance order in Definition 6.6 and generalize classical results on Kostka numbers to the twisted case. This is a key definition novel to this paper.

Signed Weight Lemma and stable partition systems (§7–§9). In §7 we prove the critical Signed Weight Lemma (Lemma 7.3): this lemma specifies the overall strategy of the proof of the main theorems and is motivated by §2. To apply the lemma we require the idea of a stable partition system, as defined in §7.1. We give two motivating examples of stable partition systems in §8, and then in §9 we construct the stable partition systems used to prove Theorems 1.1 and 1.2. Also in §8 we show some of the main ideas in the proofs of Theorems 1.1 and Theorem 1.2 by examples using the three key results proved by the end of §9, namely

- Proposition 5.6 on plethystic signed Kostka numbers, stating that $\langle s_\nu \circ s_{\mu/\mu_\star}, e_{\alpha^-} h_{\alpha^+} \rangle = |\mathrm{PSSYT}(\nu, \mu/\mu_\star)_{(\alpha^-,\alpha^+)}|$;
- Lemma 7.3, the Signed Weight Lemma;
- Corollary 9.20, that intervals for the ℓ^--twisted dominance order define stable partition systems.

Proof of Theorem 1.1 (§10–§12). In §10 we prove Proposition 10.7 giving an upper bound in the ℓ^--twisted dominance order on the constituents of an arbitrary plethysm $s_\nu \circ s_{\mu/\mu_\star}$. This is the final technical preliminary needed to apply Corollary 9.20, and hence the Signed Weight Lemma (Lemma 7.3), to prove Theorem 1.1 in §11. We give the important special case of this theorem when all tableaux have only positive entries in §12.

Proof of Theorem 1.2 (§13–§15). In §13 we prove the analogous upper bound in Corollary 13.24 on the constituents in the plethysms in Theorem 1.2, and in §14 we prove Theorem 1.2. In §15 we give many applications of this theorem, including its important special case when all tableaux have partition shape and only positive entries.

1.9. **Computer software.** MAGMA [2] code that can be used to verify all of our examples and compute with the ℓ^--twisted dominance order in Definition 6.6 may be downloaded as part of the arXiv submission of this paper. Example 4.26 is most easily checked using the second author's Haskell [20] code [24]. Computer algebra is not essential to any of our proofs or examples.

2. Overview of proof

The original Law–Okitani stability result [12, Proposition 5.3], later generalized in the main theorem of [13], is that, for any partition ν and any partition λ of $2|\nu|$, the sequence of plethysm coefficients

$$\langle s_{\nu \sqcup (1^M)} \circ s_{(2)}, s_{\lambda + (M) \sqcup (1^M)} \rangle \tag{2.1}$$

is ultimately constant. This is the special case of Theorem 1.2 for the strongly maximal signed weight $\big((1),(1)\big)$ of shape (2), size 1 and sign -1. (This weight is strongly maximal by Example 4.18(i); see §4.6 for motivation for strongly maximal weights.) Here we use the special case $\nu = (3,1)$ and $\lambda = (6,2)$ of (2.1) that

$$\langle s_{(3,1,1^M)} \circ s_{(2)}, s_{(6+M,2,1^M)} \rangle$$

is ultimately constant to sketch the overall strategy of the proofs of the two main results in this paper, indicating why certain steps cannot, we believe, be simplified. In particular, in §2.8 we give the bijection on plethystic semistandard signed tableaux (see Definition 3.10) used to prove this stability result; it is generalized in Theorem 14.7.

2.1. **Elementary-homogeneous products.** The first key idea is *to approximate Schur functions as products of elementary and homogeneous symmetric functions.* For (2.1), we set $\alpha = \lambda - (1^{\ell(\lambda)})$ and decompose the partition $\lambda + (M) \sqcup (1^M)$ as $(1^{\ell(\lambda)+M}) + \big(\alpha + (M)\big)$. It then follows from Young's rule (see the start of §5) that $s_{\lambda+(M)\sqcup(1^M)}$ is a summand of $e_{(\ell(\lambda)+M)} h_{\alpha+(M)}$. In our specific example, $\lambda = (6,2)$, $\alpha = (5,1)$, and so, when $M = 0$, the product is

$$e_{(2)} h_{(5,1)} = s_{(6,2)} + s_{(7,1)} + 2s_{(6,1,1)} + s_{(5,2,1)} + s_{(5,1,1,1)}. \tag{2.2}$$

As expected, this has $s_{(6,2)}$ as a summand, but also, of course, some Schur functions labelled by extra partitions. For general $M \in \mathbb{N}$, (2.2) becomes

$$\begin{aligned} e_{(2+M)} h_{(5+M,1)} = s_{(6+M,2,1^M)} + s_{(7+M,1,1^M)} + 2s_{(6+M,1,1,1^M)} \\ + s_{(5+M,2,1,1^M)} + s_{(5+M,1,1,1,1^M)}. \end{aligned} \tag{2.3}$$

Note that the summands in (2.3) are in bijection with the summands in (2.2) and the coefficients are independent of M. This points to a potential inductive proof, provided all the partitions in (2.2) are 'inductively smaller' than $(6,2)$ in some sense. However, we must consider not just the partitions appearing in (2.2), but the new partitions that arise when we apply this 'approximation' strategy to them. For instance, $s_{(5,1,1,1)}$ appears in (2.2) and $(5,1,1,1) = (1,1,1,1) + (4)$, so we must also consider the product

$$e_{(4)} h_{(4)} = s_{(5,1,1,1)} + s_{(4,1,1,1,1)}, \tag{2.4}$$

where we see $s_{(4,1,1,1,1)}$ for the first time. It therefore appears we need an order in which $(6,2)$, $(5,1,1,1)$ and $(4,1,1,1,1)$ form a chain. This is a property of the 1-twisted dominance order, defined by taking $\ell = 1$ in Definition 6.6. The up-set of $(6,2)$ (as defined in §6.5), in this order is

$$\begin{aligned}(6,2)^{\trianglelefteq} =& \{(6,2),(5,2,1),(4,2,1,1),(3,2,1^3),(2,2,1^4)\} \\ & \cup \{(7,1),(6,1,1),(5,1,1,1),(4,1^4),(3,1^5),(2,1^6),(1^8)\}.\end{aligned} \tag{2.5}$$

Note that both $(5,1,1,1)$ and $(4,1,1,1,1)$ appear. See Figure 6.2 for the Hasse diagram of the order. Here we mention that each of the two subsets in the decomposition above is a chain, increasing when read left to right. (Thus 'inductively smaller' means 'bigger in the 1-twisted dominance order'.) By Lemma 6.12, for every $\sigma \in (6,2)^{\trianglelefteq}$, the summands of $e_{(\ell(\sigma))}h_{\sigma-(1^{\ell(\sigma)})}$ are in $\sigma^{\trianglelefteq}$ and so in $(6,2)^{\trianglelefteq}$; for example, this is clear for $\sigma = (6,2)$ and $\sigma = (5,1,1,1)$ from the products $e_{(2)}h_{(5,1)}$ and $e_{(4)}h_{(4)}$ given in (2.2) and (2.4) above.

Remark 2.1. Many other strategies for 'approximating' $s_{(6+M,2,1^M)}$ by a product of more tractable symmetric functions, for example any strategy using homogeneous symmetric functions alone, would fail at the point of (2.3) by giving an expansion with a growing number of Schur functions, or with non-constant coefficients.

2.2. **Rough inductive hypothesis.** We now suppose inductively — but see §2.5 below for a difficulty here — that $\langle s_{(3,1,1^M)} \circ s_{(2)}, s_{\sigma+(M)\sqcup(1^M)}\rangle$ is ultimately constant for each of the partitions $\sigma \in (6,2)^{\trianglelefteq}$ except perhaps for $(6,2)$. Since stability is known inductively for each summand of $e_{(2+M)}h_{(5+M,1)}$, except $s_{(6+M,2,1^M)}$, to show that the plethysm coefficients $\langle s_{(3,1,1^M)} \circ s_{(2)}, s_{(6+M,2,1^M)}\rangle$ are ultimately constant, it suffices to show that

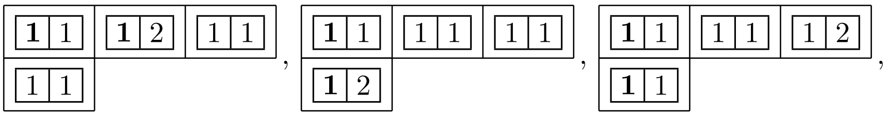
$$\langle s_{(3,1,1^M)} \circ s_{(2)}, e_{(2+M)}h_{(5+M,1)}\rangle \tag{2.6}$$

is ultimately constant.

2.3. **Plethystic semistandard signed tableaux.** To show that the plethysm coefficients $\langle s_{(3,1,1^M)} \circ s_{(2)}, e_{(2+M)}h_{(5+M,1)}\rangle$ in (2.6) are ultimately constant we need the second key idea: *there is an appealing combinatorial interpretation of $s_\nu \circ s_{\mu/\mu_\star}$ as the generating function enumerating the plethystic semistandard signed tableaux defined in Definition 3.10.* Moreover, by Proposition 5.6, the inner product of $s_\nu \circ s_{\mu/\mu_\star}$ with $e_{\pi^-}h_{\pi^+}$ is the number of plethystic semistandard signed tableaux of signed weight (π^-,π^+), in the sense of Definition 3.11. For instance,

$$\langle s_{(3,1)} \circ s_{(2)}, e_{(2)}h_{(5,1)}\rangle = \big|\mathrm{PSSYT}\big((3,1),(2)\big)_{((2),(5,1))}\big| \tag{2.7}$$

is the number of plethystic semistandard signed tableaux of shape $\big((3,1),(2)\big)$ and signed weight $\big((2),(5,1)\big)$. The three such plethystic semistandard signed tableaux are:



where **1** stands for the negative entry -1. More generally,

$$\langle s_{(3,1,1^M)} \circ s_{(2)}, e_{(2+M)}h_{(5+M,1)}\rangle = \bigl|\mathrm{PSSYT}\bigl(3,1,1^M),(2)\bigr)_{((2+M),(5+M,1)}\bigr|.$$

Thus $\langle s_{(3,1,1^M)} \circ s_{(2)}, e_{(2+M)}h_{(5+M,1)}\rangle$ is ultimately constant if and only if

$$\bigl|\mathrm{PSSYT}\bigl((3,1,1^M),(2)\bigr)_{((2+M),(5+M,1))}\bigr|$$

is ultimately constant. Hence proving the stability of the plethysm coefficient $\langle s_{(3,1,1^M)} \circ s_{(2)}, s_{(6+M,2,1^M)}\rangle$ reduces to the combinatorial problem of enumerating certain plethystic semistandard signed tableaux. We solve this problem in §2.8 below by exhibiting explicit bijections between the sets $\mathrm{PSSYT}\bigl((3,1,1^M),(2)\bigr)_{((2+M),(5+M))}$ for M sufficiently large. (The proof of Theorem 14.7 has the general argument.) In our specific example, $M = 0$ is already sufficiently large and the constant multiplicity is 3.

2.4. **Why the inductive step as described fails in general.** This is an honest sketch of the proof, except for one problem. We saw in §2.1 that we have to consider all the partitions in the up-set $(6+M,2,1^M)^{\unlhd}$, not just those in the support of $e_{(2+M)}h_{(5+M,1)}$. If all these partitions were of the form $\sigma + (M) \sqcup (1^M)$ for $\sigma \in (6,2)^{\unlhd}$, then nothing new would be needed, and the inductive step would go through. The difficulty is that this is not the case: for instance

$$(7,2,1)^{\unlhd} = \{\sigma + (1) \sqcup (1) : \sigma \in (6,2)^{\unlhd}\} \cup \{(2,2,1^6),(1^{10})\}$$

where the union is disjoint, and there is no way to deduce from the inductive assumptions for partitions in $(6,2)^{\unlhd}$ that $\langle s_{(3,1,1^{K+1})} \circ s_{(2)}, s_{(2+K,2,1^6,1^K)}\rangle$ is ultimately constant, as required in the inductive step.

2.5. **Cut up-sets.** We get around this obstacle to the inductive strategy as presented in §2.2 by the third key idea: *we do not need to consider every partition appearing in the up-set* $(6+M,2,1^M)^{\unlhd}$, *only those that appear in the plethysm* $s_{(3,1,1^M)} \circ s_{(2)}$. It follows from the Littlewood–Richardson rule that only partitions with at most $4+M$ parts appear in this plethysm, so rather than work with $(6+M,2,1^M)^{\unlhd}$, we can instead take the 'cut' up-set

$$\mathcal{P}^{(M)} = \bigl\{\sigma \in \mathrm{Par}(8+2M) : \sigma \trianglerighteq (6+M,2,1^M),\ \ell(\sigma) \le 4+M\bigr\}.$$

Thus $\mathcal{P}^{(0)} = \{(6,2),(5,2,1),(4,2,1,1),(7,1),(6,1,1),(5,1,1,1)\}$ and in general we have

$$\begin{aligned}\mathcal{P}^{(M)} = \bigl\{&(6+M,2,1^M),(5+M,2,1,1^M),(4+M,2,1,1,1^M),\\ &(7+M,1,1^M),(6+M,1,1,1^M),(5+M,1,1,1,1^M)\bigr\}\end{aligned}$$

for each $M \in \mathbb{N}_0$. When $M = 1$ the 'cut' removes the two partitions $(2,2,1^6)$ and (1^{10}) blocking the inductive argument, and in general, every partition in $\mathcal{P}^{(M)}$ is of the form $\sigma + (M) \sqcup (1^M)$ for $\sigma \in \mathcal{P}^{(0)}$. Note however that $\mathcal{P}^{(0)}$ contains $(4,2,1,1)$ and so $\mathcal{P}^{(0)}$ is not contained in the support (see

Definition 5.2) of $e_{(2)}h_{(5,1)}$. Thus we must still consider more partitions than are immediately required by (2.2).

2.6. **Signed Weight Lemma.** As we show by proving the Signed Weight (Lemma 7.3), after this refinement, the inductive step goes through. Because of our use of this critical lemma, our proofs are not explicitly inductive. Instead, each proof specifies the relevant way to apply the Signed Weight Lemma, and verifies its hypotheses: the most technical part of the argument is captured in the notion of a stable partition system, as defined in Definition 7.1.

2.7. **Twisted dominance order.** The definition of a stable partition system is deliberately quite general. This generality is needed for other applications of the Signed Weight Lemma (Lemma 7.3) beyond the scope of this paper, and, in any case, seems to us to be the clearest way to present the proof. In practice, the stable partition systems we use are certain families of intervals for the twisted dominance order on partitions (see Definition 6.6). For instance $\mathcal{P}^{(M)}$ above is the interval $[(6,2) \oplus (M,M), (5,1,1,1) \oplus (M,M)]_{\unlhd}$ for the 1-twisted dominance order. Definition 6.6 is a key definition in this paper; more broadly, the attractive interplay between the ℓ-decomposition $\langle \pi^-, \pi^+ \rangle$ defined in Definition 6.1, the partition π, and the symmetric function $e_{\pi^-}h_{\pi^+}$ is seen in many results and proofs below, notably Lemma 6.12 and Proposition 10.7.

2.8. **Bijections between plethystic tableaux.** In §2.3 we claimed that $|\,\mathrm{PSSYT}\big((3,1,1^M),(2)\big)_{((2+M),(5+M,1))}| = 3$ for all $M \in \mathbb{N}_0$. To illustrate that this stability result is non-obvious, we back up one step, and note that $|\,\mathrm{PSSYT}\big((3,1^K),(2)\big)_{((1+K),(4+K,1))}| = 2$ when $K = 0$: the relevant plethystic semistandard signed tableaux are

$$\boxed{\boxed{\mathbf{1}\,|\,1}\ \boxed{1\,|\,1}\ \boxed{1\,|\,2}}\,,\quad \boxed{\boxed{\mathbf{1}\,|\,2}\ \boxed{1\,|\,1}\ \boxed{1\,|\,1}}$$

(To show the tight connection between symmetric functions and plethystic semistandard signed tableaux, we note this may also be proved algebraically by using Young's rule to write $e_1h_{(4,1)} = s_{(6)} + 2s_{(5,1)} + s_{(4,2)} + s_{(4,1,1)}$ and taking an inner product with the known decomposition $s_3 \circ s_2 = s_{(6)} + s_{(4,2)} + s_{(2,2,2)}$. See Lemma 5.5 and Proposition 5.6 for why this works.) Observe that two of the plethystic semistandard tableaux for $K = 1$ are obtained by inserting $\boxed{\mathbf{1}\,|\,1}$ as a new entry in position $(1,1)$, moving the existing entry down to row 2. But the plethystic semistandard signed tableaux shown in the margin is not obtained in this way, because by the row semistandard condition in Definition 3.10, $\boxed{1\,|\,1}$ cannot appear left of $\boxed{\mathbf{1}\,|\,2}$. Generally this insertion map defines an injection between the sets for K and $K+1$ and, by the part of the proof of Theorem 14.7 dealing with condition (ii) in the Signed Weight Lemma (Lemma 7.3), it is surjective for $K \geq 1$, proving the claimed stability result. For a larger example see Example 13.26.

$$\begin{array}{|c|c|c|}\hline \boxed{\mathbf{1}\,|\,1} & \boxed{\mathbf{1}\,|\,2} & \boxed{1\,|\,1} \\ \hline \boxed{1\,|\,1} & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}$$

3. Partitions, tableaux and plethystic tableaux

In this section we give numbered definitions for the key terms novel to this paper. Other than these, we believe our notation is standard; we hope that the reader will be able to skim this section and then treat it as a reference. For essential preliminaries on symmetric functions see the start of §5.

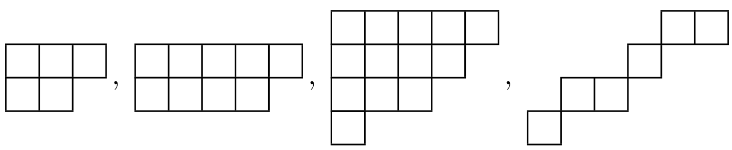
Weights, compositions and partitions. A *weight*, also sometimes called a *composition*, is an infinite sequence of non-negative integers with finite sum, called its *size*. The *length* of a *weight* α, denoted $\ell(\alpha)$, is the maximum ℓ such that $\alpha_\ell \neq 0$. (We set $\ell(\varnothing) = 0$.) Dually, we often write $a(\alpha)$ for α_1. A weight is a *partition* if α is non-increasing. The terms in a weight or partition are called *parts*. We always omit the infinite tail of zero parts when writing weights or partitions. Let $\mathcal{W}$ be the set of weights, let Par be the set of partitions, and let $\mathrm{Par}(n)$ be the set of partitions of n.

Young diagrams and skew partitions. We write $[\lambda]$ for the *Young diagram* of a partition λ, defined by

$$[\lambda] = \big\{(i,j) : 1 \leq i \leq \ell(\lambda), 1 \leq j \leq \lambda_i\big\}.$$

The elements of $[\lambda]$ are called *boxes*. A *skew partition* is a pair of partitions, denoted $\lambda/\lambda^\star$, such that $[\lambda^\star] \subseteq [\lambda]$. The *size* of a skew partition $\lambda/\lambda^\star$, denoted $|\lambda/\lambda^\star|$, is $|\lambda| - |\lambda^\star|$. We extend the definition of Young diagrams to skew partitions in the obvious way, by setting $[\lambda/\lambda^\star] = [\lambda]\backslash[\lambda^\star]$. We draw Young diagrams in the 'English' convention with box $(1,1)$ in the top-left of the page. The *conjugate partition* to λ, denoted λ', is the unique partition with Young diagram $\big\{(j,i) : (i,j) \in [\lambda]\big\}$. For example $(3,2)' = (2,2,1)$. The conjugate of a skew partition $\mu/\mu_\star$ is $\mu'/\mu'_\star$.

Operations on partitions. The sum and difference of partitions is defined componentwise by $(\alpha+\beta)_i = \alpha_i + \beta_i$, and $(\alpha-\beta)_i = \alpha_i - \beta_i$ when β is a subpartition of α. Let $\alpha \sqcup \beta$ be the partition whose multiset of non-zero parts is the disjoint union of the multisets of non-zero parts of α and β; equivalently $(\alpha \sqcup \beta)' = \alpha' + \beta'$. We say that $\alpha \sqcup \beta$ is the *join* of α and β. For instance,



are the Young diagrams of $(3,2)$, $(3,2)+(2,2)$, $(3,2)+(2,2)\sqcup(3,1)$ and $(6,4,3,1)\backslash(4,3,1)$, respectively. As already seen in the statements of the two main theorems, we define

$$\mu/\mu_\star \oplus (\gamma,\delta) = \big((\mu+\delta)\sqcup\gamma'\big)/\mu_\star \tag{3.1}$$

with the special case that for partitions that $\lambda \oplus (\gamma,\delta) = (\lambda+\delta)\sqcup\gamma'$. Note the conjugation of γ. (In examples we often omit the parentheses, writing instead $\lambda + \delta \sqcup \gamma'$.) We suggest '$\oplus$' be read as 'adjoin'. For example, $(3,2) \oplus \big((2,1,1),(2,2)\big) = (5,4,3,1)$, was seen above and, thanks to the conjugation of $(2,1,1)$, we have $(3,2) \oplus 2\big((2,1,1),(2,2)\big) = (7,6,3,3,1,1)$. Note this agrees with $(3,2) \oplus \big((2,1,1),(2,2)\big) \oplus \big((2,1,1),(2,2)\big)$; for instance, in either case we insert two new parts of size 3. There is one annoyingly technical point, seen by comparing $\varnothing \oplus \big((1),(2)\big) = \varnothing + (2) \sqcup (1) = (2) \sqcup (1) = (2,1)$ with $\varnothing \sqcup (1) + (2) = (1) + (2) = (3)$, which we address in the following definition.

Definition 3.1. Let $\mu/\mu_\star$ be a skew partition. Given ℓ^- and $\ell^+ \in \mathbb{N}_0$, we say that $\mu/\mu_\star$ is (ℓ^-,ℓ^+)-*large* if either $\ell^- = 0$ or $\ell^+ = 0$ or $\mu_{\ell^+} \geq \ell^-$.

Equivalently, $\mu/\mu_\star$ is (ℓ^-,ℓ^+)-*large* if (ℓ^+,ℓ^-) either has a zero coordinate or is a box of $[\mu]$: see Figure 3.1 for an example. It is deliberate that $\mu_\star$ does not enter in the body of this definition.

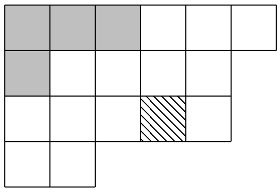


FIGURE 3.1. The skew partition $(6,5,5,2)/(3,1)$ above is $(4,3)$-large in the sense of Definition 3.1 because $(3,4) \in [(6,5,5,2))]$. It is also $(5,3)$-large, but not $(5,4)$-large.

Remark 3.2. Fix partitions κ^- and κ^+ and let $\ell^- = \ell(\kappa^-)$, $\ell^+ = \ell(\kappa^+)$. For any partition μ, the adjoining map $\mu \mapsto \mu \oplus (\kappa^-,\kappa^+)$ increases μ_{ℓ^-} by at least $\kappa^+_{\ell^-}$ and μ'_{ℓ^+} by at least $\kappa^-_{\ell^+}$. (The example $\varnothing \oplus \big((1),(2)\big) = (2,1)$ above shows that 'at least' cannot be replaced with 'exactly'.) If $\kappa^- = \varnothing$ then μ is already (ℓ^-,ℓ^+)-large; otherwise μ becomes (ℓ^-,ℓ^+)-large after at most $\lceil \ell^+/\kappa^-_{\ell^-} \rceil$ adjoinings. The dual result holds for κ^+, now with $\lceil \ell^-/\kappa^+_{\ell^+} \rceil$ adjoinings. Thus there exists K such that $\mu/\mu_\star \oplus K(\kappa^-,\kappa^+)$ is (ℓ^-,ℓ^+)-large. For later use, for instance in the context of Lemma 9.9, we remark that one further application of the adjoining map gives an (ℓ^-+1,ℓ^+)-large partition. By Lemma 9.6, when λ is (ℓ^-,ℓ^+)-large, adding κ^+ and joining $\kappa^{-\prime}$ to λ are commuting operations. Hence setting $\sigma/\sigma_\star = \mu/\mu_\star \oplus K(\kappa^-,\kappa^+)$, we have for all $M \geq K$,

$$\begin{aligned}\mu/\mu_\star \oplus M(\kappa^-,\kappa^+) &= \sigma/\sigma_\star \oplus (M-K)(\kappa^-,\kappa^+) \\ &= \sigma/\sigma_\star \oplus (\kappa^-,\kappa^+) \oplus \overset{M-K}{\cdots} \oplus (\kappa^-,\kappa^+)\end{aligned}$$

By this remark, there is no loss of generality in assuming in our main theorems that all the partitions involved are large, and so, in practice, there is no need to worry about whether to add κ^+ or join $\kappa^{-\prime}$ first in the map $\lambda \mapsto \lambda \oplus (\kappa^-,\kappa^+)$. (By (3.1), adding first is our definition.) In the important special case where $\kappa^- = \varnothing$, this technicality does not arise.

Dominance order. We partially order partitions of the same size by the *dominance order*, defined as usual by $\kappa \trianglelefteq \lambda$ if and only if $\kappa_1 + \cdots + \kappa_i \leq \lambda_1 + \cdots + \lambda_i$ for all i. We use the obvious extension of the dominance order to compositions and to partitions of different size: in the latter case, to indicate that the partitions may have different sizes, we write $\underline{\blacktriangleleft}$ rather than $\trianglelefteq$.

Signed tableaux and signed weights. We work throughout with tableaux having entries from $\mathbb{Z}\backslash\{0\}$.

Definition 3.3 (Signed tableau)**.** Let $\mu/\mu_\star$ be a skew partition. A *signed tableau* of *shape* $\mu/\mu_\star$ is a function $t : [\mu/\mu_\star] \rightarrow \mathbb{Z}\backslash\{0\}$. If $t(i,j) = x$ then we say that t has *entry* x in box (i,j).

We write $\mathrm{YT}(\mu/\mu_\star)$ for the set of signed tableaux of shape $\mu/\mu_\star$.

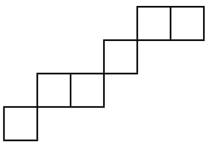
Definition 3.4 (Signed weight)**.** A signed weight is an element of $\mathcal{W} \times \mathcal{W}$.

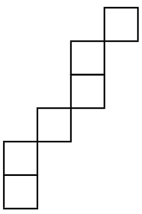
Definition 3.5 (Signed weight of a signed tableau)**.** The *signed weight* of a signed tableau t is the pair $(\alpha^-, \alpha^+) \in \mathcal{W} \times \mathcal{W}$ where, for each $i \in \mathbb{N}$, α^-_i is the number of entries of t equal to $-i$, and α^+_i is the number of entries of t equal to i.

If a tableau t has only positive entries then its signed weight is $(\varnothing, \alpha)$ for some weight α, and in this case we say, as usual, that α is the *weight* of t and write $\alpha = \mathrm{wt}(t)$.

Definition 3.6 (Sign of a signed tableau)**.** The *sign* of a signed tableau t, denoted $\mathrm{sgn}(t)$, is -1 if t has an odd number of negative entries and $+1$ if t has an even number of negative entries.

Equivalently, the sign of a signed tableau of weight (α^-, α^+) is $(-1)^{|\alpha^-|}$.





Semistandard signed tableaux. Recall that a *horizontal strip* is a skew partition whose Young diagram has at most one box in each column and a *vertical strip* is a skew partition whose Young diagram has at most one box in each row. For instance the skew partition $(6,4,3,1)/(4,3,1)$ seen earlier in this section is a horizontal strip but not a vertical strip, and its conjugate $(4,3,3,2,1,1)/(3,2,2,1)$ is a vertical strip but not a horizontal strip. (The diagrams are shown in the margin.)

Definition 3.7 (Semistandard signed tableau)**.** Let t be a signed tableau. We say t is *semistandard* if

(a) equal positive entries of T lie in horizontal strips
(b) equal negative entries of T lie in vertical strips,
(c) all entries are weakly increasing when rows are read left-to-right and columns are read top-to-bottom with respect to the total order on $\mathbb{Z}\backslash\{0\}$ defined by

$$-1 < -2 < \ldots < 1 < 2 \ldots.$$

Note that -1 is the least element in this order. We write $\mathrm{SSYT}^{\pm}(\mu/\mu_\star)$ for the set of all semistandard signed $\mu/\mu_\star$-tableaux and $\mathrm{SSYT}(\mu/\mu_\star)_{(\alpha^-,\alpha^+)}$

for the subset of those signed $\mu/\mu_\star$-tableaux of signed weight (α^-,α^+). We omit $\pm$ in the second case since it is clear from the signed weight that signed tableaux are required. As already seen, we adopt the convention that negative entries are shown in tableaux by bold numbers. For example, $\mathrm{SSYT}\big((5,4,3,1)\big)_{((2,2),(5,3,1))}$ has size two, containing the two semistandard signed tableaux

$$\begin{array}{|c|c|c|c|c|}\hline \mathbf{1}&\mathbf{2}&1&1&1\\\hline \mathbf{1}&\mathbf{2}&2&2\\\cline{1-4} 1&1&3\\\cline{1-3} 2\\\cline{1-1}\end{array}\;,\qquad \begin{array}{|c|c|c|c|c|}\hline \mathbf{1}&\mathbf{2}&1&1&1\\\hline \mathbf{1}&1&2&2\\\cline{1-4} \mathbf{2}&2&3\\\cline{1-3} 1\\\cline{1-1}\end{array}$$

and $\mathrm{SSYT}\big((5,4,3,1)\big)_{((3,1),(5,3,1))}$ has a unique semistandard signed tableau, obtained from the second semistandard signed tableau above by changing the entry of -2 in box $(3,1)$ to -1.

Definition 3.8 (Signed colexicographic order)**.** Let s and t be distinct semistandard signed tableaux of the same shape. We set $s<t$ if and only if *either*

(i) $\mathrm{sgn}(s)=-1$ and $\mathrm{sgn}(t)=1$ *or*

(ii) $\mathrm{sgn}(s)=\mathrm{sgn}(t)$ and considering the largest entry, m say, that appears in a different position in s and t, in the rightmost column in which the multiplicity of m differs between s and t, the multiplicity is less in s than in t.

The *sign-reversed colexicographic order* is defined identically except that if $\mathrm{sgn}(s)=-1$ and $\mathrm{sgn}(t)=1$ then now $s>t$.

We emphasise that here 'largest entry' is with respect to the order in Definition 3.7 in which $-1<-2<\ldots<1<2<\ldots$. For example, the signed colexicographic order restricted to semistandard signed tableaux of shape $(2,1)$ having entries from $\{1,2,3\}$ is

$$\begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array}<\begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array}<\begin{array}{|c|c|}\hline 1&1\\\hline 3\\\cline{1-1}\end{array}<\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}<\begin{array}{|c|c|}\hline 2&2\\\hline 3\\\cline{1-1}\end{array}<\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}<\begin{array}{|c|c|}\hline 1&3\\\hline 3\\\cline{1-1}\end{array}<\begin{array}{|c|c|}\hline 2&3\\\hline 3\\\cline{1-1}\end{array}$$

and the total order on $\mathrm{SSYT}^{\pm}\big((1^2)\big)$ is

$$\begin{array}{|c|}\hline \mathbf{1}\\\hline 1\\\hline\end{array}<\begin{array}{|c|}\hline \mathbf{2}\\\hline 1\\\hline\end{array}<\begin{array}{|c|}\hline \mathbf{3}\\\hline 1\\\hline\end{array}<\begin{array}{|c|}\hline \mathbf{4}\\\hline 1\\\hline\end{array}<\ldots<\begin{array}{|c|}\hline \mathbf{1}\\\hline 2\\\hline\end{array}<\begin{array}{|c|}\hline \mathbf{2}\\\hline 2\\\hline\end{array}<\begin{array}{|c|}\hline \mathbf{3}\\\hline 2\\\hline\end{array}<\begin{array}{|c|}\hline \mathbf{4}\\\hline 2\\\hline\end{array}<\ldots$$

$$\ldots<\begin{array}{|c|}\hline \mathbf{1}\\\hline \mathbf{1}\\\hline\end{array}<\begin{array}{|c|}\hline \mathbf{1}\\\hline \mathbf{2}\\\hline\end{array}<\begin{array}{|c|}\hline \mathbf{2}\\\hline \mathbf{2}\\\hline\end{array}<\begin{array}{|c|}\hline \mathbf{1}\\\hline \mathbf{3}\\\hline\end{array}<\begin{array}{|c|}\hline \mathbf{2}\\\hline \mathbf{3}\\\hline\end{array}<\begin{array}{|c|}\hline \mathbf{3}\\\hline \mathbf{3}\\\hline\end{array}<\begin{array}{|c|}\hline \mathbf{1}\\\hline \mathbf{4}\\\hline\end{array}<\ldots<\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}<\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}<\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}<\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}<\ldots.$$

Changing the order to the sign-reversed colexicographic order, the positive tableaux seen in the bottom row instead come first, and the order within each line is unchanged. In either order we have $\begin{array}{|c|}\hline \mathbf{1}\\\hline \mathbf{1}\\\hline\end{array}<\begin{array}{|c|}\hline \mathbf{1}\\\hline \mathbf{2}\\\hline\end{array}$; the greatest entry that has different multiplicity is -2 and it appears only in the tableau that is larger. More generally, the signed colexicographic order on (1^m)-tableaux with only positive entries agrees with the colexicographic order on m-subsets of $\mathbb{N}$, whence its name. It is notable that the signed colexicographic order could be replaced with any other total order on semistandard tableaux in which negative tableaux precede positive tableaux without changing any of our results: we explain this in Remark 5.7 below and use this freedom in the proof of Theorem 1.2 (see Definition 13.2).

Plethystic semistandard signed tableaux. We can now define our key combinatorial objects.

Definition 3.9 (Plethystic signed tableau)**.** A *plethystic signed tableau* T of *outer shape* ν and *inner shape* $\mu/\mu_\star$ is a function $T : [\nu] \to \mathrm{YT}(\mu/\mu_\star)$. If $T(i,j) = t$ then we say that T has *entry* t in box (i,j). We call the entries of T *inner tableaux*.

Let $\mathrm{PYT}(\nu, \mu/\mu_\star)$ denote the set of plethystic signed tableaux of outer shape ν and inner shape $\mu/\mu_\star$. For example three elements of $\mathrm{PYT}\big((3,2),(2)\big)$ are shown below

$$\begin{array}{|c|c|c|}\hline \boxed{1\,|\,\mathbf{1}} & \boxed{\mathbf{1}\,|\,2} & \boxed{2\,|\,2} \\ \hline \boxed{\mathbf{1}\,|\,\mathbf{2}} & \boxed{1\,|\,3} \\ \cline{1-2}\end{array}\;,\quad \begin{array}{|c|c|c|}\hline \boxed{\mathbf{1}\,|\,1} & \boxed{2\,|\,2} & \boxed{1\,|\,2} \\ \hline \boxed{\mathbf{1}\,|\,\mathbf{2}} & \boxed{1\,|\,3} \\ \cline{1-2}\end{array}\;,\quad \begin{array}{|c|c|c|}\hline \boxed{\mathbf{1}\,|\,1} & \boxed{2\,|\,2} & \boxed{2\,|\,2} \\ \hline \boxed{\mathbf{1}\,|\,\mathbf{2}} & \boxed{1\,|\,3} \\ \cline{1-2}\end{array}\;.$$

Note that each inner $\mu/\mu_\star$-tableau in a plethystic signed tableau has a sign defined by Definition 3.6. Moreover, if these inner $\mu/\mu_\star$-tableaux are semistandard in the sense of Definition 3.7, as in the second and third examples above, then they are totally ordered by the signed and sign-reversed colexicographic orders in Definition 3.8. We use this to lift Definition 3.7 almost verbatim to the plethystic setting.

Definition 3.10 (Plethystic semistandard signed tableau)**.** Let T be a plethystic signed tableau with semistandard inner tableau entries. We say that T is *semistandard* if

(a) equal positive entries of T lie in horizontal strips
(b) equal negative entries of T lie in vertical strips,
(c) all entries are weakly increasing when rows are read left-to-right and columns are read top-to-bottom with respect to the signed colexicographic order.

We say that T is *sign-reversed semistandard* if the same holds with respect to the sign-reversed colexicographic order.

Let $\mathrm{PSSYT}^{\pm}(\nu, \mu/\mu_\star)$ and $\mathrm{PSSYT}^{\mp}(\nu, \mu/\mu_\star)$ denote the sets of all plethystic semistandard signed tableaux and sign-reversed plethystic semistandard signed tableaux of outer shape ν and inner shape $\mu/\mu_\star$. Thus the first two tableaux displayed above are in $\mathrm{PYT}\big((3,2),(2)\big)$ but not in either of these subsets, because in the first $\boxed{1\,|\,\mathbf{1}}$ is not semistandard and in the second $\boxed{2\,|\,2} > \boxed{1\,|\,2}$ violates condition (c) above. The third tableau is in $\mathrm{PSSYT}^{\pm}\big((3,2),(2)\big)$ but not in $\mathrm{PSSYT}^{\mp}\big((3,2),(2)\big)$.

Definition 3.11 (Signed weight of a plethystic signed tableau)**.** The *signed weight* of a plethystic signed tableau T is the sum of the signed weights of its inner tableaux.

We denote by $\mathrm{PSSYT}(\nu, \mu/\mu_\star)_{(\alpha^-,\alpha^+)}$ and $\mathrm{PSSYT}^{\mp}(\nu, \mu/\mu_\star)_{(\alpha^-,\alpha^+)}$ the subsets of those plethystic semistandard signed tableaux of signed weight

(α^-, α^+). For instance the signed weights of the elements of $\mathrm{PYT}\big((3,2),(2)\big)$ shown above are $\big((3,1),(2,3,1)\big)$, $\big((2,1),(3,3,1)\big)$ and $\big((2,1),(2,4,1)\big)$.

The definition of the signed colexicographic order (Definition 3.8) applies to both these subsets, since the inner $\mu/\mu_\star$-tableau entries are totally ordered. The three elements of $\mathrm{PSSYT}\big((2,2),(3)\big)_{((3),(7,2))}$ are, ordered by the signed colexicographic order,

$$\begin{array}{|c|c|}\hline \boxed{\mathbf{1}\,|\,1\,|\,1} & \boxed{\mathbf{1}\,|\,1\,|\,2} \\ \hline \boxed{\mathbf{1}\,|\,1\,|\,2} & \boxed{1\,|\,1\,|\,1} \\ \hline \end{array}\,, \quad \begin{array}{|c|c|}\hline \boxed{\mathbf{1}\,|\,1\,|\,1} & \boxed{\mathbf{1}\,|\,2\,|\,2} \\ \hline \boxed{\mathbf{1}\,|\,1\,|\,1} & \boxed{1\,|\,1\,|\,1} \\ \hline \end{array}\,, \quad \begin{array}{|c|c|}\hline \boxed{\mathbf{1}\,|\,1\,|\,1} & \boxed{\mathbf{1}\,|\,1\,|\,2} \\ \hline \boxed{\mathbf{1}\,|\,1\,|\,1} & \boxed{1\,|\,1\,|\,2} \\ \hline \end{array}\,.$$

For instance, the third plethystic semistandard signed tableau is greater than the second because the greatest (3)-tableau entry of the third, namely $\boxed{1\,|\,1\,|\,2}$, is not in the second. To explain one feature that may at first seem surprising, note that since $\boxed{\mathbf{1}\,|\,1\,|\,1}$ has negative sign, it may appear multiple times in the same column of a plethystic semistandard tableau, but it cannot be repeated within the same row. See before Remark 5.7 for the analogous example using sign-reversed plethystic semistandard signed tableaux and also Example 11.10 for another example showing repeated inner tableaux.

4. Maximal and strongly maximal signed weights

The results and definitions in §4.1 are needed throughout; the remainder of this section has the definitions needed in Theorem 1.2. In §4.4, §4.5 and §4.6 we give motivating examples: these final three subsections are not logically essential.

4.1. Greatest signed weights.

We begin with a partial order on signed weights. Let $\mathcal{W}_{\ell^-}$ be the set of weights of length at most $\ell^- \in \mathbb{N}_0$.

Definition 4.1 (ℓ^--Signed dominance order)**.** Let $\ell^- \in \mathbb{N}_0$. The ℓ^--signed dominance order is the partial order on $\mathcal{W}_{\ell^-} \times \mathcal{W}$ defined by $(\alpha^-, \alpha^+) \trianglelefteq (\beta^-, \beta^+)$ if

$$(\alpha^-_1, \ldots, \alpha^-_{\ell^-}, \alpha^+_1, \alpha^+_2, \ldots) \trianglelefteq (\beta^-_1, \ldots, \beta^-_{\ell^-}, \beta^+_1, \beta^+_2, \ldots).$$

For example we have $\big((1,1,1),(2,1)\big) \trianglelefteq \big((2,1),(3)\big)$ in the 3-signed dominance order because $(1,1,1,2,1) \trianglelefteq (2,1,0,3,0)$, whereas $\big((3),(2,1)\big)$ and $\big((2,1),(3)\big)$ are incomparable in the 2-signed dominance order since $(3,0,2,1)$ and $(2,1,3,0)$ are incomparable in the dominance order. This example should make it clear that no ambiguity arises from using the same symbol $\trianglelefteq$ for both the dominance and ℓ^--signed dominance order. The value of ℓ^- will always be clear from context; in all our main theorems, ℓ^- is the length of the partition κ^-.

Definition 4.2. Let $\ell^- \in \mathbb{N}_0$. Given a skew partition $\tau/\tau_\star$, let t^- be the semistandard signed tableau with only negative entries defined by putting $\max(\ell^-, \tau_i - \tau_{\star i})$ entries from $-1, \ldots, -\ell^-$ into row i of $[\tau/\tau_\star]$. Supposing that t^- has shape $\sigma/\tau_\star$, let $t_{\ell^-}(\tau/\tau_\star)$ be the semistandard signed tableau of shape $\tau/\tau_\star$ obtained from t^- by putting $\tau'_j - \sigma'_j$ entries from $1, 2, \ldots$ into column j.

Definition 4.3 (Greatest signed weight)**.** Let $\ell^- \in \mathbb{N}_0$. Given a skew partition $\tau/\tau_\star$ we define the ℓ^--*greatest signed weight* of shape $\tau/\tau_\star$, denoted $\omega_{\ell^-}(\tau/\tau_\star)$, to be the signed weight of $t_{\ell^-}(\tau/\tau_\star)$.

We immediately justify calling $\omega_{\ell^-}(\tau/\tau_\star)$ 'greatest'. An example is given following this lemma.

Lemma 4.4. *Let $\tau/\tau_\star$ be a skew partition and let $\ell^- \in \mathbb{N}_0$. The tableau $t_{\ell^-}(\tau/\tau_\star)$ is the greatest signed tableau of shape $\tau/\tau_\star$ when signed weights are ordered by the ℓ^--signed dominance order. Moreover, given any $\tau/\tau_\star$-tableau t with negative entries from $\{-1, \ldots, -\ell^-\}$ we have, writing* $\mathrm{swt}(t)$ *for the signed weight of t,*

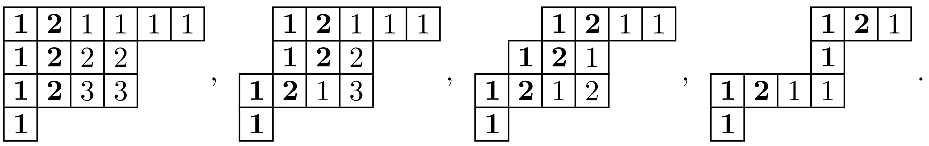
$$(\mathrm{swt}(t)^-, \mathrm{swt}(t)^+) \trianglelefteq \big(\omega_{\ell^-}(\tau/\tau_\star)^-, \omega_{\ell^-}(\tau/\tau_\star)^+\big)$$

where both $\omega_{\ell^-}(\tau/\tau_\star)^-$ and $\omega_{\ell^-}(\tau/\tau_\star)^+$ are partitions.

Proof. It is clear from the construction of $t_{\ell^-}(\tau/\tau_\star)$ that, amongst all semistandard signed $\tau/\tau_\star$-tableaux, $t_{\ell^-}(\tau/\tau_\star)$ greedily maximizes first the number of -1s, then the number of -2s, and so on, until all the negative entries in $\{-1, \ldots, -\ell^-\}$ are placed, and then the number of 1s, then the number of 2s, and so on, until all positive entries are placed. The displayed inequality is therefore obvious from the definition of the ℓ^--signed dominance order in Definition 4.1. By the construction of $t_{\ell^-}(\tau/\tau_\star)$, each entry $-k$ for $k \geq 2$ has an entry $-(k-1)$ to its left, and each entry k for $k \geq 2$ has an entry $k-1$ above it. Therefore $\omega_{\ell^-}(\tau/\tau_\star)^-$ and $\omega_{\ell^-}(\tau/\tau_\star)^+$ are partitions. □

See Lemma 6.4 for a strengthening of the final part of this lemma.

Example 4.5. The 2-greatest tableaux $t_2\big((6,4,4,1)/\tau_\star\big)$ for the four choices $\varnothing$, $(1,1)$, $(2,1)$, $(3,3)$ of $\tau_\star$ are shown below



Their greatest signed weights $\omega_2\big((6,4,4,1)/\tau_\star\big)$ are $\big((4,3),(4,2,2)\big)$, $\big((4,3),(4,1,1)\big)$, $\big((4,3),(4,1)\big)$ and $\big((4,2),(3)\big)$. We continue this example in Example 10.2.

In general it is quite fiddly to specify $\omega_{\ell^-}(\tau/\tau_\star)$ except by the algorithmic construction above. In the partition case however there is an simple formula, which the reader will easily guess from the previous example. We postpone it to (6.1) since it is an example of the ℓ^--decomposition of partitions in §6.

The final remark below is not logically essential, but will help orient the reader, while addressing one potential confusion.

Remark 4.6. Let $\ell^- \in \mathbb{N}_0$ and let $\tau/\tau_\star$ be a skew partition. Recall from Definition 3.8 that negative tableaux precede positive tableaux in the signed colexicographic order and *vice versa* in the sign-reversed colexicographic order. It follows, by a similar argument to Lemma 4.4, that $t_{\ell^-}(\tau/\tau_\star)$ is the least tableau in the signed colexicographic order if $|\omega_{\ell^-}(\tau/\tau_\star)^-|$ is odd, and in the sign-reversed colexicographic order if $|\omega_{\ell^-}(\tau/\tau_\star)^-|$ is even. More generally signed tableaux of *large* signed weight (in the ℓ^--signed dominance order) are *small* in the sign and sign-reversed colexicographic orders.

4.2. **Semistandard signed tableau families.** For Theorem 1.2 we must extend these ideas to families of semistandard signed tableaux.

Definition 4.7 (Semistandard signed tableau families)**.** Let $\tau/\tau_\star$ be a skew partition and let $R \in \mathbb{N}$.

(a) A *row-type semistandard signed tableau family* of shape $\tau/\tau_\star$ and size R is the multiset of entries in a plethystic semistandard signed tableau of outer shape (R) and inner shape $\tau/\tau_\star$.

(b) A *column-type semistandard signed tableau family* of shape $\tau/\tau_\star$ and size R is the multiset of entries in a plethystic semistandard signed tableau of outer shape (1^R) and inner shape $\tau/\tau_\star$.

The *signed weight* of a semistandard signed tableau family is the sum of the signed weights of its $\tau/\tau_\star$-tableau elements.

Signed weights are ordered by the ℓ-signed dominance order of Definition 4.1.

Definition 4.8 (Maximal signed weights)**.** A semistandard signed tableau family of signed weight (κ^-, κ^+) is *maximal* if its signed weight is maximal in the $\ell(\kappa^-)$-signed dominance order amongst all semistandard signed tableau families of its type, shape and size, *considering only* those families whose negative entries come from $\{-1, \ldots, -\ell(\kappa^-)\}$. A *maximal signed weight* is the signed weight of a maximal semistandard signed tableau family. A tableau family is *singleton* if it has a single element.

For example, the maximal singleton semistandard signed tableau families of shape $(2, 2)$ have as their unique elements the tableaux $t_{\ell^-}\big((2, 2)\big)$ for $\ell^- = 2$, 1 and 0, shown below:

$$\begin{array}{|c|c|}\hline \mathbf{1} & \mathbf{2} \\ \hline \mathbf{1} & \mathbf{2} \\ \hline \end{array}\,, \quad \begin{array}{|c|c|}\hline \mathbf{1} & 1 \\ \hline \mathbf{1} & 2 \\ \hline \end{array}\,, \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 2 \\ \hline \end{array}\,. \tag{4.1}$$

Their maximal signed weights are $\big((2, 2), \varnothing\big)$, $\big((2), (1, 1)\big)$ and $\big(\varnothing, (2, 2)\big)$, respectively. This shows that 'maximal' must be interpreted using the appropriate value of ℓ^-: for instance, while $\big((2), (1, 1)\big) \trianglerighteq \big(\varnothing, (2, 2)\big)$ in the 1-signed dominance order of Definition 4.1, the signed weight $\big(\varnothing, (2, 2)\big)$ is still maximal according to Definition 4.8, because it is compared only with other signed weights of the form $\big(\varnothing, \tau^+\big)$ using the 0-signed dominance order. Note also that the tableau shown in the margin of signed weight $\big((1), (2, 1)\big)$ is not maximal, because $\big((1), (2, 1)\big) \triangleleft \big((2), (1, 1)\big)$ in the 1-signed dominance order; this illustrates that Definition 4.8 requires a comparison with tableaux of *both possible signs*.

$$\begin{array}{|c|c|}\hline \mathbf{1} & 1 \\ \hline 1 & 2 \\ \hline \end{array}$$

More generally Lemma 4.17 classifies all singleton maximal semistandard signed tableau families. In these singleton examples, the row/column-type is irrelevant. We now give an example showing all features of Definition 4.8.

Example 4.9. The five maximal row-type semistandard signed tableau families of shape (2) and size 3 are

$$\Big\{\begin{array}{|c|c|}\hline \mathbf{1} & \mathbf{2} \\ \hline\end{array}, \begin{array}{|c|c|}\hline \mathbf{1} & \mathbf{2} \\ \hline\end{array}, \begin{array}{|c|c|}\hline \mathbf{1} & \mathbf{2} \\ \hline\end{array}\Big\}, \Big\{\begin{array}{|c|c|}\hline \mathbf{1} & 1 \\ \hline\end{array}, \begin{array}{|c|c|}\hline \mathbf{1} & 2 \\ \hline\end{array}, \begin{array}{|c|c|}\hline \mathbf{1} & 3 \\ \hline\end{array}\Big\}, \Big\{\begin{array}{|c|c|}\hline \mathbf{1} & 1 \\ \hline\end{array}, \begin{array}{|c|c|}\hline \mathbf{1} & 2 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 1 & 1 \\ \hline\end{array}\Big\},$$
$$\Big\{\begin{array}{|c|c|}\hline \mathbf{1} & 1 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 1 & 1 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 1 & 1 \\ \hline\end{array}\Big\}, \Big\{\begin{array}{|c|c|}\hline 1 & 1 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 1 & 1 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 1 & 1 \\ \hline\end{array}\Big\}$$

of signed weights $\big((3,3),\varnothing\big), \big((3),(1,1,1)\big), \big((2),(3,1)\big), \big((1),(5)\big)$ and $\big(\varnothing,(6)\big)$, respectively. Note that two of the families have tableaux of both signs and three have a repeated positive tableau. The seven maximal column-type semistandard signed tableau families of shape (2) and size 3 are

$$\Big\{\begin{array}{|c|c|}\hline \mathbf{1} & \mathbf{2} \\ \hline\end{array}, \begin{array}{|c|c|}\hline \mathbf{1} & \mathbf{3} \\ \hline\end{array}, \begin{array}{|c|c|}\hline \mathbf{1} & \mathbf{4} \\ \hline\end{array}\Big\}, \Big\{\begin{array}{|c|c|}\hline \mathbf{1} & \mathbf{2} \\ \hline\end{array}, \begin{array}{|c|c|}\hline \mathbf{1} & \mathbf{3} \\ \hline\end{array}, \begin{array}{|c|c|}\hline \mathbf{2} & \mathbf{3} \\ \hline\end{array}\Big\},$$
$$\Big\{\begin{array}{|c|c|}\hline \mathbf{1} & 1 \\ \hline\end{array}, \begin{array}{|c|c|}\hline \mathbf{1} & \mathbf{2} \\ \hline\end{array}, \begin{array}{|c|c|}\hline \mathbf{1} & \mathbf{3} \\ \hline\end{array}\Big\}, \Big\{\begin{array}{|c|c|}\hline \mathbf{1} & 1 \\ \hline\end{array}, \begin{array}{|c|c|}\hline \mathbf{1} & 1 \\ \hline\end{array}, \begin{array}{|c|c|}\hline \mathbf{1} & \mathbf{2} \\ \hline\end{array}\Big\}, \Big\{\begin{array}{|c|c|}\hline \mathbf{1} & 1 \\ \hline\end{array}, \begin{array}{|c|c|}\hline \mathbf{1} & 1 \\ \hline\end{array}, \begin{array}{|c|c|}\hline \mathbf{1} & 1 \\ \hline\end{array}\Big\},$$
$$\Big\{\begin{array}{|c|c|}\hline 1 & 1 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 2 & 2 \\ \hline\end{array}\Big\}, \Big\{\begin{array}{|c|c|}\hline 1 & 1 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 1 & 3 \\ \hline\end{array}\Big\}$$

of signed weights $\big((3,1,1,1),\varnothing\big)$, $\big((2,2,2),\varnothing\big)$, $\big((3,1,1),(1)\big)$, $\big((3,1),(2)\big)$, $\big((3),(3)\big)$, $\big(\varnothing,(3,3)\big)$ and $\big(\varnothing,(4,1,1)\big)$, respectively. Again note that two families have a repeated negative tableau. We continue this example in Example 4.12.

We use Definition 4.8 at a critical point in the proof of Lemma 13.14; it is also needed in Definition 4.10 shortly below.

4.3. **Strongly maximal signed weights.** We define the maximal semistandard signed tableau families in the statement of Theorem 1.2 as follows. Say that a $\tau/\tau_\star$-tableau is ℓ^--*negative greatest* if it agrees with $t_{\ell^-}(\tau/\tau_\star)$ in its negative entries. Let $\max \mathcal{M}$ denote the maximum integer entry of all the tableaux in a semistandard signed tableau family $\mathcal{M}$.

Definition 4.10 (Strongly maximal)**.** Let $\tau/\tau_\star$ be a non-empty skew partition and let $R \in \mathbb{N}$. Let $c^+ \in \mathbb{N}_0$. Let (κ^-, κ^+) be a signed weight and set $\ell^- = \ell(\kappa^-)$. Let $\varepsilon \in \{-1, +1\}$ be the sign of $t_{\ell^-}(\tau/\tau_\star)$. A semistandard signed tableau family $\mathcal{M}$ of shape $\tau/\tau_\star$ and signed weight (κ^-, κ^+) is *strongly* c^+*-maximal* if

(a) each $t \in \mathcal{M}$ is $\ell(\kappa^-)$-negative greatest;
(b) if $\varepsilon = +1$ then $\mathcal{M}$ has column-type; if $\varepsilon = -1$ then $\mathcal{M}$ has row-type;
(c) if (ϕ^-, ϕ^+) is the signed weight of a maximal semistandard signed tableau family $\mathcal{T}$ of the same shape, size and type as $\mathcal{M}$, such that each member of $\mathcal{T}$ is ℓ^--negative greatest and $\max \mathcal{T} \leq \max \mathcal{M}$, then $\sum_{i=1}^{c^+} \phi_i^+ \leq \sum_{i=1}^{c^+} \kappa_i^+$ with equality if and only if $\mathcal{T} = \mathcal{M}$.

The *sign* of $\mathcal{M}$ is ε. A signed weight is *strongly* c^+*-maximal* if it is the signed weight of a strongly c^+-maximal semistandard signed tableau family; its *shape* and *sign* is the common shape and sign of the tableaux in the family and its *size* is the size of the family.

See §4.6 for motivation for this definition. See also Remark 13.7 for how tableau families are used to define a bijection on plethystic semistandard signed tableaux. Later in §13 we give a running example using the strongly 1-maximal signed weight $(\varnothing, (4,1,1))$ of the tableau family $\bigl\{\begin{array}{|c|c|}\hline 1&1\\\hline\end{array}\,, \begin{array}{|c|c|}\hline 1&2\\\hline\end{array}\,, \begin{array}{|c|c|}\hline 1&3\\\hline\end{array}\bigr\}$ found in Example 4.12(0) using Example 4.9; this running example illustrates the significance of condition (c).

As an immediate example of Definition 4.10, it is routine to check that the three singleton tableau families of shape $(2,2)$ in (4.1) are strongly 0-, 2- and 1-maximal respectively. The relevant values of $\ell(\kappa^-)$, specifying the least negative entry, are 2, 1 and 0 respectively. The final tableau family is also strongly 2-maximal.

Lemma 4.11. *If (κ^-, κ^+) is a strongly maximal signed weight of shape $\mu/\mu_\star$ then there is a unique semistandard signed tableau family $\mathcal{M}$ of shape $\mu/\mu_\star$ and the same size and type as (κ^-, κ^+). The $\mu/\mu_\star$-tableau entries of $\mathcal{M}$ are distinct and agree in their negative entries.*

Proof. By (a) in Definition 4.10, the tableaux in $\mathcal{M}$ are equal in their negative entries. If the sign is $+1$ then by (b), $\mathcal{M}$ has column-type, and since the inner tableaux of sign $+1$ in a plethystic semistandard tableau of shape (1^R) are distinct, the tableaux in $\mathcal{M}$ are distinct. The proof is similar if the sign is -1. The uniqueness of $\mathcal{M}$ is obvious from (c). □

Example 4.12. Using Lemma 4.11 we find all strongly maximal signed tableau families and signed weights of shape (2) and size 3, considering each possibility for ℓ^-, the length of the negative part of the signed weight, in turn.

(0) Take $\ell^- = 0$. Then all integer entries are positive and, by (b) the family has column-type. The two relevant maximal signed weights of shape (2) and size 3 seen in Example 4.9 are $\bigl(\varnothing, (3,3)\bigr)$ and $\bigl(\varnothing, (4,1,1)\bigr)$. Now $\bigl(\varnothing, (4,1,1)\bigr)$ is strongly 1-maximal, satisfying (c) because when compared to $\bigl(\varnothing, (3,3)\bigr)$, we have $(4,1,1)_1 > (3,3)_1$. Similarly $\bigl(\varnothing(3,3),\bigr)$ is strongly 1- and 2-maximal. Note that to verify (c), $\bigl(\varnothing, (3,3)\bigr)$ should not be compared to $\bigl(\varnothing, (4,1,1)\bigr)$ because $(4,1,1)$ has strictly greater length, corresponding to a larger maximum integer entry.

(1) Take $\ell^- = 1$. By Lemma 4.11, a strongly maximal semistandard signed tableau family of shape (2) and size 3 has the form

$$\bigl\{\begin{array}{|c|c|}\hline \mathbf{1}&x\\\hline\end{array}\,, \begin{array}{|c|c|}\hline \mathbf{1}&y\\\hline\end{array}\,, \begin{array}{|c|c|}\hline \mathbf{1}&z\\\hline\end{array}\bigr\}$$

where, since by (b) the family has row-type, $x < y < z$. Taking $x = 1$, $y = 2$ and $z = 3$ we obtain $\bigl\{\begin{array}{|c|c|}\hline \mathbf{1}&1\\\hline\end{array}\,, \begin{array}{|c|c|}\hline \mathbf{1}&2\\\hline\end{array}\,, \begin{array}{|c|c|}\hline \mathbf{1}&3\\\hline\end{array}\bigr\}$. None of the four other maximal row-type weights (ϕ^-, ϕ^+) of shape (2) and size 3 seen in Example 4.9 are of a family all of whose members are 1-greatest. Therefore (c) holds and so the tableau family above is strongly 3-maximal, of strongly maximal signed weight $\bigl((3), (1,1,1)\bigr)$. It is clear that this choice of x, y and z defines the unique strongly 3-maximal signed weight of shape (2) and size 3.

(2) There is no strongly maximal semistandard signed tableau family with $\ell^- = 2$ because by (a) each (2)-tableau element is $\boxed{\mathbf{1}\,|\,\mathbf{2}}$, but, as observed above, by (b) the three (2)-tableaux in the family are distinct. In particular, while we saw in Example 4.9 that $\{\boxed{\mathbf{1}\,|\,\mathbf{2}}, \boxed{\mathbf{1}\,|\,\mathbf{3}}, \boxed{\mathbf{2}\,|\,\mathbf{3}}\}$ is a maximal semistandard signed tableau family of shape (2) and size 3 in the 3-signed dominance order, it is *not* strongly maximal in either the 2-signed or 3-signed dominance orders. (If instead $R = 1$ then $\{\boxed{\mathbf{1}\,|\,\mathbf{2}}\}$ is strongly 0-maximal in the 2-signed dominance order.)

Note we do *not* assume in Definition 4.10 that $\mathcal{M}$ is maximal; instead, as we now show, this follows from the three hypotheses, as the reader may have guessed from the previous example.

Lemma 4.13. *Let $\mathcal{M}$ be a strongly c^+-maximal semistandard signed tableau family of signed weight (κ^-, κ^+). Let (ψ^-, ψ^+) be the signed weight of a maximal semistandard signed tableau family $\mathcal{S}$ of the same shape, size and type as $\mathcal{M}$ with negative entries from $\{-1, \ldots, -\ell(\kappa^-)\}$ and such that $\mathcal{S} \neq \mathcal{M}$. Then*

$$|\psi^-| \leq |\kappa^-| \tag{4.2}$$

with equality if and only if either $\max \mathcal{S} > \max \mathcal{M}$ or

$$|\psi^-| + \textstyle\sum_{i=1}^{c^+} \psi_i^+ < |\kappa^-| + \sum_{i=1}^{c^+} \kappa_i^+. \tag{4.3}$$

Moreover (κ^-, κ^+) is a maximal signed weight in the sense of Definition 4.8.

Proof. Set $\ell^- = \ell(\kappa^-)$. Let $\tau/\tau_\star$ be the shape of $\mathcal{S}$ and $\mathcal{M}$. Suppose there exists $t \in \mathcal{S}$ such that t is not ℓ^--negative greatest. By Lemma 4.4, if t has signed weight (α^-, α^+) then $|\alpha^-| < |\omega_{\ell^-}(\tau/\tau_\star)^-|$ and, by summing over all $t \in \mathcal{S}$, we see that (4.2) holds. Moreover $\psi^- \lhd \kappa^-$ and so $(\kappa^-, \kappa^+) \not\trianglelefteq (\psi^-, \psi^+)$, as required in the final claim.

In the remaining case every element of $\mathcal{S}$ is ℓ^--negative greatest. Hence $\psi^- = \kappa^-$. If $\max \mathcal{S} > \max \mathcal{M}$ then we need only verify the final claim. Since $\max \mathcal{S} = \ell(\psi^+)$ and $\max \mathcal{M} = \ell(\kappa^+)$, we have

$$|\psi^-| + \sum_{i=1}^{\ell(\kappa^+)} \psi_i^+ < |\psi^-| + |\psi^+| = |\kappa^-| + |\kappa^+| = |\kappa^-| + \sum_{i=1}^{\ell(\kappa^+)} \kappa_i^+.$$

Hence, by definition of the ℓ^--signed dominance order in Definition 4.1, we have $(\kappa^-, \kappa^+) \not\trianglelefteq (\psi^-, \psi^+)$, as required. We have now reduced further to the case where $\max \mathcal{S} \leq \max \mathcal{M}$. By (c) in Definition 4.10, noting that $|\psi^-| = |\kappa^-|$, we now have (4.3). It now follows from the definition of the ℓ^--signed dominance order in Definition 4.1, as in the previous paragraph, that $(\kappa^-, \kappa^+) \not\trianglelefteq (\psi^-, \psi^+)$. This completes the proof. □

Remark 4.14. We remark that the converse to this lemma also holds: if $\mathcal{M}$ is a maximal semistandard signed tableau family such that either (4.2) or (4.3) holds when $\mathcal{M}$ is compared with a maximal semistandard signed tableau family $\mathcal{S}$ then all tableaux in $\mathcal{M}$ have the same sign and are ℓ^--negative greatest; given this, *provided* $\mathcal{M}$ has the type specified by (b), we have (b) and (4.3) implies that (c) holds. This gives an equivalent definition of 'strongly maximal'; in this paper we prefer Definition 4.10 since, while it has a technical flavour, examples can easily be given straight from the definition, rather than via the argument of Lemma 4.13 and this remark.

If (κ^-, κ^+) is a strongly maximal signed weight then κ^- and κ^+ are partitions. This is implicitly assumed in the statement of Theorem 1.2 because we have only defined the adjoining operation $\oplus$ for partitions. To prove this fact we use the Bender–Knuth involution on semistandard tableaux, of general skew shape, but having only positive entries. (For a textbook presentation of the method see the proof of Theorem 7.10.2 in [21].) We remark that the proof of the following lemma generalizes (but with much more work) to show that *any* maximal signed weight is a pair of partitions.

Lemma 4.15. *If (κ^-, κ^+) is a strongly maximal signed weight then κ^- and κ^+ are partitions.*

Proof. Set $\ell^- = \ell(\kappa^-)$. Suppose that (κ^-, κ^+) has shape $\mu/\mu_\star$ and size R. Let $\mathcal{M}$ be the unique strongly maximal semistandard signed tableau family of signed weight (κ^-, κ^+). By Lemma 4.4, $\omega_{\ell^-}(\mu/\mu_\star)^-$ is a partition. By (a) in Definition 4.10 we have $\kappa^- = R\omega_{\ell^-}(\mu/\mu_\star)^-$, and so κ^- is a partition.

Fix $i < \ell(\kappa^+)$. Let $t^+_{(1)} \ldots, t^+_{(R)}$ be the subtableaux of skew shape defined by taking the positive entries of each tableau in $\mathcal{M}$. By (a) and (b) in Definition 4.10, $t^+_{(1)}, \ldots, t^+_{(R)}$ are distinct semistandard tableaux of the same shape. Applying the Bender–Knuth involution swapping i and $i+1$ to each $t^+_{(k)}$ gives distinct semistandard tableaux $u^+_{(1)}, \ldots, u^+_{(R)}$. Let $\mathcal{U}$ be the semistandard signed tableau family obtained by replacing the subtableau $t^+_{(i)}$ with $u^+_{(i)}$ in each inner tableau in $\mathcal{M}$. Observe that $\mathcal{U}$ has signed weight (κ^-, λ^+) where

$$\lambda^+_k = \begin{cases} \kappa^+_{i+1} & \text{if } k = i \\ \kappa^+_i & \text{if } k = i+1 \\ \kappa^+_k & \text{if } k \neq i, i+1. \end{cases}$$

By the 'moreover' part of Lemma 4.13, (κ^-, κ^+) is a maximal signed weight in the ℓ^--signed dominance order. Comparing it with (κ^-, λ^+), we see that *either* $\kappa^+_i = \kappa^+_{i+1}$ and so $\kappa^+_i = \lambda^+_i = \kappa^+_{i+1}$, *or* $\kappa^+ \not\trianglelefteq \lambda^+$ and so $\kappa^+_i > \kappa^+_{i+1}$. This completes the proof. □

We use this lemma later to prove Proposition 6.5 and then in the proof of Corollary 13.24 and in Lemma 14.2.

Remark 4.16. Definition 4.10 is deliberately asymmetric with respect to positive and negative entries. The effect of this is seen most obviously in Example 4.12(2) and in Definition 6.6 below. This asymmetry ultimately reflects our decision in Definition 4.1 to order the negative part of signed weights first. For this reason, while applying the ω-involution to Theorem 1.1

gives no new results, as we show in Example 8.3, this is not the case for Theorem 1.2.

4.4. **Further examples of strongly maximal signed weights.** This section is not logically essential: it is included to show that Definition 4.10 is not overly restrictive, and so there is a rich supply of strongly maximal signed weights to which Theorem 1.2 may be applied. (Many more illustrative examples are shown in Table 4.23 in §4.5.) We begin with singleton semistandard signed tableau families, generalizing the small example immediately after Definition 4.8.

Lemma 4.17. *Let $\tau/\tau_\star$ be a skew partition. The maximal singleton semistandard signed tableau families are precisely $\{t_{\ell^-}(\tau/\tau_\star)\}$ for $0 \leq \ell^- \leq \max\{\tau_i - \tau_{\star i} : 1 \leq i \leq \ell(\tau)\}$. If c^+ is the greatest positive entry of $t_{\ell^-}(\tau/\tau_\star)$ then $\{t_{\ell^-}(\tau/\tau_\star)\}$ is strongly c^+-maximal.*

Proof. It is obvious that (a) holds in Definition 4.10 and we stipulate that the singleton family has row-type or column-type according to the sign of $t_{\ell^-}(\tau/\tau_\star)$ so that (b) holds. Finally, by Lemma 4.4, if t is a $\tau/\tau_\star$-tableau of signed weight $\mathrm{swt}(t)$ with negative entries from $\{-1,\ldots,-\ell^-\}$ then

$$\bigl|\mathrm{swt}(t)^-\bigr| + \sum_{i=1}^{c^+} \mathrm{swt}(t)_i^+ \leq \bigl|\omega_{\ell^-}(\tau/\tau_\star)\bigr| + \sum_{i=1}^{c^+} \omega_{\ell^-}(\tau/\tau_\star)_i$$

so we have (c). □

In particular, when $\tau_\star = \varnothing$, by taking $\ell^- = 0$ we find that $(\varnothing, \tau)$ is strongly $\ell(\tau)$-maximal and by taking $\ell^- = a(\tau)$ that $(\tau', \varnothing)$ is strongly 0-maximal. This gives the strongly maximal signed weights mentioned in the introduction.

Example 4.18. Let $m \in \mathbb{N}$ and let $0 \leq d \leq m$. The greatest tableau $t_d\bigl((m)\bigr)$ is

1	**2**	$\cdots$	**d**	1	$\cdots$	1

.

It has signed weight $\omega_d\bigl((m)\bigr) = \bigl((1^d), (m-d)\bigr)$.

(i) By Lemma 4.17, $\bigl\{t_d\bigl((m)\bigr)\bigr\}$ is a strongly 1-maximal semistandard signed tableau family. (To satisfy (b) in Definition 4.10 we stipulate that it has row-type if d is odd and column-type if d is even.) This can also be seen directly from Definition 4.10: since $t_d\bigl((m)\bigr)$ has leftmost d boxes **1** **2** $\cdots$ **d** it is d-negative greatest, and clearly it has the greatest possible number of 1s of all such tableaux. Therefore $\bigl((1^d), (m-d)\bigr)$ is a strongly 1-maximal signed weight. Note this holds even when $d = 0$. By Theorem 1.2, if ν and λ are any partition, then $\langle s_{\nu^{(M)}} \circ s_{(m)}, s_{\lambda + M(m-d) \sqcup (d^M)}\rangle$ is ultimately constant where $\nu^{(M)} = \nu + (M)$ if d is even and $\nu^{(M)} = \nu \sqcup (1^M)$ if d is odd, proving (1.3) and (1.4) in §1.7; as discussed earlier, these results were first proved in [13]. See Proposition 15.6 for explicit bounds deduced from our Theorem 14.7, together with a sufficient condition for the constant value of the plethysm coefficient to be zero.

(ii) Suppose that $d < m$. For $h \in \mathbb{N}$, let $u^{(h)}$ be the (m)-tableau obtained from $t_d\big((m)\big)$ by changing the rightmost 1 to h. Thus $t = u^{(1)}$. Fix $R \in \mathbb{N}$. We claim that the tableau family

$$\{u^{(1)}, \ldots, u^{(R)}\}$$

of shape (m) and size R is strongly 1-maximal, of row-type if d is odd and column-type if d is even. Clearly it satisfies conditions (a) and (b) in Definition 4.10. For (c), we observe that the family has the maximum possible number of entries of 1 of all families of size R formed from d-negative greatest tableaux. The corresponding strongly 1-maximal signed weight of shape (d) and size R is

$$\big((R^d), ((m-d)R - (R-1), 1^{R-1})\big).$$

By Theorem 1.2, if ν and λ are any partitions then

$$\big\langle s_{\nu^{(M)}} \circ s_{(m)}, s_{\lambda \oplus M((R^d),((m-d)R-(R-1),1^{R-1}))}\big\rangle$$

is ultimately constant, where $\nu^{(M)} = \nu + (M^R)$ if d is even and $\nu^{(M)} = \nu \sqcup (R^M)$ if d is odd.

After Corollary 14.9 we give some notable special cases of (i) and (ii) above in which the plethysm coefficient is constant for all $M \geq 0$. See also Example 15.7 for some explicit stability bounds obtained using Proposition 15.6.

Example 4.19. The plethystic semistandard signed tableau

$$\begin{array}{|c|c|c|}\hline \begin{array}{|c|c|}\hline \mathbf{1} & 1 \\ \hline \mathbf{1} & 2 \\ \hline\end{array} & \begin{array}{|c|c|}\hline \mathbf{1} & 1 \\ \hline \mathbf{1} & 3 \\ \hline\end{array} & \begin{array}{|c|c|}\hline \mathbf{1} & 1 \\ \hline \mathbf{1} & 4 \\ \hline\end{array} \\ \hline\end{array}\,.$$

of shape $\big((3),(2,2)\big)$ has entries from the semistandard signed $(2,2)$-tableau family

$$\mathcal{M} = \left\{ \begin{array}{|c|c|}\hline \mathbf{1} & 1 \\ \hline \mathbf{1} & 2 \\ \hline\end{array}\,, \begin{array}{|c|c|}\hline \mathbf{1} & 1 \\ \hline \mathbf{1} & 3 \\ \hline\end{array}\,, \begin{array}{|c|c|}\hline \mathbf{1} & 1 \\ \hline \mathbf{1} & 4 \\ \hline\end{array} \right\}$$

of size 3, sign $+1$ and signed weight $\big((6),(3,1,1,1)\big)$. Suppose that $\mathcal{T}$ is a row-type semistandard signed tableau family of size 3, shape $(2,2)$ in which each member of $\mathcal{T}$ is 1-negative greatest. Then each $(2,2)$-tableau in T has two entries of -1 in its first column and since box $(2,2)$ cannot contain either -1 or 1, the inequality $\tau_1^+ \leq \kappa_1^+ = 3$ required by Definition 4.10(c) when $c^+ = 1$ holds. Moreover, we have equality if and only if every $(2,2)$-tableau has the form shown in the margin, and in this case it is easy to see that $\mathcal{T}$ is the family $\mathcal{M}$. Therefore $\mathcal{M}$ is strongly 1-maximal and $\big((6),(3,1,1,1)\big)$ is a strongly maximal signed weight of shape $(2,2)$, size 3 and sign $+1$. By Theorem 1.2, $\langle s_{\nu+(M,M,M)} \circ s_{(2,2)}, s_{\lambda \oplus M((6),(3,1,1,1))}\rangle$ is ultimately constant for any partitions ν and λ.

$$\begin{array}{|c|c|}\hline \mathbf{1} & 1 \\ \hline \mathbf{1} & \star \\ \hline\end{array}$$

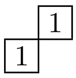
The special case $\mu_\star = \varnothing$ of the following lemma was used in §1.7 to show that (1.1), taken from [4, (9)], is a special case of Theorem 1.2.

Lemma 4.20. *Let $\mu/\mu_\star$ be a skew partition. Fix $\ell \in \mathbb{N}$ and let $\mathcal{M}$ be the set of all semistandard $\mu/\mu_\star$-tableaux having entries from $\{1,\ldots,\ell\}$. The signed weight of $\mathcal{M}$ is $\bigl(\varnothing,(q^\ell)\bigr)$ where $q = |\mathcal{M}|/\ell|\mu/\mu_\star|$; it is a strongly c^+-maximal signed weight of sign $+1$ for all $c^+ \in \{1,\ldots,\ell\}$.*

Proof. Clearly each $t \in \mathcal{M}$ is 0-negative greatest and $\mathcal{M}$ has column-type since its entries are distinct. Hence (a) and (b) in Definition 4.10 hold. Let $R = |\mathcal{M}|$. If $\mathcal{T}$ is another tableau family of shape $\mu/\mu_\star$ and size R then $\mathcal{T}$ contains a $\mu/\mu_\star$-tableau with maximum entry strictly greater than ℓ. Hence $\max\mathcal{M} < \max\mathcal{T}$, and condition (c) holds vacuously for any permitted c^+. Since the skew Schur function $s_{\mu/\mu_\star}$ is symmetric, each element of $\{1,\ldots,\ell\}$ appears equally often as an entry in a tableau $t \in \mathcal{M}$, and so the signed weight of $\mathcal{T}$ is $\bigl(\varnothing,(q^\ell)\bigr)$ for some $q \in \mathbb{N}$. Since each tableau has $|\mu/\mu_\star|$ entries, the common multiplicity q is $|\mathcal{M}|/\ell|\mu/\mu_\star|$, as claimed. □

The following example shows the usefulness of skew partitions in Theorem 1.2.

Example 4.21. Take $\mu/\mu_\star = (2,1)/(1)$. By Definition 4.10, or alternatively by Lemma 4.17, the signed weight $\bigl(\varnothing,(2)\bigr)$ of the tableau shown in the margin is strongly 1-maximal. Since it is defined by a single semistandard signed tableau of sign $+1$, the size is 1 and the sign is $+1$. It therefore follows from Theorem 1.2 that $\langle s_{\nu+(M)} \circ s_{(2,1)/(1)}, s_{\lambda+(2M)}\rangle$ is ultimately constant, for all partitions ν and λ such that $2|\nu| = |\lambda|$; using $s_{(2,1)/(1)} = s_{(2)} + s_{(1,1)} = s_{(1)}^2$ an equivalent formulation is that $\langle s_{\nu+(M)} \circ s_{(1)}^2, s_{\lambda+(2M)}\rangle$ is ultimately constant. Since the plethysm product is not distributive over addition in its second component, this result is already non-trivial to prove by other methods.



The final example in this subsection is included to give an idea of the rich behaviour of maximal signed weights of large size. It is is instructive but not logically essential, and so we omit far more details than usual.

Example 4.22. As seen earlier in Example 4.9, the column-type semistandard signed tableau families of shape (2) and size 3, namely

$$\bigl\{\boxed{1}\boxed{1},\ \boxed{1}\boxed{2},\ \boxed{1}\boxed{3}\bigr\},\ \bigl\{\boxed{1}\boxed{1},\ \boxed{1}\boxed{2},\ \boxed{2}\boxed{2}\bigr\},$$

are maximal, of signed weights $\bigl(\varnothing,(4,1,1)\bigr)$ and $\bigl(\varnothing,(3,3)\bigr)$ respectively. Correspondingly $s_{(1^3)} \circ s_{(2)}$ has maximal constituents $s_{(4,1,1)}$ and $s_{(3,3)}$, and in fact $s_{(1^3)} \circ s_{(2)} = s_{(4,1,1)} + s_{(3,3)}$. More generally, one can show (see for instance [15, page 138, Example 6] or [17, §8.5]) that $s_{(1^n)} \circ s_2 = \sum_\lambda s_{2[\lambda]}$ where the sum is over all partitions of n having distinct parts and $2[\lambda]$ is the partition whose main diagonal hook lengths are $2\lambda_1,\ldots,2\lambda_{\ell(\lambda)}$ such that $2[\lambda]_i = \lambda_i + i$ for $1 \le i \le \ell(\lambda)$. For each such partition $2[\lambda]$ there is a unique maximal column-type semistandard tableau family of shape (2) and size R and signed weight $(\varnothing, 2[\lambda])$. In particular, any two partitions $2[\lambda]$ are either equal or incomparable in the dominance order, and so every constituent of the plethysm $s_{(1^n)} \circ s_{(2)}$ is *both* maximal and minimal. Two examples have already been given and the tableau in the margin indicates how

$$\Big\{\boxed{1}\boxed{1}\,,\ \boxed{1}\boxed{2}\,,\ \boxed{1}\boxed{3}\Big\} \cup \Big\{\boxed{2}\boxed{2}\,,\ \boxed{2}\boxed{3}\Big\}$$

is constructed from $2[(3,2)]$ (shown below in the margin) by forming the three (2)-tableaux in the first set of total signed weight $\big(\varnothing,(4,1,1)\big)$ from the entries $\{1,1,1,1,2,3\}$ in the hook on the box $(1,1)$ and two tableaux in the second set of total signed weight $\big(\varnothing,(0,3,1)\big)$ from the entries $\{2,2,2,3\}$ in the hook on the box $(2,2)$ on the main diagonal boxes. Summing $(4,1,1)$ and $(0,3,1)$ we obtain a maximal semistandard tableau family of shape (2) and size 5 and signed weight $\big(\varnothing,(4,4,2)\big)$, corresponding to the partition $2[(3,2)]$.

1	1	1	1
2	2	2	2
3	3		

6			
	4		

We invite the reader to check that if α is a partition of n the maximal signed weight $(\varnothing, 2[\alpha])$ of shape (2), size n and column-type is strongly 1-maximal if and only if $\alpha \in \{(n),(n-1,1),(n-2,2),(3,2,1)\}$ and strongly 2-maximal if and only if $\alpha = (k+1,k-1)$ or $\alpha = (k+1,k)$ where $k = \lfloor n/2 \rfloor$, according to the parity of n. If α is the least distinct parts partition in the lexicographic order on partitions of n then $\alpha = (\ell,\ell-1,\ldots,b+2,b,\ldots,1)$ for some b and one can show that the corresponding maximal semistandard tableau family has strongly $(\ell-1)$-maximal signed weight $(\varnothing, 2[\alpha])$. (It may also be strongly c^+-maximal for other c^+: for instance if $b = \ell$ so that α is $(\ell,\ell-1,\ldots,1)$ then $2[\alpha] = (\ell+1, \overset{\ell}{\ldots}, \ell+1)$ and the maximal semistandard tableau family is the initial segment of the colexicographic order ending at $\boxed{\ell}\boxed{\ell}$ and Lemma 4.20 applies.) These remarks imply that if $n \le 7$ then all maximal signed weights of shape (2), size n and sign $+1$ are strongly maximal. When $n = 8$, we have $2[5,2,1] = (6,4,4,1,1)$ and $\big(\varnothing,(6,4,4,1,1)\big)$ is maximal, but comparison with the signed weights from $2[5,3] = (6,5,2,2,1)$ and $2[4,3,1] = (5,5,4,2)$ show that it is not strongly c^+-maximal for any value of c^+.

4.5. **Tables of strongly maximal signed weights.** Table 4.23 overleaf shows column-type strongly maximal signed weights of shape $\mu/\mu_\star$ and size R with $2 \le R \le 5$. (Singleton strongly maximal signed weights of size 1 are classified in Lemma 4.17.) The entries in the column c^+ show all the values for which the weight is strongly c^+-maximal. The 'unsigned' weights with $\ell^- = 0$ may be used in Corollary 15.9 as well as Theorem 1.2, or its sharp version Theorem 14.7.

Many further strongly maximal signed weights, including those of row-type, can be found using the Haskell software [24] mentioned in the introduction: see the module `MaximalTableauFamily.hs` for instructions.

4.6. **Why maximal weights are not sufficient.** In this subsection we show that, while Theorem 1.2 certainly requires maximal signed weights, this is not a sufficient hypothesis for this theorem, and so some stronger notion, such as the strongly maximal signed weights in Definition 4.10 is required. Again this section is not logically necessary, but we believe it is important to explain what we *cannot* hope to prove. In the following example we shall need to use Proposition 5.6.

$\mu/\mu_\star$	ℓ^-	R	(κ^-,κ^+)	c^+
(3)	0	2	$(\varnothing,(5,1))$	1
		3	$(\varnothing,(6,3))$	1, 2
			$(\varnothing,(7,1,1))$	1
		4	$(\varnothing,(6,6))$	1, 2
			$(\varnothing,(8,3,1))$	1
			$(\varnothing,(9,1,1,1))$	1
		5	$(\varnothing,(8,6,1))$	2
			$(\varnothing,(9,4,2))$	1
			$(\varnothing,(10,3,1,1))$	1
			$(\varnothing,(11,1,1,1,1))$	1
	2	R	$((2R),(1^R))$	$1,\ldots,R$
(2,1)	0	2	$(\varnothing,(3,3))$	1, 2
			$(\varnothing,(4,1,1))$	1
		3	$(\varnothing,(5,3,1))$	1
			$(\varnothing,(6,1,1,1))$	1
		4	$(\varnothing,(7,3,1,1))$	1
			$(\varnothing,(8,1,1,1,1))$	1
		5	$(\varnothing,(7,5,3))$	1
			$(\varnothing,(9,3,1,1,1))$	1
	1	R	$((2R),(1^R))$	$1,\ldots,R$

$\mu/\mu_\star$	ℓ^-	R	(κ^-,κ^+)	c^+
(4)	0	2	$(\varnothing,(7,1))$	1, 2
		3	$(\varnothing,(9,3))$	1, 2
			$(\varnothing,(10,1,1))$	1
		4	$(\varnothing,(10,6))$	1, 2
			$(\varnothing,(12,3,1))$	1
			$(\varnothing,(13,1,1,1))$	1
		5	$(\varnothing,(10,10))$	1, 2
			$(\varnothing,(14,4,2))$	1
			$(\varnothing,(15,3,1,1))$	1
			$(\varnothing,(16,1,1,1,1))$	1
	2	2	$((2,2),(3,1))$	1, 2
		3	$((3,3),(3,3))$	1, 2
			$((3,3),(4,1,1))$	1
		4	$((4,4),(4,3,1))$	1, 2, 3
			$((4,4),(5,1,1,1))$	1
		5	$((5,5),(4,4,2))$	1, 2, 3
			$((5,5),(5,3,1,1))$	1
			$((5,5),(6,1,1,1,1))$	1

$\mu/\mu_\star$	ℓ^-	R	(κ^-,κ^+)	c^+
(3,1)	0	2	$(\varnothing,(5,3))$	1, 2
			$(\varnothing,(6,1,1))$	1, 2
		3	$(\varnothing,(8,3,1))$	1
			$(\varnothing,(9,1,1,1))$	1
		4	$(\varnothing,(11,3,1,1))$	1, 2
			$(\varnothing,(12,1,1,1,1))$	1, 2, 3
		5	$(\varnothing,(12,5,3))$	1
			$(\varnothing,(14,3,1,1,1))$	1
	1	2	$((4),(3,1))$	1, 2
		3	$((6),(3,3))$	1, 2, 3
			$((6),(4,1,1))$	1
		4	$((8),(4,3,1))$	1, 2, 3
			$((8),(5,1,1,1))$	1
		5	$((10),(4,4,2))$	1, 2, 3, 4
			$((10),(5,3,1,1))$	1
			$((10),(6,1,1,1,1))$	1
(2,2)	0	2	$(\varnothing,(4,3,1))$	1, 2, 3
		3	$(\varnothing,(6,3,3))$	1
			$(\varnothing,(5,5,2))$	1
		4	$(\varnothing,(7,5,4))$	1, 2
			$(\varnothing,(8,4,3,1))$	1
			$(\varnothing,(7,6,2,1))$	2
			$(\varnothing,(7,5,3))$	3

$\mu/\mu_\star$	ℓ^-	R	(κ^-,κ^+)	c^+
(2,2)	0	5	$(\varnothing,(8,6,6))$	1, 2, 3
			$(\varnothing,(10,4,4,2))$	1
			$(\varnothing,(8,8,2,2))$	2
(2,2)	1	2	$((4),(2,1,1))$	1, 2, 3
		3	$((6),(2,2,2))$	1, 2, 3
			$((6),(3,1,1,1))$	1
		4	$((8),(3,2,2,1))$	1, 2, 3, 4
			$((8),(4,1,1,1,1))$	1
		5	$((10),(3,3,2,2))$	1, 2, 3, 4
			$((10),(4,2,2,1,1))$	1
(3,2)/(1)	0	2	$(\varnothing,(6,1,1))$	1
		3	$(\varnothing,(7,5))$	1, 2
			$(\varnothing,(9,1,1,1))$	1
		4	$(\varnothing,(8,8))$	1, 2
			$(\varnothing,(12,1,1,1,1))$	1
		5	none	
	1	2	none	
		3	$((6),(4,2))$	1, 2
		4	$((8),(4,4))$	1, 2
		5	$((10),(6,2,2))$	1

TABLE 4.23 Column-type strongly c^+-maximal signed weights for certain shapes $\mu/\mu_\star$ and size R with $2 \leq R \leq 5$.

Example 4.24. Taking $\mu/\mu_\star = (2,1)/(1)$ as in Example 4.21, suppose instead we take the non-maximal signed weight $\bigl((1),(1)\bigr)$, dominated in the 1-signed dominance order (see Definition 4.1) by $\bigl((2),\varnothing\bigr)$, and, to give the simplest possible example, $\nu = (1)$ and $\lambda = (2)$. Since this signed weight has sign -1 and $(2) \oplus (N-1)\bigl((1),(1)\bigr) = \bigl((2)+(N-1)\bigr) \sqcup (1^{N-1}) = (N+1, 1^{N-1})$, the prediction of Theorem 1.2 — wrongly applied with a weight that is not even maximal — is that $\langle s_{(1^N)} \circ s_{(2,1)/(1)}, s_{(N+1,1^{N-1})}\rangle$ is ultimately constant. To see this is false, let t_{+-}, t_{-+} and t_{++} be the three semistandard signed tableaux of shape $(2,1)/(1)$ shown below

$$\begin{array}{|c|c|}\cline{2-2} \multicolumn{1}{c|}{} & 1 \\ \hline \mathbf{1} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\,, \quad \begin{array}{|c|c|}\cline{2-2} \multicolumn{1}{c|}{} & \mathbf{1} \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\,, \quad \begin{array}{|c|c|}\cline{2-2} \multicolumn{1}{c|}{} & 1 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\,.$$

(Again $\mathbf{1}$ stands for -1.) For each $N \in \mathbb{N}_0$ and $L \in \{0,\ldots,N-1\}$ there is a unique plethystic semistandard signed tableau of outer shape (1^N) and inner shape $(2,1)/(1)$ which has L inner tableaux t_{+-}, $N-1-L$ inner tableaux t_{-+} and a final inner tableau t_{++}. (Note that only inner tableaux of negative sign are repeated.) These are all the plethystic semistandard signed tableaux of signed weight $\bigl((N+1),(N-1)\bigr)$ and so $\bigl|\mathrm{PSSYT}\bigl((1^N),(2,1)/(1)\bigr)_{(N+1),(N-1)}\bigr| = N$. By Proposition 5.6 (Plethystic Signed Kostka Numbers) it follows that

$$\langle s_{(1^N)} \circ s_{(2,1)/(1)}, e_{(N-1)}h_{(N+1)}\rangle = N.$$

Thus condition (ii) in the the Signed Weight Lemma (Lemma 7.3) does not hold when the lemma is applied (as is usual in this paper) with the twisted symmetric functions defined in Definition 6.11; this is the first point where the proof can be seen to fail. Moreover, by a very similar enumeration of plethystic tableaux one can show that $\langle s_{(1^N)} \circ s_{(2,1)/(1)}, e_{(N+d)}h_{(N-d}\rangle = 0$ for each $d > 1$ and $N \geq d$; it now follows from the identity

$$s_{(N+1,1^{N-1})} = e_{(N-1)}h_{(N+1)} - e_{(N-2)}h_{(N+2)} + \cdots + (-1)^{N-1}h_{(2N)}$$

that $\langle s_{(1^N)} \circ s_{(2,1)/(1)}, s_{(N+1,1^{N-1})}\rangle = N$ for all $N \in \mathbb{N}_0$, showing that in fact the plethysm coefficient is not stable.

The non-uniqueness seen in Example 4.24 is completely typical of the non-maximal case, and always leads to a similar obstruction to our proof strategy; indeed in most such cases, there is no stability result to be proved. But as the following example shows, mere maximality is not enough.

Example 4.25. The two tableau families of shape $(2,1)$, size 4 and signed weight $\bigl(\varnothing,(6,4,2)\bigr)$ are

$$\mathcal{S} = \left\{ \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\,, \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\,, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\,, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \right\}$$

$$\mathcal{T} = \left\{ \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\,, \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\,, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\,, \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \right\}.$$

Each of $\mathcal{S}$ and $\mathcal{T}$ is the set of entries of a unique plethystic semistandard signed tableau of outer shape (1^4) and inner shape $(2,1)$, and so $\bigl|\mathrm{PSSYT}\bigl((1^4),(2,1)\bigr)_{(\varnothing,(6,4,2))}\bigr| = 2$. Moreover, there is no tableau family of shape $(2,1)$ and size 4 with signed weight strictly dominating $\bigl(\varnothing,(6,4,2)\bigr)$. (Note that such a family has only positive entries.) But, by the uniqueness part of Lemma 4.11, the signed weight $\bigl(\varnothing,(6,4,2)\bigr)$ is not strongly maximal. Theorem 1.2 is therefore inapplicable. If, ignoring that one of the hypotheses fails to hold, we nonetheless take $\nu=\varnothing$, $\mu=(2,1)$ and $\lambda=\varnothing$, we wrongly conclude that $\langle s_{(M,M,M,M)}\circ s_{(2,1)}, s_{(6M,4M,2M)}\rangle$ is ultimately constant.

To see this is false, first note, analogously to Example 4.24, that given $0\le L\le M$, there is a unique plethystic semistandard tableau T_L of shape (M,M,M,M) whose first L columns have entries $\mathcal{S}$ and whose final $M-L$ columns have entries $\mathcal{T}$; the families occur in this order because, as seen after Definition 3.8, we have

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} < \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$$

in the signed colexicographic order. The maximal tableau families of shape $(2,1)$, size 4 and sign $+1$ have weights, in the usual sense for unsigned tableaux, as defined after Definition 3.4, $(8,1,1,1,1), (7,3,1,1)$ and $(6,4,2)$. Since $(6,4,2)$ has the least number of parts, it need not be compared with $(8,1,1,1,1)$ or $(7,3,1,1)$ in Definition 4.10(c), and so the *only* reason why $\bigl(\varnothing,(6,4,2)\bigr)$ fails to be a strongly maximal signed weight is that $\mathcal{S}$ and $\mathcal{T}$ have the same weight. Since the signed weight $\bigl(\varnothing,(6,4,2)\bigr)$ *is* maximal, *any* tableau family of shape $(2,1)$, size 4, sign $+1$ and entries from $\{1,2,3\}$ has weight dominated by $\bigl(\varnothing,(6,4,2)\bigr)$. Hence

$$\mathrm{PSSYT}\bigl((1^4),(2,1)\bigr)_{(6M,4M,2M)} = \{T_L : 0\le L\le M\} \tag{4.4}$$

and $\mathrm{PSSYT}\bigl((1^4),(2,1)\bigr)_\pi=\varnothing$ if $\pi\rhd(6M,4M,2M)$. By the basic result on Kostka numbers mentioned before Lemma 5.3 we have $s_{(6M,4M,2M)} = h_{(6M,4M,2M)}+f$ where f is a linear combination of complete homogeneous symmetric functions h_π with $\pi\rhd(6M,4M,2M)$. Hence

$$\begin{aligned}\langle s_{(M,M,M,M)}\circ s_{(2,1)}, s_{(6M,4M,2M)}\rangle &= \langle s_{(M,M,M,M)}\circ s_{(2,1)}, h_{(6M,4M,2M)}\rangle \\ &= \bigl|\mathrm{PSSYT}\bigl((M,M,M,M),(2,1)\bigr)_{(\varnothing,(6M,4M,2M))}\bigr| = M+1.\end{aligned}$$

(Alternatively this is a corollary of the Signed Weight Lemma (Lemma 7.3), applied with the singleton sets $\mathcal{P}^{(M)}=\bigl\{(6M,4M,2M)\bigr\}$.) In particular, the multiplicity is unbounded.

The previous paragraph also indicates where the proof of Theorem 1.2 breaks down. Applying Definition 13.10 with respect to the signed weight $\bigl(\varnothing,(6,4,2)\bigr)$ — which according to this definition is illegitimate as the signed weight is not strongly maximal — each non-exceptional column of a plethystic semistandard tableau $T\in\mathrm{PSSYT}\bigl((M,M,M,M),(2,1)\bigr)$ may be either $\mathcal{S}$ or $\mathcal{T}$. Thus there is no canonical tableau family that can be inserted as the entries in a new column of height 4 to define a bijection between the sets of plethystic semistandard signed tableaux for M and $M+1$ and the key result Lemma 13.14(i) fails in this attempt to adapt the proof of Theorem 1.2 in §14.5.

We remark that an alternative way to see that the conclusion of Theorem 1.2 is false in the previous example uses the highest-weight vector methods in [8]. Let ∇^γ denote the Schur functor for the partition γ, let E be a 3-dimensional vector space, let T_L be as defined in the previous example, and let $F(T_L)$ be as defined in Definition 2.3 of [8]. Generalizing Example 7.5 in [8], for each L, the vector

$$F(T_L) \in \nabla^{(M,M,M,M)}\big(\nabla^{(2,1)}(E)\big)$$

is highest weight of weight $(6M, 4M, 2M)$. The vectors $F(T_L)$ for $0 \le L \le M$ are linearly independent because the multisets of semistandard $(2,1)$-tableau entries of each T_L are distinct. It again follows that $\langle s_{(M,M,M,M)} \circ s_{(2,1)}, s_{(6M,4M,2M)}\rangle \ge M+1$ for each $M \in \mathbb{N}_0$.

Example 4.26. In the previous example the problem was that there were two semistandard signed tableau families of the same maximal weight. The other potential problem solved by Definition 4.10 is seen only in relatively large examples, such as the following. Take $\mu = (3)$. There are three maximal semistandard signed tableau families of shape (3) and size 17 having only positive entries, each obtained from

$$\boxed{1|1|1},\ \boxed{1|1|2},\ \boxed{1|2|2},\ \boxed{2|2|2},\ \boxed{1|1|3},\ \boxed{1|2|3},\ \boxed{2|2|3},\ \boxed{1|3|3},\ \boxed{2|3|3}$$

by taking the union with the eight tableaux shown in the table below.

Signed weight	Extend by
$\big(\varnothing, (17,16,11,4,3)\big)$	$\boxed{3\|3\|3}$, $\boxed{1\|1\|4}$, $\boxed{1\|2\|4}$, $\boxed{2\|2\|4}$, $\boxed{1\|3\|4}$, $\boxed{1\|1\|5}$, $\boxed{1\|2\|5}$, $\boxed{2\|2\|5}$
$\big(\varnothing, (18,15,10,5,3)\big)$	$\boxed{1\|1\|4}$, $\boxed{1\|2\|4}$, $\boxed{2\|2\|4}$, $\boxed{1\|3\|4}$, $\boxed{2\|3\|4}$, $\boxed{1\|1\|5}$, $\boxed{1\|2\|5}$, $\boxed{1\|3\|5}$
$\big(\varnothing, (19,14,9,6,3)\big)$	$\boxed{1\|1\|4}$, $\boxed{1\|2\|4}$, $\boxed{2\|2\|4}$, $\boxed{1\|3\|4}$, $\boxed{1\|4\|4}$, $\boxed{1\|1\|5}$, $\boxed{1\|2\|5}$, $\boxed{1\|3\|5}$

That these families *are* maximal can be checked by hand, or more quickly, using the Haskell software [24] mentioned in the introduction using `display $ maximalTableauFamilies ColType Closed 17 (ssyts 5 [3])`. Let T, U and V be the plethystic semistandard signed tableaux of outer shape (1^{17}) and inner shape 3 having as their entries the three families above. From the table above which lists the (3)-tableau entries in the signed colexicographic order from Definition 3.8, one can see that any plethystic tableau of the form

$$\boxed{T \mid \cdots \mid T \mid U \mid \cdots \mid U \mid V \mid \cdots \mid V} \tag{4.5}$$

is semistandard. Observe that

$$2(18,15,10,5,3) = (19,14,9,6,3) + (17,16,11,4,3).$$

Thus two of the $2N$ columns $\boxed{U|U}$ of length 17 in a plethystic semistandard signed tableau of outer shape $(2N, \overset{17}{\ldots}, 2N)$ and inner shape (3) may be replaced with two columns $\boxed{T|V}$ without changing the weight. (The columns must then be reordered as in (4.5) to respect the semistandard condition). Hence are at least 2^N plethystic semistandard signed tableaux of outer shape whose signed weight is $2(18N, 15N, 10N, 5N, 3N)$. Similar arguments to the previous Example 4.25 now show the plethysm coefficients $\langle s_{(M^{17})} \circ s_{(3)}, s_{(18M,15M,10M,5M,3M)}\rangle$ for even M do not stabilise, even though the relevant signed weight is maximal.

We remark that each tableau family in the previous example is a downset for the majorization partial order $\preceq$ defined by comparing tableaux entry by entry; this is a necessary, but not in general sufficient, condition for maximality. For instance the family of weight $(17, 16, 11, 4, 3)$ is $\boxed{2\,|\,2\,|\,5}^{\preceq} \cup \boxed{1\,|\,3\,|\,4}^{\preceq} \cup \boxed{3\,|\,3\,|\,3}^{\preceq}$ with three incomparable maximals in the dominance order. This leads to an efficient algorithm implemented in [24] for finding maximal, and so strongly maximal, tableau families.

5. Symmetric functions and plethystic semistandard signed tableaux

5.1. **Basic results.** We refer the reader to Stanley's textbook [21, Ch. 7] for an introduction to the Hopf algebra Λ of symmetric functions and to [14] for a careful development of plethysm and the formalism of plethystic substitutions. We define the elementary and homogeneous symmetric functions e_π and h_π for arbitrary weights $\pi \in \mathcal{W}$ (as defined at the start of §3) while Schur functions s_λ are labelled by partitions as usual. Beyond very basic results, the minimum we require is:

- Young's rule (horizontal strip addition) and Pieri's rule (vertical strip addition) as stated in (7.65) and after Example 7.15.8 in [21];
- the coproduct Δ on Λ satisfies $\Delta s_{\lambda/\lambda_\star} = \sum_\tau s_{\tau/\lambda_\star} \otimes s_{\lambda/\tau}$ and is compatible with the inner product on Λ (see the proof of Lemma 5.3);
- the formal definition of substitution by an alphabet with mixed signs, namely
$$f[-x_1, -x_2, \ldots, y_1, y_2, \ldots] = \sum_i f_i^-[-x_1, -x_2, \ldots] f_i^+[y_1, y_2, \ldots] \quad (5.1)$$
where $\Delta f = \sum_i f_i^- \otimes f_i^+$ (this follows from the equation for $s_{\lambda/\nu}[A - B]$ on page 177 of [14], using the result on the coproduct just mentioned, and that the Schur functions s_λ are a basis for Λ);
- the *negation rule*
$$s_\lambda[-x_1, -x_2, \ldots] = (-1)^{|\lambda|} s_{\lambda'}[x_1, x_2, \ldots] \quad (5.2)$$
which is a special case of [14, Theorem 6];
- the rule for a general plethystic substitution into a Schur function given in [14, Theorem 10].

We shall not state the final rule here, since it is lengthy and we only need it once, in the proof of Lemma 5.5; the reader may then either refer to [14], or take it on trust that it has the effect we claim.

Remark 5.1. Let $\nu/\nu_\star$ be a skew partition. By the adjointness relation in Corollary 7.15.4 of [21] we have $s_{\nu/\nu_\star} = \sum_\sigma \langle s_\nu, s_{\nu_\star} s_\sigma\rangle s_\sigma$ where the sum is over all partitions σ of $|\nu/\nu_\star|$. Since the plethysm product is linear in its first component, i.e. $(f+g)\circ h = f\circ h + g\circ h$ for all $f,g,h\in\Lambda$, it follows that for any skew partition $\mu/\mu_\star$,

$$s_{\nu/\nu_\star}\circ s_{\mu/\mu_\star} = \sum_\sigma \langle s_\nu, s_{\nu_\star}s_\sigma\rangle s_\sigma \circ s_{\mu/\mu_\star}$$

with the same condition on the sum. This reduces an arbitrary plethysm of skew Schur functions to the case dealt with in this paper. On the other hand, since the plethysm product is *not* linear in its second component (this is already clear from the negation rule) there is no further reduction to plethysm products where both factors are Schur functions.

The following definition is standard.

Definition 5.2. Given a symmetric function f expressed in the Schur basis as $\sum_\lambda c_\lambda s_\lambda$ we define the *support* of f by $\mathrm{supp}(f) = \{\lambda \in \mathrm{Par} : c_\lambda \neq 0\}$.

For example by (2.2), we have $\mathrm{supp}(e_{(2)}h_{(5,1)}) = \big\{(6,2),(7,1),(6,1,1),(5,2,1),(5,1,1,1)\big\}$.

5.2. **Enumerating semistandard signed tableaux.** Our first result is the twisted generalization of the basic result, see for instance [21, (7.30), (7.36)] that $\langle s_{\beta/\beta_\star}, h_\lambda\rangle$ is the number of semistandard tableaux of shape $\beta/\beta_\star$ and weight λ. When $\beta_\star = \varnothing$, this quantity may also be familiar as the Kostka number $K_{\beta\lambda}$. In our signed weight notation it is $|\,\mathrm{SSYT}(\beta/\beta_\star)_{(\varnothing,\lambda)}|$. (Semistandard signed Young tableaux are defined in Definition 3.7 and their signed weights in Definition 3.5.)

Lemma 5.3 (Twisted Kostka numbers for skew shapes)**.** *Let $\beta/\beta_\star$ be a skew partition and let (γ^-,γ^+) be a signed weight of size $|\beta/\beta_\star|$. Then*

$$\langle s_{\beta/\beta_\star}, e_{\gamma^-}h_{\gamma^+}\rangle = |\mathrm{SSYT}(\beta/\beta_\star)_{(\gamma^-,\gamma^+)}|.$$

In particular $\beta \in \mathrm{supp}(e_{\gamma^-}h_{\gamma^+})$ if and only if $\mathrm{SSYT}(\beta)_{(\gamma^-,\gamma^+)}$ is non-empty.

Proof. The inner product on $\Lambda\otimes\Lambda$ is defined in the natural way by linear extension of $\langle f\otimes f', g\otimes g'\rangle = \langle f,g\rangle\langle f',g'\rangle$. In the following two steps we use subscripts to indicate the relevant inner product. By the identity $\langle f, gh\rangle_\Lambda = \langle \Delta f, g\otimes h\rangle_{\Lambda\otimes\Lambda}$ we have

$$\langle s_{\beta/\beta_\star}, e_{\gamma^-}h_{\gamma^+}\rangle_\Lambda = \langle \Delta s_{\beta/\beta_\star}, e_{\gamma^-}\otimes h_{\gamma^+}\rangle_{\Lambda\otimes\Lambda} = \sum_\tau \langle s_{\tau/\beta_\star}\otimes s_{\beta/\tau}, e_{\gamma^-}\otimes h_{\gamma^+}\rangle_{\Lambda\otimes\Lambda}$$

where the sum is over all partitions τ of $|\beta_\star| + |\gamma^-|$. The right-hand side is $\sum_\tau \langle s_{\tau/\beta_\star}, e_{\gamma^-}\rangle_\Lambda \langle s_{\beta/\tau}, h_{\gamma^+}\rangle_\Lambda$. By the remark before the proof, the second factor is $|\mathrm{SSYT}(\beta/\tau)_{(\varnothing,\gamma^+)}|$. Applying the omega-involution (see [21, Theorem 7.14.5]) to the first factor gives

$$\langle s_{\tau/\beta_\star}, e_{\gamma^-}\rangle = \langle s_{\tau'/\beta'_\star}, h_{\gamma^-}\rangle.$$

The right-hand side is the number of semistandard tableaux of shape $\tau'/\beta'_\star$ with *positive* entries of weight γ^-, and so equal to $|\mathrm{SSYT}(\tau/\beta_\star)_{(\gamma^-,\varnothing)}|$ by the obvious bijection conjugating tableaux and switching signs of the integer entries. Since negative entries always precede positive entries in the order in Definition 3.7, the pairs of tableaux enumerated by the two factors are in bijection with $\mathrm{SSYT}(\beta/\beta_\star)_{(\gamma^-,\gamma^+)}$. The final claim is now immediately obvious on taking $\beta_\star = \varnothing$. □

Lemma 5.4. *Let f be a symmetric function and let (α^-,α^+) be a signed weight of size $\deg f$ where α^-, α^+ are partitions. Then $\langle f, e_{\alpha^-} h_{\alpha^+}\rangle$ is the coefficient of $(-x)^{\alpha^-} y^{\alpha^+}$ in $f[-x_1,-x_2,\ldots,y_1,y_2,\ldots]$.*

Proof. Let $\Delta(f) = \sum_i f_i^- \otimes f_i^+$. Since f is a symmetric function, so are each f_i^- and f_i^+. By [21, (7.30)], the complete homogeneous and monomial symmetric functions are dual bases of Λ. Hence the coefficient of y^{α^+} in $f_i^+[y_1,y_2,\ldots]$ is $\langle f_i^+, h_{\alpha^+}\rangle$. Similarly, now also using the negation rule (5.2), the coefficient of $(-x)^{\alpha^-}$ in $f_i^-[-x_1,-x_2,\ldots]$ is $\langle f_i^-, e_{\alpha^-}\rangle$. The lemma now follows by applying (5.1). □

5.3. **Enumerating plethystic semistandard signed tableaux.** We can now extend Lemma 5.3 (Twisted Kostka Numbers) to plethystic signed tableaux.

Lemma 5.5. *Let ν be a partition and let $\mu/\mu_\star$ be a skew partition. Then*

$$\begin{aligned}(s_\nu \circ s_{\mu/\mu_\star})&[-x_1,-x_2,\ldots,y_1,y_2,\ldots]\\ &= \sum_{(\alpha^-,\alpha^+)} |\mathrm{PSSYT}(\nu,\mu/\mu_\star)_{(\alpha^-,\alpha^+)}|\,(-x)^{\alpha^-}y^{\alpha^+}\end{aligned}$$

where the sum is over all signed weights (α^-,α^+) of size $|\nu||\mu/\mu_\star|$. Moreover, $\bigl|\mathrm{PSSYT}\bigl(\nu,\mu/\mu_\star)_{(\alpha^-,\alpha^+)}\bigr| = \bigl|\mathrm{PSSYT}^{\mp}\bigl(\nu,\mu/\mu_\star)_{(\alpha^-,\alpha^+)}\bigr|$.

Proof. By Lemma 5.3 (Twisted Kostka Numbers) and Lemma 5.4 we have

$$s_{\mu/\mu_\star}[-x_1,-x_2,\ldots,y_1,y_2,\ldots] = \sum_{(\alpha^-,\alpha^+)} |\mathrm{SSYT}(\mu/\mu_\star)_{(\alpha^-,\alpha^+)}|\,(-x)^{\alpha^-}y^{\alpha^+}$$

where the sum is over all signed weights (α^-,α^+) of size $|\mu/\mu_\star|$. Therefore $(s_\nu \circ s_{\mu/\mu_\star})[-x_1,-x_2,\ldots,y_1,y_2,\ldots] = s_\nu[\mathcal{A}]$ where the plethystic alphabet $\mathcal{A}$ is all semistandard tableaux of shape $\mu/\mu_\star$ having entries from $\mathbb{Z}\backslash\{0\}$ ordered by the signed colexicographic order in Definition 3.8. Note this alphabet has formal symbols (i.e. tableaux) of both positive and negative sign and that the sign of a semistandard signed $\mu/\mu_\star$-tableau of weight (α^-,α^+) from the displayed equation above, namely $(-1)^{|\alpha^-|}$, agrees with the sign defined by Definition 3.6. Moreover, negative tableaux are less than positive tableaux. Therefore, by the definition of general plethystic substitution [14, Theorem 8], taking D to be the negative tableaux in $\mathcal{A}$ and E to be the positive tableaux in $\mathcal{A}$, $S_\nu[\mathcal{A}]$ is the generating function enumerating, by their signed weight, the ν-tableau T having entries from $\mathcal{A}$, such that for some subpartition β of ν,

(i) the negative entries in T form a subtableau of shape β and are strictly increasing along rows and weakly increasing down columns;
(ii) the positive entries in T form a subtableau of skew shape ν/β and are weakly increasing along rows and strictly increasing down columns.

Since (i) implies that all negative entries of T are in boxes above or left of the positive entries of T, it follows that T is a plethystic semistandard signed tableau of outer shape ν and inner shape $\mu/\mu_\star$, as defined in Definition 3.10. Moreover, since the weight of a plethystic tableau is, by Definition 3.11, the sum of the weights of its $\mu/\mu_\star$-tableau entries, the sign attached to each plethystic tableau in $\mathrm{PSSYT}(\nu, \mu/\mu_\star)_{(\alpha^-,\alpha^+)}$ is $(-1)^{|\alpha^-|}$. This completes the proof of the displayed equation in the statement of the lemma. For the second claim, observe that we could instead order $\mathcal{A}$ by the sign-reversed colexicographic order, and take D to be the positive tableaux in $\mathcal{A}$ and E to be the negative tableaux in $\mathcal{A}$. We then obtain the displayed equation, modified by replacing $\big|\mathrm{PSSYT}\big(\nu, \mu/\mu_\star\big)_{(\alpha^-,\alpha^+)}\big|$ with $\big|\mathrm{PSSYT}^{\mp}\big(\nu, \mu/\mu_\star\big)_{(\alpha^-,\alpha^+)}\big|$. □

Proposition 5.6 (Plethystic Signed Kostka Numbers)**.** *Let ν be a partition and let $\mu/\mu_\star$ be a skew partition. Let (α^-, α^+) be a signed weight of size $|\nu||\mu/\mu_\star|$.*

$$\langle s_\nu \circ s_{\mu/\mu_\star}, e_{\alpha^-} h_{\alpha^+}\rangle = |\mathrm{PSSYT}(\nu, \mu/\mu_\star)_{(\alpha^-,\alpha^+)}|.$$

Proof. This is immediate from Lemma 5.5 and Lemma 5.4. □

The special case $\nu = (1)$ of the proposition just proved recovers Lemma 5.3. Note also that, by the final part of Lemma 5.5, Proposition 5.6 implies that $\langle s_\nu \circ s_{\mu/\mu_\star}, e_{\alpha^-} h_{\alpha^+}\rangle = \big|\mathrm{PSSYT}^{\mp}\big(\nu, \mu/\mu_\star\big)_{(\alpha^-,\alpha^+)}\big|$. For example, the three elements of $\mathrm{PSSYT}\big((2,2),(3)\big)_{((3),(7,2))}$ were seen after Definition 3.10; these may be compared with the three elements of $\mathrm{PSSYT}^{\mp}\big((2,2),(3)\big)_{((3),(7,2))}$ shown below in the sign-reversed colexicographic order

$$\begin{array}{|c|c|}\hline \boxed{1\,|\,1\,|\,2} & \boxed{\mathbf{1}\,|\,1\,|\,1} \\ \hline \boxed{\mathbf{1}\,|\,1\,|\,1} & \boxed{\mathbf{1}\,|\,1\,|\,2} \\ \hline \end{array}\,,\quad \begin{array}{|c|c|}\hline \boxed{1\,|\,1\,|\,1} & \boxed{\mathbf{1}\,|\,1\,|\,2} \\ \hline \boxed{\mathbf{1}\,|\,1\,|\,1} & \boxed{\mathbf{1}\,|\,1\,|\,2} \\ \hline \end{array}\,,\quad \begin{array}{|c|c|}\hline \boxed{1\,|\,1\,|\,1} & \boxed{\mathbf{1}\,|\,1\,|\,1} \\ \hline \boxed{\mathbf{1}\,|\,1\,|\,1} & \boxed{\mathbf{1}\,|\,2\,|\,2} \\ \hline \end{array}\,.$$

Remark 5.7. The only property of the signed colexicographic order we used in the proof of Lemma 5.5 was that negative tableaux are always less than positive tableaux. We are therefore free to use any other order $\preceq$ that has this property, obtaining plethystic semistandard signed tableaux defined as in Definition 3.10, but whose inner tableaux are instead semistandard with respect to $\preceq$. An analogous remark holds for the sign-reversed colexicographic order. We use this freedom in the proof of Theorem 1.2 (see Definition 13.2).

We end this section with an immediate application.

5.4. **A generalized Cayley–Sylvester formula.** By Proposition 5.6, using that $s_{(k-\ell,\ell)} = h_{(k-\ell,\ell)} - h_{(k-\ell+1,\ell-1)}$, for any ℓ with $1 \le \ell \le k/2$, we have

$$\begin{aligned}&\langle s_\nu \circ s_{\mu/\mu_\star}, s_{(mn-\ell,\ell)}\rangle \\ &\quad = |\mathrm{PSSYT}(\nu, \mu/\mu_\star)_{(\varnothing,(mn-\ell,\ell))}| - |\mathrm{PSSYT}(\nu, \mu/\mu_\star)_{(\varnothing,(mn-\ell+1,\ell-1))}|\end{aligned} \tag{5.3}$$

for $1 \le \ell \le mn/2$. Special cases of (5.3) have appeared throughout the literature on plethysms, especially in the context of representations of $\mathrm{SL}_2(\mathbb{C})$.

The most important case occurs when $\nu = (n)$ and $\mu = (m)$. Observe that an element of of $\mathrm{PSSYT}\big((n),(m)\big)_{(\varnothing,(mn-\ell,\ell))}$ is determined by the number of 2s in each of its n (m)-tableau entries. The corresponding non-negative sequence of length n may be interpreted as a partition of ℓ having at most n parts, each part having size at most m. For example when $n = 4$ and $m = 4$,

$$\boxed{\boxed{1\,|\,1\,|\,1\,|\,1}\;\boxed{1\,|\,1\,|\,2\,|\,2}\;\boxed{1\,|\,1\,|\,2\,|\,2}\;\boxed{1\,|\,2\,|\,2\,|\,2}} \longleftrightarrow (3,2,2).$$

This bijection shows that $\langle s_{(n)} \circ s_{(m)}, s_{(mn-\ell,\ell)}\rangle$ is the number of partitions of ℓ contained in an $m \times n$-box *minus* the number of partitions of $\ell - 1$ contained in an $m \times n$ box. This is one form of the Cayley–Sylvester identity. The bijective proof just given is similar to that in [11], where it is derived using symmetric group methods. When m and n are large, and so $\ell \leq \min(m,n)$, this simplifies to the number of partitions of ℓ having no parts of size 1: see [3, Proposition 8.4] for an earlier proof of this fact. For many further applications of (5.3), and related results such as Stanley's Hook Content Formula, see [19].

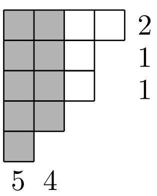
6. Twisted dominance order and twisted symmetric functions

In this section we define the ℓ^--twisted dominance order and twisted symmetric functions. Twisted symmetric functions interpolate between the homogeneous and elementary symmetric functions, in an analogous way (see Remark 6.8) to to the way the ℓ^--twisted dominance orders interpolate between the dominance order and its opposite order. This is made precise by Lemma 6.12. The 1-twisted dominance order was seen informally in the overview in §2.

6.1. ℓ^--decomposition.

The following definition and notation is shown diagrammatically in Figure 6.1.

Definition 6.1. Fix $\ell^- \in \mathbb{N}_0$. Given a partition σ, we set $\sigma^- = (\sigma'_1, \ldots, \sigma'_{\ell^-})$ and $\sigma^+ = \sigma - \sigma^{-\prime}$. We say that the ordered pair $\langle \sigma^-, \sigma^+\rangle$ is the ℓ^--*decomposition of* σ and write $\sigma \leftrightarrow \langle \sigma^-, \sigma^+\rangle$.



The relevant ℓ^- will always be clear from context. The ℓ^--decomposition of a partition may be used as a signed weight (see Definition 3.4), but since this is not always the case, we use angled brackets to make a visual distinction. For example, the 0-, 1-, 2-, 3- and 4-decompositions of $(4,3,3,2,1)$ are $\big\langle \varnothing, (4,3,3,2,1)\big\rangle$, $\big\langle (5),(3,2,2,1)\big\rangle$, $\big\langle (5,4),(2,1,1)\big\rangle$, $\big\langle (5,4,3),(1)\big\rangle$ and $\big\langle (5,4,3,1),\varnothing\big\rangle$; the 2-decomposition is shown in the margin.

Remark 6.2. Not all ordered pairs of partitions are ℓ^--decompositions: in fact $\big\langle \alpha^-, \alpha^+\big\rangle$ is an ℓ^--decomposition if and only if $\ell(\alpha^-) \leq \ell^-$ and $\alpha^-_{\ell^-} \geq \ell(\alpha^+)$. Note also that if this condition holds then the partition α such that $\alpha \leftrightarrow \big\langle \alpha^-, \alpha^+\big\rangle$ is $\big(\ell(\alpha^-)+1, \ell(\alpha^+)\big)$-large in the sense of Definition 3.1, and hence, as is sometimes all we need, $\big(\ell(\alpha^-), \ell(\alpha^+)\big)$-large.

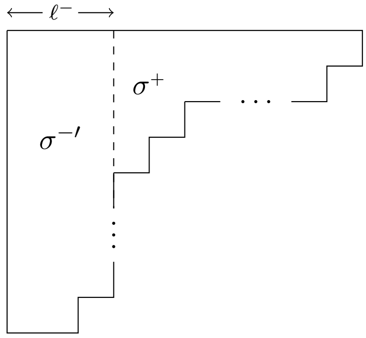


FIGURE 6.1. The partitions in the ℓ^--decomposition $\langle \sigma^-, \sigma^+ \rangle$ of $\sigma \in \mathrm{Par}$. Note that σ^- has at most ℓ^- parts and that $\sigma^-_{\ell^-} \geq \ell(\sigma^+)$, so σ is $\bigl(\ell(\sigma^-), \ell(\sigma^+)\bigr)$-large in the sense of Definition 3.1.

Example 6.3. Let μ be a partition. By Definition 4.2, row i of $t_{\ell^-}(\mu)$ has the first μ_i entries from the infinite sequence $-1, \ldots, -\ell^-, i, i, \ldots$. The greatest semistandard signed tableau $t_{\ell^-}(\mu)$ has signed weight

$$\bigl(\omega_{\ell^-}(\mu)^-, \omega_{\ell^-}(\mu)^+\bigr) = (\mu^-, \mu^+). \tag{6.1}$$

In particular, if $\ell^- = 0$ then we have $\omega_{\ell^-}(\mu) = \mu$; this is the well-known fact that the greatest weight of a semistandard μ-tableau is μ.

Note that in the previous remark, $\bigl\langle \mu^-, \mu^+ \bigr\rangle$ is the ℓ^--decomposition of the partition μ. More generally, we have the following lemma which is critical in §10; its generalization in Proposition 6.5 is important in §13.

Lemma 6.4. *Let $\tau/\tau_\star$ be a skew partition. Then the signed weight*

$$\bigl(\omega_{\ell^-}(\tau/\tau_\star)^-, \omega_{\ell^-}(\tau/\tau_\star)^+\bigr)$$

is the ℓ^--decomposition of a partition.

Proof. The greatest semistandard signed tableau $t_{\ell^-}(\tau/\tau_\star)$ of signed weight $\bigl(\omega_{\ell^-}(\tau/\tau_\star), \omega_{\ell^+}(\tau/\tau_\star)\bigr)$ is defined in Definition 4.2. Let d be the length of the partition $\omega_{\ell^-}(\tau/\tau_\star)^+$. If $d = 0$ then the result is immediate, so we may suppose that $d \in \mathbb{N}$. Thus d is the greatest positive entry of $t_{\ell^-}(\tau/\tau_\star)$. Choose the leftmost box of $t_{\ell^-}(\tau/\tau_\star)$ containing d, in position (i, j) say. By the construction of $t_{\ell^-}(\tau/\tau_\star)$ in Definition 4.2, $t_{\ell^-}(\tau/\tau_\star)$ has entries $1, \ldots, d$ in positions $(i - d + 1, j), \ldots, (i, j)$. In particular, each such row has a positive entry. By the construction of $t_{\ell^-}(\tau/\tau_\star)$ in which negative entries from $-1, \ldots, \ell^-$ are placed before positive entries, each of the rows $i - d + 1, \ldots, i$ of $t_{\ell^-}(\tau/\tau_\star)$ begins, after skipping any boxes not considered in $[\tau]$ because they are in $[\tau_\star]$, with $\boxed{\mathbf{1}}\boxed{\mathbf{2}}\boxed{\cdots}\boxed{\boldsymbol{\ell^-}}$. Hence $\omega_{\ell^-}(\tau/\tau_\star)^-_{\ell^-}$, which counts the number of entries of $-\ell^-$ in $t(\tau/\tau_\star)$, is at least d. Equivalently, $\omega_{\ell^-}(\tau/\tau_\star)^- \geq \ell\bigl(\omega_{\ell^-}(\tau/\tau_\star)^+\bigr)$. The lemma follows. $\square$

We have already seen in Lemma 4.15 that if (κ^-, κ^+) is a strongly maximal signed weight then κ^- and κ^+ are partitions. We now build on this to show that one potentially nasty technicality does not arise: in fact $\bigl\langle \kappa^-, \kappa^+ \bigr\rangle$

is an $\ell(\kappa^-)$-decomposition. This result generalizes Lemma 6.4 since, by Lemma 4.17, $\bigl(\omega_{\ell^-}(\mu/\mu_\star)^-, \omega_{\ell^-}(\mu/\mu_\star)^+\bigr)$ is the strongly maximal signed weight of the singleton strongly maximal tableau family $\{t_{\ell^-}(\mu/\mu_\star)\}$. We use this result in the proof of Lemma 13.23, part of the proof of Theorem 1.2.

Proposition 6.5. *Let (κ^-, κ^+) be a strongly maximal signed weight. Then $\langle \kappa^-, \kappa^+ \rangle$ is a well-defined $\ell(\kappa^-)$-decomposition of an $\bigl(\ell(\kappa^-)+1, \ell(\kappa^+)\bigr)$-large partition.*

Proof. Set $\ell^- = \ell(\kappa^-)$; thus (κ^-, κ^+) is a strongly c^+-maximal signed weight for some c^+. (The value of c^+ will not be relevant in this proof.) If $\ell^- = 0$ then $\kappa^- = \varnothing$ and the result holds trivially. Similarly if there are no positive entries then $\kappa^+ = \varnothing$ and the result is obvious. Therefore we may assume that $\kappa^+ \neq \varnothing$. As we noted before this proposition, by Lemma 4.15, κ^- and κ^+ are partitions.

Let $\mu/\mu_\star$ be the shape and let R be the size of (κ^-, κ^+). By (a) in Definition 4.10 we have $\kappa^- = R\omega_{\ell^-}(\mu/\mu_\star)^-$. Let $d = \ell\bigl(\omega_{\ell^-}(\mu/\mu_\star)^+\bigr)$ and let $e = \ell(\kappa^+)$; note that d is the greatest positive entry in $t_{\ell^-}(\mu/\mu_\star)$ and e is the greatest positive entry in the unique tableau family of shape $\mu/\mu_\star$, size R and signed weight (κ^-, κ^+). Denote this tableau family $\mathcal{T}$.

By maximality (Lemma 4.13), $t_{\ell^-}(\mu/\mu_\star)$ is an element of $\mathcal{T}$. Again by maximality, there is a tableau in $\mathcal{T}$ containing $d+1$ obtained by incrementing the entry in a single box of $t_{\ell^-}(\mu/\mu_\star)$. Repeating this argument, we see that there exist distinct tableaux $t_{(d)}, t_{(d+1)}, \ldots, t_{(e)} \in \mathcal{T}$ such that, for each $k \in \{d+1, \ldots, e\}$, the tableau $t_{(k)}$ has k as an entry. (For instance this can be seen in Example 4.18(ii), by considering the entries in the box $(1, m)$.) Therefore $R \geq e - d + 1$. By Lemma 6.4, every tableau in $\mathcal{T}$ agrees with $t_{\ell^-}(\mu/\mu_\star)$ in its negative entries. Hence the number of entries of tableaux in $\mathcal{T}$ equal to ℓ^- is $R\omega_{\ell^-}(\mu/\mu_\star)_{\ell^-}$. Using this for the first equality below, and recalling that by definition $d = \ell\bigl(\omega_{\ell^-}(\mu/\mu_\star)^+\bigr)$, we obtain

$$\kappa^-_{\ell^-} = R\omega_{\ell^-}(\mu/\mu_\star)_{\ell^-} \geq R\ell(\omega_{\ell^-}(\mu/\mu_\star)^+) = Rd \geq d + (R-1) \geq d + (e-d) = e.$$

Hence $\kappa^-_{\ell^-} \geq \ell(\kappa^+)$ and by Remark 6.2, $\langle \kappa^-, \kappa^+ \rangle$ is a well-defined $\ell(\kappa^-)$-decomposition of an $\bigl(\ell(\kappa^-)+1, \ell(\kappa^+)\bigr)$-large partition. □

6.2. **The ℓ^--twisted dominance order.** The sets used in applications of the critical Signed Weight Lemma (see Lemma 7.3 below) are subsets of intervals for a partial order on partitions defined using the ℓ^--decomposition in Definition 6.1 and the signed dominance order in Definition 4.1.

Definition 6.6 (ℓ^--twisted dominance order)**.** Fix $\ell^- \in \mathbb{N}_0$. The ℓ^--*twisted dominance order* is the partial order defined on partitions of the same size by $\pi \trianglelefteq \sigma$ if and only if $\langle \pi^-, \pi^+ \rangle \trianglelefteq \langle \sigma^-, \sigma^+ \rangle$, where $\trianglelefteq$ is the signed dominance order on the set $\mathcal{W}_{\ell^-} \times \mathcal{W}$.

The value of ℓ^- will always be clear from context. An example is given following Remark 6.8 below. In practice we shall often use the following lemma to work with the ℓ^--twisted dominance order. Recall that $\blacktriangleleft$ denotes the dominance order on partitions of arbitrary size.

Lemma 6.7 (Characterization of the ℓ^--dominance order)**.** *Let π and σ be partitions of the same size. We have $\pi \trianglelefteq \sigma$ in the ℓ^--signed dominance order if and only if both*

(a) $\pi^- \blacktriangleleft\!\!\!\!\!\_\; \sigma^-$ *and*
(b) $|\pi^+| \geq |\sigma^+|$ *and* $\pi^+ \trianglelefteq \sigma^+ + (|\pi^+| - |\sigma^+|)$.

Proof. From the equation

$$\sum_{i=1}^{\ell^-} \sigma_i^- - \sum_{i=1}^{\ell^-} \pi_i^- = |\sigma^-| - |\pi^-| = (|\sigma| - |\sigma^+|) - (|\pi| - |\pi^+|) = |\pi^+| - |\sigma^+|$$

we have $\sum_{i=1}^{\ell^-} \pi_i^- + \sum_{i=1}^{k} \pi_i^+ \leq \sum_{i=1}^{\ell^-} \sigma_i^- + \sum_{i=1}^{k} \sigma_i^+$ if and only if $\sum_{i=1}^{k} \pi_i^+ \leq (|\pi^+| - |\sigma^+|) + \sum_{i=1}^{k} \sigma_i^+$. The lemma now follows from the definition of the dominance order and the ℓ^--twisted dominance order. $\square$

Remark 6.8. It is obvious from Definition 6.6 that the 0-twisted dominance order is the ordinary dominance order. If $\ell^- \geq p$ then the ℓ^--twisted dominance order on partitions of size p is the reverse of the usual dominance order. Whenever $\ell^- \geq 1$, the greatest partition in the ℓ^--signed dominance order is (1^p).

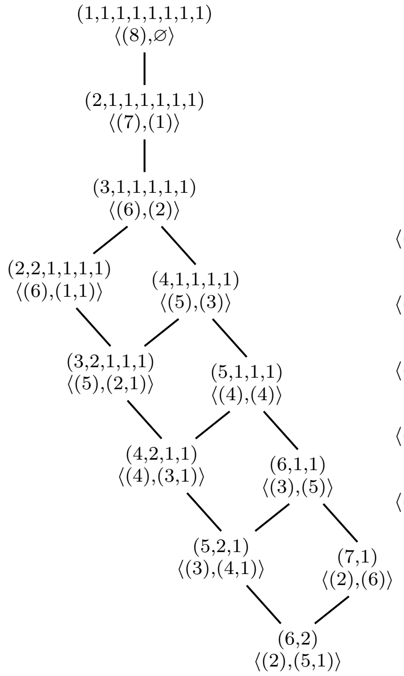
Example 6.9. In the 1-twisted dominance order on partitions of 8, the negative component σ^- of each partition σ has exactly one part. Let $\sigma^- = (b)$ where $1 \leq b \leq 8$. By Lemma 6.7, $\sigma \trianglerighteq (6,2)$ if and only if $(b) \blacktriangleright\!\!\!\!\!\_\; (2)$ and $\sigma^+ + (6 - |\sigma^+|) \trianglerighteq (5,1)$. If $b \leq 6$ then $\sigma^+ \in \{(8-b), (7-b,1)\}$; if $b = 7$ then $\sigma^+ = (1)$ and if $b = 8$ then $\sigma^+ = \varnothing$. The up-set of $(6,2)$ is therefore as claimed in (2.5) in the overview in §2. It is shown in the Hasse diagram in Figure 6.2. See §8.2 for a continuation to the 'cut' up-set used in §2.5.

For a further example of the twisted dominance order see Example 6.13. In practice, we find the following informal interpretation, using the standing notation shown in Figure 6.1 is helpful: *the partitions larger than π in the ℓ^--signed dominance order are exactly those obtained from π by a combination of box moves that are either: down and left within π^-, up and right within π^+, or from π^+ to π^-.* The final possibility is responsible for the equalization of sizes in condition (b) in Lemma 6.7.

In particular we have the analogue of the well-known property of the normal dominance order that $\alpha \trianglelefteq \beta$ implies $\ell(\alpha) \geq \ell(\beta)$.

Lemma 6.10. *Let α and β be partitions. If $\alpha \trianglelefteq \beta$ in the ℓ^--twisted dominance order then $\ell(\alpha^+) \geq \ell(\beta^+)$.*

Proof. By Lemma 6.7(b) we have $\alpha^+ \trianglelefteq \beta^+ + (|\alpha^+| - |\beta^+|)$. Hence by the property of the dominance order just mentioned, $\ell(\alpha^+) \geq \ell(\beta^+)$. $\square$



	(1^8)	$(2,1^6)$	$(3,1^5)$	$(2,2,1^4)$	$(4,1^4)$	(3,2,1,1,1)	(5,1,1,1)	(4,2,1,1)	(6,1,1)	(5,2,1)	(7,1)	(6,2)
⟨(8),∅⟩	1	·	·	·	·	·	·	·	·	·	·	·
⟨(7),(1)⟩	1	1	·	·	·	·	·	·	·	·	·	·
⟨(6),(2)⟩	0	1	1	·	·	·	·	·	·	·	·	·
⟨(6),(1,1)⟩	1	2	1	1	·	·	·	·	·	·	·	·
⟨(5),(3)⟩	0	0	1	0	1	·	·	·	·	·	·	·
⟨(5),(2,1)⟩	0	1	2	1	1	1	·	·	·	·	·	·
⟨(4),(4)⟩	0	0	0	0	1	0	1	·	·	·	·	·
⟨(4),(3,1)⟩	0	0	1	2	0	1	1	1	·	·	·	·
⟨(3),(5)⟩	0	0	0	0	0	0	1	0	1	·	·	·
⟨(3),(4,1)⟩	0	0	0	0	1	0	2	1	1	1	·	·
⟨(2),(6)⟩	0	0	0	0	0	0	0	0	1	0	1	·
⟨(2),(5,1)⟩	0	0	0	0	0	0	1	0	2	1	1	1

FIGURE 6.2. Hasse diagram of the up-set $(6,2)^{\trianglelefteq}$ in the 1-twisted dominance order on Par(8), as seen in (2.5) in the overview of the proof in §2. By Remark 6.8, this up-set is also the interval $[(6,2),(1^8)]_{\trianglelefteq}$. The total order $\leq$ refining $\trianglelefteq$ is indicated by vertical height. The matrix with entries $\mathrm{SSYT}(\sigma)_{(\pi^-,\pi^+)}$ for π, $\sigma \in (6,2)^{\trianglelefteq}$ relevant to condition (b) in the definition of a stable partition system (Definition 7.1) is shown to the right, with row and column labels ordered by the total order in Definition 6.14. It is lower unitriangular by Lemma 6.12. We use $\cdot$ to show a zero implied by this lemma.

6.3. **Twisted symmetric functions and twisted Kostka numbers.**

Definition 6.11 (ℓ^--twisted symmetric function)**.** Fix $\ell^- \in \mathbb{N}_0$. We define the *ℓ^--twisted symmetric function* g_π for a partition π by $g_\pi = e_{\pi^-} h_{\pi^+}$.

For example if $\ell^- = 0$ then $g_\pi = h_{\pi^+}$, or equivalently, $g_\pi = h_\pi$, and if $\ell^- \geq a(\pi)$ then $g_\pi = e_{\pi^-}$, or equivalently, $g_\pi = e_{\pi'}$. Thus as claimed at the start of this section, the ℓ^--twisted symmetric functions interpolate between the homogeneous and elementary symmetric functions.

The following lemma is vital when verifying condition (i) in the Signed Weight Lemma (Lemma 7.3). Example 6.13 following illustrates the iterative part of the proof. We require Young's rule and Pieri's rule: see references in §5.1. The support of a symmetric function is defined in Definition 5.2.

Lemma 6.12 (Twisted Kostka matrix)**.** *Let* $\pi \in \mathrm{Par}(n)$ *have* ℓ^-*-decomposition* $\langle \pi^-, \pi^+ \rangle$ *where* $\ell(\pi^-) = \ell^-$*. If* $\sigma \in \mathrm{supp}(e_{\pi^-} h_{\pi^+})$ *then* $\sigma \trianglerighteq \pi$*. Moreover we have* $\langle e_{\pi^-} h_{\pi^+}, s_\pi \rangle = 1$.

Proof. We describe the summands of $e_{\pi^-}h_{\pi^+}$ combinatorially. By Pieri's rule, if $s_\beta \in \mathrm{supp}(e_{\pi^-})$ then $\beta \trianglelefteq \pi^{-\prime}$, or equivalently, $\beta' \trianglerighteq \pi^-$. Since β' has at most ℓ^- parts, we have $\beta^- = \beta'$ and hence $\beta^- \trianglerighteq \pi^-$. Set $k = \ell(\pi^+)$. The product $s_\beta h_{\pi^+}$ may be computed by repeated applications of Young's rule: starting with $\gamma(0) = \beta$, let $s_{\gamma(i+1)}$ be a chosen Schur function summand of $s_{\gamma(i)}h_{\pi_i^+}$ for each i such that $0 \le i < k$. To find the possible $\gamma(i+1)$, we fix b_i^- and $b_i^+ \in \mathbb{N}_0$ with $b_i^- + b_i^+ = \pi_i^+$, then

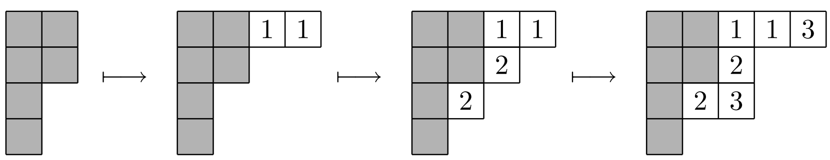
- add a horizontal strip of length b_i^- to $[\gamma(i)^{-\prime}]$ to obtain $[\gamma(i+1)^{-\prime}]$;
- add a horizontal strip of length b_i^+ to $[\gamma(i)]\backslash[\gamma(i+1)^{-\prime}]$.

Let $\sigma = \gamma(k)$ be the Schur function obtained after iteratively applying this procedure to all parts of π^+. Since $\beta^- \trianglerighteq \pi^-$, and subsequent steps add boxes to each $[\gamma(i)^{-\prime}]$, we have $\sigma^- \blacktriangleright \pi^-$ and condition (a) in Lemma 6.7 holds. The horizontal additions to each successive $[\gamma(i)^{-\prime}]$ in this sequence used in total $|\pi^+| - |\sigma^+|$ boxes. Moreover, at step i, the b_i^+ boxes added to $[\gamma(i)]\backslash[\gamma(i)^{-\prime}]$ lie in rows 1 up to i of $[\gamma(i)]\backslash[\gamma(i)^{-\prime}]$. It follows that σ^+ satisfies $\sigma^+ + (|\pi^+| - |\sigma^+|) \trianglerighteq \pi^+$. This gives (b) in Lemma 6.7. Hence, by this lemma, $\sigma \trianglerighteq \pi$. Finally, if $\sigma = \pi$ then $\beta = \pi^-$ and $\gamma(i) = (\pi_1^+, \ldots, \pi_i^+)$ for each i. Since the sequence $\gamma(0), \ldots, \gamma(k)$ is uniquely determined, we have $\langle e_{\pi^-}h_{\pi^+}, s_\pi\rangle = 1$, as required. $\square$

Example 6.13. Take $\ell^- = 2$ and let $\pi = (4,4,4)$ with 2-decomposition

$$\langle \pi^-, \pi^+\rangle = \big\langle (3,3), (2,2,2)\big\rangle.$$

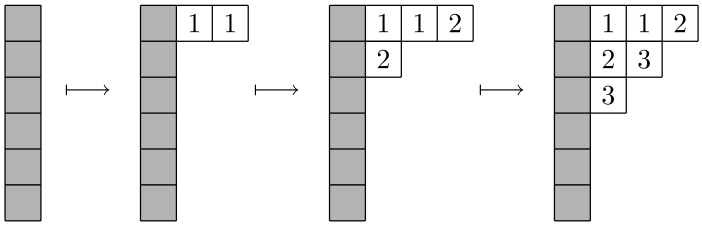
The Schur function summands of e_{π^-} are all s_β such that $\beta \trianglelefteq (3,3)'$. For this example we take $\beta = (2,2,1,1)$. The partition π^+ specifies three Young's rule additions of two boxes $\square\square$. The sequence of partitions $\gamma(0)$, $\gamma(1)$, $\gamma(2)$, $\gamma(3)$ in the proof is, for one particular choice of Young's rule additions, $(2,2,1,1)$, $(4,2,1,1)$, $(4,3,2,1)$, $(5,3,3,1)$. The final partition $\sigma = \gamma(3)$ has 2-decomposition $\big\langle (4,3), (3,1,1)\big\rangle$.



At step 2 we added one box to $[\gamma(1)^{-\prime}]$ and one box to $[\gamma(1)^+]$, taking $b_2^- = b_2^+ = 1$; in the other two steps $b_1^- = b_3^- = 0$. As expected, conditions (a) and (b) in Lemma 6.7 hold, with $(3,3) \blacktriangleleft (4,3)$ and $(2,2,2) \trianglelefteq (3,1,1) + (1)$. Moreover,

$$\big\langle (4,3), (3,1,1)\big\rangle \leftrightarrow (5,3,3,1) \trianglerighteq \pi \leftrightarrow \big\langle (3,3), (2,2,2)\big\rangle$$

as expected from the conclusion of Lemma 6.12. If instead we had chosen $\beta = (1^6)$ then a possible sequence ending with $\sigma = (4,3,2,1,1,1)$ is



in which $b_1^- = b_2^- = b_3^- = 3$ and correspondingly $|\pi^+| - |\sigma^+| = 3$. Again conditions (a) and (b) in Lemma 6.7 hold, now with $(3,3) \blacktriangleleft (6,3)$ and $(2,2,2) \trianglelefteq (2,1) + (3)$ and again the conclusion of Lemma 6.12 holds since $\big\langle (6,3),(2,1)\big\rangle \leftrightarrow (4,3,2,1,1,1) \trianglerighteq \pi$.

Figure 6.2 has an example of the matrix $\langle e_{\pi^-} h_{\pi^+}, s_\sigma\rangle$ in Lemma 6.12. It is an instructive exercise to show that the many zeros in this matrix correspond to pairs of partitions incomparable in the 1-twisted dominance order. For a further example of the conclusion of Lemma 6.12, calculation shows that, cut to partitions of length at most 3, $e_{(3,3)}h_{(3,3)}$ and $e_{(3,3)}h_{(4,1,1)}$ have supports

$$\{(5,5,2),(6,4,2),(7,3,2),(8,2,2)\},$$
$$\{(6,3,3),(6,4,2),(7,3,2),(8,2,2)\}$$

respectively, corresponding to the part of the up-sets seen in Figure 8.1 lying above $(5,5,2)$ and $(6,3,3)$, respectively.

6.4. **Twisted total order.** While not logically essential, it is useful to have a total order that makes the twisted Kostka matrix seen in Figure 6.2 lower-triangular. These matrices are used in (b) in the critical Signed Weight Lemma (Lemma 7.3).

Definition 6.14. Fix $\ell^- \in \mathbb{N}_0$. We define the *ℓ^--twisted total order* by $\pi \underline{\lessdot} \sigma$ if and only if $(\pi^-, \pi^+) \le (\sigma^-, \sigma^+)$ where $\le$ is the lexicographic order on compositions.

Equivalently, $\pi \underline{\lessdot} \sigma$ if and only if $\pi^- < \sigma^-$ or $\pi^- = \sigma^-$ and $\pi^+ \le \sigma^+$, where $<$ and $\le$ are the lexicographic order on partitions (now possibly of different sizes). It is easily seen that $\underline{\lessdot}$ is a total order refining the ℓ^--twisted dominance order. For example, in the total order on compositions we have

$$\big\langle (3,3),(3,2,1)\big\rangle \lessdot \big\langle (3,3),(3,3)\big\rangle \lessdot \big\langle (3,3),(4,1,1)\big\rangle \lessdot \big\langle (3,3),(4,2)\big\rangle$$

and hence in the 2-twisted total order we have $(5,4,3) \lessdot (5,5,2) \lessdot (6,3,3) \lessdot (6,4,2)$. (See Figure 8.1 for the relevant Hasse diagram.) Moreover

$$\big\langle (3,3),(4,2)\big\rangle \le \big\langle (4,2),\pi^+\big\rangle \le \big\langle (4,3),\sigma^+\big\rangle$$

and hence $(6,4,2) \lessdot (2,2,1,1) + \pi^+ \lessdot (2,2,2,1) + \sigma^+$ for any partitions π^+ of 6 and σ^+ of 5 with $\ell(\pi^+) \le 2$ and $\ell(\sigma^+) \le 3$.

6.5. **Up-sets and twisted intervals.** For a fixed $\ell^- \in \mathbb{N}_0$, and partitions γ, δ of the same size we define the *twisted interval* $[\gamma, \delta]_{\trianglelefteq}$ by

$$[\gamma, \delta]_{\trianglelefteq} = \{\sigma \in \mathrm{Par}(p) : \gamma \trianglelefteq \sigma \trianglelefteq \delta\}.$$

where $\trianglelefteq$ is the ℓ^--twisted dominance order. We define the *up-set* of a partition λ of size p by

$$\lambda^{\trianglelefteq} = \{\sigma \in \mathrm{Par}(p) : \sigma \trianglerighteq \lambda\}.$$

As ever, the value of ℓ^- will be clear from context. Equivalently, by Remark 6.8, $\lambda^{\trianglelefteq} = [\lambda, (p)]_{\trianglelefteq}$ when $\ell^- = 0$ and $\lambda^{\trianglelefteq} = [\lambda, (1^p)]_{\trianglelefteq}$ when $\ell^- \geq 1$.

Example 6.15 (Length bound recast as an interval)**.** In the overview in §2 we used (without giving full details) the stable partition system $(\mathcal{P}^{(M)})_{M \in \mathbb{N}_0}$ defined using the 1-twisted dominance order by

$$\mathcal{P}^{(M)} = \big\{\sigma \in \mathrm{Par}(8 + 2M) : \sigma \trianglerighteq (6 + M, 2, 1^M),\ \ell(\sigma) \leq 4 + M\big\}.$$

Let $\sigma \in \mathrm{Par}(8 + 2M)$. Observe that

$$\ell(\sigma) \leq 4 + M \iff \sigma^- \blacktriangleleft\!\!\!\!\underline{\ \ } (4 + M) \iff \sigma \trianglelefteq (5 + M, 1^{3+M}),$$

where the final implication holds since $(5 + M, 1^{3+M}) \leftrightarrow \big\langle (4 + M), (4 + M)\big\rangle$ is the greatest partition in the 1-twisted dominance order with negative part $(4 + M)$. Therefore an equivalent definition of $\mathcal{P}^{(M)}$ is

$$\begin{aligned}\mathcal{P}^{(M)} &= [(6 + M, 2, 1^M), (5 + M, 1^{3+M})]_{\trianglelefteq} \\ &= [(6, 2) \oplus M\big((1), (1)\big), (5, 1, 1, 1) \oplus M\big((1), (1)\big)]_{\trianglelefteq}\end{aligned}$$

where $\trianglelefteq$ is the 1-twisted dominance order, as claimed in §2.6.

It is a special feature of the 1-twisted dominance order that the *only* restriction imposed by the comparison on negative parts is a bound on the length of the partition. See §8.1 for an extended example more typical of the general case.

7. Signed Weight Lemma

In this section we prove the critical Signed Weight Lemma (Lemma 7.3) and give the related results and definitions needed to apply it to prove our main theorems.

7.1. **Stable partition systems.** We isolate the two more technical hypotheses of the Signed Weight Lemma in the following definition.

Definition 7.1. A *partition system* is a sequence $(\mathcal{P}^{(M)})_{M \in \mathbb{N}_0}$ of sets of partitions such that all partitions in each $\mathcal{P}^{(M)}$ have the same size, together with a function $F : \mathrm{Par} \to \mathrm{Par}$ such that $F(\mathcal{P}^{(M)}) \subseteq \mathcal{P}^{(M+1)}$ for all $M \in \mathbb{N}_0$. For each $\pi \in \mathrm{Par}$, let g_π be a symmetric function of degree $|\pi|$. We say the partition system is *stable* with respect to the family g_π if

(a) $F : \mathcal{P}^{(M)} \to \mathcal{P}^{(M+1)}$ is a bijection for all M sufficiently large;
(b) if M is sufficiently large then $\langle g_\pi, s_\sigma\rangle = \langle g_{F(\pi)}, s_{F(\sigma)}\rangle$ for all π, $\sigma \in \mathcal{P}^{(M)}$, and moreover, the matrix $K(M)$ with rows and columns labelled by $\mathcal{P}^{(M)}$ and entries $K(M)_{\pi\sigma} = \langle g_\pi, s_\sigma\rangle$ is invertible.

If (a) and (b) hold for $M \geq L$ then we say the system is *stable for* $M \geq L$. Given $k \in \mathbb{N}$, the *k-subsystem* of $(\mathcal{P}^{(M)})_{M\in\mathbb{N}_0}$ is $(\mathcal{P}^{(kM)})_{M\in\mathbb{N}_0}$ with function F^k, i.e. the k-fold composition of F.

Note in particular that the conditions imply that the matrix $K(M)$ is constant for M sufficiently large. It is routine to check that a k-subsystem of a stable partition system for the family g_π is a stable partition system, again for the family g_π. For an extended example of a stable partition system, see §8. The general results we need on stable partition systems are in §9.

Example 7.2. Let $g_\pi = h_\pi$ for all $\pi \in \mathrm{Par}$ and let $F : \mathrm{Par} \to \mathrm{Par}$ be defined by $F(\sigma) = \sigma + (1)$. Fix a partition λ and let

$$\mathcal{P}^{(M)} = \big\{\sigma \in \mathrm{Par}(|\lambda| + M) : \sigma \trianglerighteq \lambda + (M)\big\}.$$

We claim that the sets $\mathcal{P}^{(M)}$ form a stable partition system. First note that, provided M is sufficiently large, every partition μ such that $\mu \trianglerighteq \lambda + (M+1)$ satisfies $\mu_1 > \mu_2$ and so is in the image of the map F. (Explicitly, it suffices to take $M \geq |\lambda| - 2a(\lambda)$; this is the bound L from Corollary 9.20; we explain why it applies after this example.) Hence (a) holds. By a special case of Lemma 5.3 (Twisted Kostka Numbers), the matrix $K(M)$ in condition (b) is the matrix of Kostka numbers:

$$K(M)_{\pi\sigma} = \langle s_\sigma, h_\pi\rangle = |\,\mathrm{SSYT}(\sigma)_{(\varnothing,\pi)}|$$

for σ, $\pi \in \mathcal{P}^{(M)}$. Provided M is sufficiently large, we have $K_{\pi+(1)\,\sigma+(1)} = K_{\pi\sigma}$ since the relevant semistandard tableaux have the form shown below with 1s in the shaded region, and so are in bijection by removing the hatched box and shifted the boxes right of it one position left. Hence (b) holds.

The argument for (b) is seen in more generality and detail in the extended example in §8.3 and the proof of Proposition 9.19, which shows that for (b) the same bound $L \geq |\lambda| - 2a(\lambda)$ as (a) suffices.

We leave it as an instructive exercise to use the Signed Weight Lemma with the stable partition system in Example 7.2 to prove the stability of the plethysm coefficients $\langle s_{(n+M)} \circ s_{(m)}, s_{\lambda+mM}\rangle$ and $\langle s_{(n)} \circ s_{(m+M)}, s_{\lambda+nM}\rangle$ in Foulkes' Conjecture. Of course this also follows from our main theorems; in the context of their proofs, one should think of $\big\{\sigma \in \mathrm{Par}(mn + M) : \sigma \trianglerighteq \lambda + (M)\big\}$ as the interval $[\lambda + (M), (|\lambda| + M)]_\trianglelefteq$ for the 0-dominance order. With this interpretation the stability of the partition system follows from Corollary 9.20 applied with $\kappa^+ = (1)$, $\kappa^- = \varnothing$ and $\omega = (|\lambda|)$, giving the bound $M \geq L\big([\lambda, (|\lambda|)], (1)\big) = |\lambda| - 2a(\lambda)$. (This is the first bound in the corollary; the remaining three impose no restriction, as is generally the case when $\ell(\kappa^-) = 0$. Alternatively since the interval is 'unsigned' one can use Proposition 9.3.) See §8.4 for a related example where we reinterpret a stable partition system as a sequence of intervals.

7.2. **Signed weight lemma.** The following key lemma specifies the overall strategy of the proofs of Theorem 1.1 and 1.2.

Lemma 7.3 (Signed Weight Lemma)**.** *Fix $\ell^- \in \mathbb{N}$. Set $g_\pi = e_{\pi^-} h_{\pi^+}$ for each $\pi \in \mathrm{Par}$. Let $\nu^{(M)}$ be a sequence of partitions and let $\mu/\mu_\star{}^{(M)}$ be a sequence of skew partitions, indexed by $M \in \mathbb{N}_0$. Let $\mathcal{P}^{(M)}$ be a stable partition system for $M \geq L$ with respect to the symmetric functions g_π and the function $F : \mathrm{Par} \to \mathrm{Par}$, such that the common size of all partitions in $\mathcal{P}^{(M)}$ is $|\nu^{(M)}||\mu/\mu_\star{}^{(M)}|$. Suppose that*

(i) *if M is sufficiently large and $\pi \in \mathcal{P}^{(M)}$ then*

$$\mathrm{supp}(g_\pi) \cap \mathrm{supp}(s_{\nu^{(M)}} \circ s_{\mu/\mu_\star{}^{(M)}}) \subseteq \mathcal{P}^{(M)},$$

(ii) *if M is sufficiently large then, for all $\pi \in \mathcal{P}^{(M)}$,*

$$\bigl|\mathrm{PSSYT}\bigl(\nu^{(M)}, \mu/\mu_\star{}^{(M)}\bigr)_{(\pi^-,\pi^+)}\bigr| = \bigl|\mathrm{PSSYT}\bigl(\nu^{(M+1)}, \mu/\mu_\star{}^{(M+1)}\bigr)_{(F(\pi)^-,F(\pi)^+)}\bigr|.$$

Then, provided $M \geq L$ and M meets the bounds required by (i) *and* (ii)*,*

$$\langle s_{\nu^{(M)}} \circ s_{\mu/\mu_\star{}^{(M)}}, s_\sigma \rangle = \langle s_{\nu^{(M+1)}} \circ s_{\mu/\mu_\star{}^{(M+1)}}, s_{F(\sigma)} \rangle$$

for all $\sigma \in \mathcal{P}^{(M)}$.

We hope to convince the reader, both by the proofs of our main theorems, and the extended example in §8 below, that Lemma 7.3 is both powerful and practical, and not as technical as it appears at first sight. In particular we note that by Lemma 6.12, if $\sigma \in \mathrm{supp}(e_{\pi^-} h_{\pi^+})$ then $\sigma \trianglerighteq \pi$ in the ℓ^--twisted dominance order, and so condition (i) can be tested in practice when, as usual, g_π is the twisted symmetric function $e_{\pi^-} h_{\pi^+}$ in Definition 6.11.

Proof of Lemma 7.3. To simplify notation we set $p^{(M)} = s_{\nu^{(M)}} \circ s_{\mu/\mu_\star{}^{(M)}}$ for $M \in \mathbb{N}_0$. Let $M \geq L$ be given and let $\pi \in \mathcal{P}^{(M)}$. Recall the matrix $K(M)$ from Definition 7.1(b). By hypothesis (i), we have

$$g_\pi = \sum_{\tau \in \mathcal{P}^{(M)}} K(M)_{\pi\tau} s_\tau + G_\pi$$

where the symmetric function G_π satisfies $\langle p^{(M)}, G_\pi \rangle = 0$. By Definition 7.1(b) $K(M)$ is invertible, hence for $\sigma \in \mathcal{P}^{(M)}$ we have

$$\sum_{\pi \in \mathcal{P}^{(M)}} K(M)^{-1}_{\sigma\pi} g_\pi = s_\sigma + \sum_{\pi \in \mathcal{P}^{(M)}} K(M)^{-1}_{\sigma\pi} G_\pi.$$

Substituting $g_\pi = e_{\pi^-} h_{\pi^+}$ we obtain $s_\sigma = \sum_{\pi \in \mathcal{P}^{(M)}} K(M)^{-1}_{\sigma\pi} e_{\pi^-} h_{\pi^+} + E_\sigma$ where, since E_σ is a linear combination of the G_π, we have $\langle p^{(M)}, E_\sigma \rangle = 0$. By this equation for s_σ and Proposition 5.6 we get

$$\langle p^{(M)}, s_\sigma \rangle = \sum_{\pi \in \mathcal{P}^{(M)}} K(M)^{-1}_{\sigma\pi} \bigl|\mathrm{PSSYT}\bigl(\nu^{(M)}, \mu/\mu_\star{}^{(M)}\bigr)_{(\pi^-,\pi^+)}\bigr|. \tag{7.1}$$

The same argument applies with M replaced with $M+1$ and $\sigma \in \mathcal{P}^{(M)}$ replaced with $F(\sigma) \in \mathcal{P}^{(M+1)}$. Hence we also have

$$\langle p^{(M')}, s_{F(\sigma)} \rangle = \sum_{\rho \in \mathcal{P}^{(M')}} K(M')^{-1}_{F(\sigma)\rho} \bigl|\mathrm{PSSYT}\bigl(\nu^{(M')}, \mu/\mu_\star{}^{(M')}\bigr)_{(\rho^-,\rho^+)}\bigr|. \quad (7.2)$$

(Here we reduce clutter by writing M' for $M+1$.) By Definition 7.1(a), the set $\mathcal{P}^{(M)}$ labelling the rows and columns of $K(M)$ and the set $\mathcal{P}^{(M')}$ labelling the rows and columns of $K(M')$ are in bijection by F. Therefore we may take (7.2) and replace each ρ with $F(\pi)$ and the sum over $\rho \in \mathcal{P}^{(M')}$ with a sum over $\pi \in \mathcal{P}^{(M)}$. By Definition 7.1(b) we have $K(M)_{\sigma\pi} = K(M')_{F(\sigma)F(\pi)}$, and so $K(M)^{-1}_{\sigma\pi} = K(M')^{-1}_{F(\sigma)F(\pi)}$ for all $\pi, \sigma \in \mathcal{P}^{(M)}$. This matches up the first factors after the sums in the right-hand sides of (7.1) and (7.2), and hypothesis (ii) immediately implies the second factors are equal. Therefore the right-hand sides agree. Comparing the left-hand sides gives the Signed Weight Lemma. □

8. Extended example of the Signed Weight Lemma

This section is not logically essential. Instead it is intended to illuminate stable partition systems defined in Definition 7.1 and the ℓ^--twisted dominance order defined in Definition 6.6, and to show the strategy in the proofs of our two main theorems using the Signed Weight Lemma (Lemma 7.3).

8.1. **A stable partition system defined by a length bound.** We continue in the setting of Example 6.13, so $\ell^- = 2$. Our aim in this subsection is to show that the partition system

$$\mathcal{P}^{(M)} = \bigl\{\sigma \in \mathrm{Par}(12+4M) : \sigma \trianglerighteq (4+2M, 4, 4, 2^M), \ell(\sigma) \le 3+M\bigr\} \quad (8.1)$$

is stable with respect to the map $F : \mathrm{Par} \to \mathrm{Par}$ defined by

$$\lambda \overset{F}{\longmapsto} \lambda \oplus \bigl((1^2), (2)\bigr) = \lambda + (2) \sqcup (2).$$

We remark that, using the idea seen in Example 6.15, there is an alternative definition of the sets $\mathcal{P}^{(M)}$ as intervals for the 2-twisted dominance order. We explain this in §8.4 at the end of this section, and hence deduce the stability of the partition system from the relevant general result, Corollary 9.20.

Stability is not immediate. Indeed, from the Hasse diagrams in Figure 8.1 we see that F is not bijective when $M = 0$: for example, the partition $(6,6,2,2)$ is clearly not in its image. Suppose that $N \ge 2$ and take $\sigma \in \mathrm{Par}(12+4N)$. By hypothesis

$$\sigma \trianglerighteq (4+2N, 4, 4, 2^N) \leftrightarrow \bigl\langle (3+N, 3+N), (2+2N, 2, 2)\bigr\rangle.$$

Since $\sigma^- \blacktriangleright (3+N, 3+N)$ and, by definition of $\mathcal{P}^{(M)}$, we have $\ell(\sigma) \le 3+N$, we have $\ell(\sigma) = 3+N$. By definition of the 2-twisted dominance order, we have $\sigma^- = (3+N, 3+N)$ and hence $\sigma^+ \trianglerighteq (2+2N, 2, 2)$. Since $N \ge 2$, it follows that $\sigma_1^+ - \sigma_2^+ \ge 2$. Therefore every partition in $\mathcal{P}^{(N)}$ is of the form $\lambda \oplus \bigl((1^2), (2)\bigr)$ and so F is bijective for $M \ge 1$. This verifies condition (i) in the definition of a stable partition system (Definition 7.1).

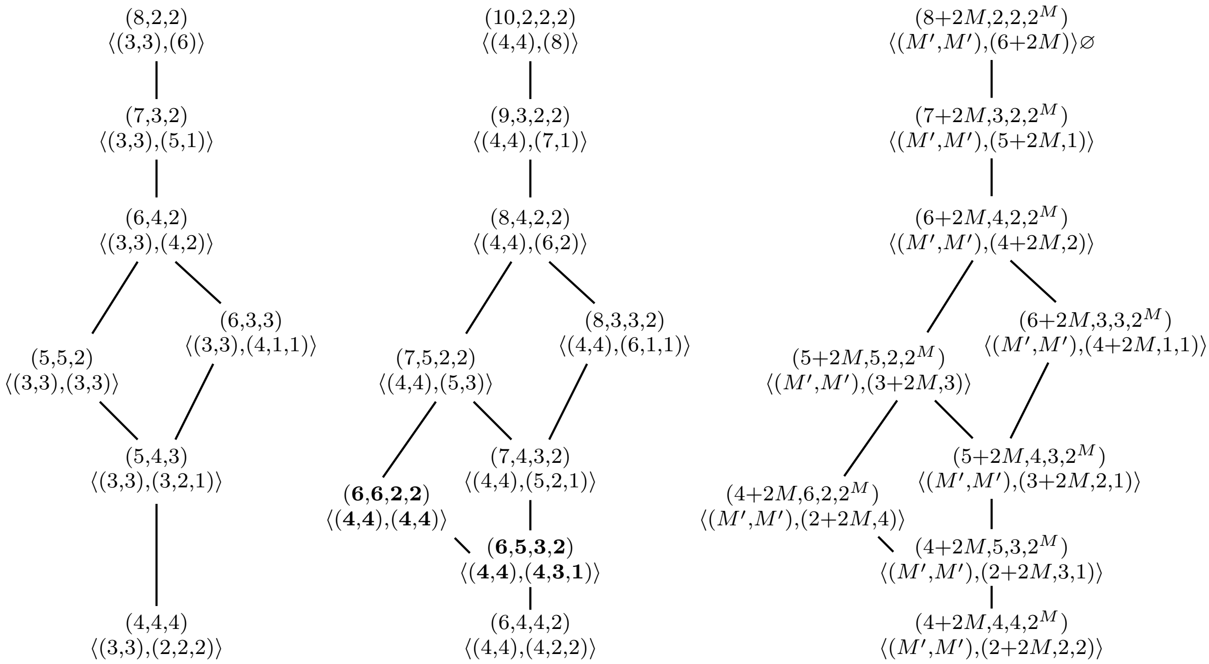


FIGURE 8.1. Hasse diagrams of up-sets in the 2-twisted dominance order. The total order $\leq$ refining $\trianglelefteq$ defined in Definition 6.14 is indicated by vertical height. On the left is the up-set of $(4,4,4) \leftrightarrow \big\langle (3,3),(2,2,2)\big\rangle$ restricted to partitions of length at most 3. (This is part of the up-set relevant to Example 6.13 and the following remark.) This poset maps under $\lambda \mapsto \lambda \oplus \big((1,1),(2)\big)$ into the up-set of $(6,4,4,2) \leftrightarrow \big\langle (4,4),(4,2,2)\big\rangle$ restricted to partitions of length at most 4, shown in the middle; the two partitions not in the image of the map are highlighted. In turn, for each $M \geq 1$, the middle poset is in bijection, by iterating this map, with the up-set of $(4,4,4) \oplus M\big((1^2),(2)\big) = (4+2M,4,4,2^M) \leftrightarrow \big\langle (3+M,3+M),(2+2M,2,2)\big\rangle$ cut to partitions of length at most $M+3$, as shown on the right. (To save space we write M' for $M+3$.)

Continuing we now check condition (b) in the definition of a stable partition system (Definition 7.1). We have $g_\pi = e_{\pi^-}h_{\pi^+}$, where π^- and π^+ are defined using the ℓ^--decomposition with $\ell^- = 2$. The key result we need is Lemma 5.3 (Twisted Kostka Numbers). By this lemma, for π, $\sigma \in \mathrm{Par}$, we have $\langle g_\pi, s_\sigma\rangle = |\,\mathrm{SSYT}(\sigma)_{(\pi^-,\pi^+)}|$. Since $K(M)_{\pi\sigma} = \langle g_\pi, s_\sigma\rangle$ by definition, the matrix $K(M)$ is invertible for all M by Lemma 6.12, and it only remains to show, if $M \geq 1$, then there is a bijection

$$\mathrm{SSYT}(\sigma)_{(\pi^-,\pi^+)} \to \mathrm{SSYT}\big(F(\sigma)\big)_{(F(\pi)^-,F(\pi)^+)} \tag{8.2}$$

for each pair σ, $\pi \in \mathcal{P}^{(M)}$.

Example 8.1. To give some idea why there is a natural bijection we take $M = 1$ and $\pi = (6,4,4,2)$, $\sigma = (7,5,2,2)$. Figure 8.2 shows the two elements of each of $\mathrm{SSYT}\big((7,5,2,2)\big)_{((4,4),(4,2,2))}$ and $\mathrm{SSYT}\big((9,5,2,2,2)\big)_{((5,5),(6,2,2))}$.

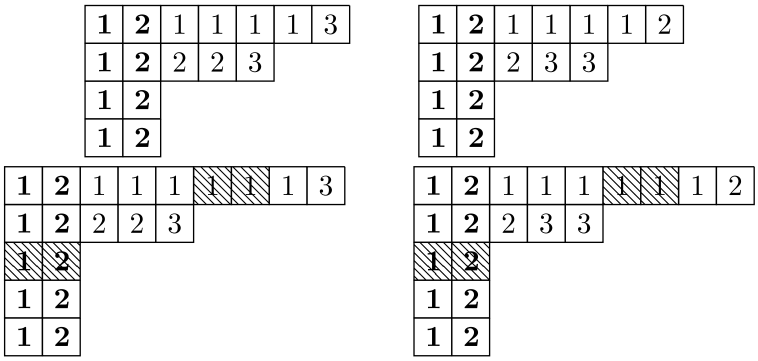
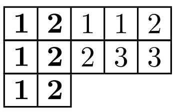


FIGURE 8.2. The two semistandard signed tableaux in the sets $\mathrm{SSYT}\big((7,5,2,2)\big)_{((4,4),(4,2,2))}$ and $\mathrm{SSYT}\big((9,5,2,2,2)\big)_{((5,5),(6,2,2))}$. The hatched boxes are inserted by the $\mathcal{F}$ insertion map.

Observe that, when $M = 2$, the two tableaux of shape $(9,5,2,2,2)$ each have a removable $\boxed{\mathbf{1}|\mathbf{2}}$ in positions $(3,1)$ and $(3,2)$ and two adjacent boxes $\boxed{1|1}$ in positions $(1,6)$ and $(1,7)$ in its top row. Removing these boxes and shifting the remaining boxes in the first column strictly below row 3 up by one row and the remaining boxes in the top row strictly right of column 5 left by two columns pairs up the sets of tableaux. (We admit it might be more natural here to define the bijection by removing $\boxed{\mathbf{1}|\mathbf{2}}$ from positions $(3+M,1)$ and $(3+M,2)$; we choose the complicated specification to agree with the proof of Proposition 9.19 (Tableau Stability) using Lemma 9.16(ii).) Note also that the two tableaux for $M = 1$ have $\boxed{1|3}$ and $\boxed{1|2}$ in boxes $(1,6)$ and $(1,7)$, so removing the four boxes from the positions $(3,1),(3,2),(1,6),(1,7)$ hatched in Figure 8.2 gives tableaux of signed weight $\big((3,3),(3,2,1)\big)$ and $\big((3,3),(3,1,2)\big)$, not $\big((3,3),(2,2,2)\big)$ as required. Correspondingly, the unique element of $\mathrm{SSYT}\big(5,5,2)\big)_{((3,3),(2,2,2))}$ is as shown in the margin, so there is no possibility of a bijection between $\mathcal{P}^{(0)}$ and $\mathcal{P}^{(1)}$ satisfying (8.2).



To generalize this example to arbitrary M it is most convenient to consider the inverse map. Fix $N \geq 2$, let σ, $\pi \in \mathcal{P}^{(N)}$ and let $t \in \mathrm{SSYT}(\sigma)_{(\pi^-,\pi^+)}$. We know that $\sigma^- = \pi^- = (3+N, 3+N)$. Hence t has $3+N$ entries of -1 and $3+N$ entries of -2 which, since negative entries cannot be repeated in a row, must form the first two columns of t. Therefore position $(1,2)$ t contains $\mathbf{2}$. Moreover, by Definition 7.1(a), σ satisfies $\sigma_1 - \sigma_2 \geq 1$ and since $\pi^- = (4+2N, 4, 4, 2^N)^-$, it is immediate from Definition 6.6 that

$$\pi^+ \trianglerighteq (4+2N, 4, 4, 2^N)^+ = (2+2N, 2, 2).$$

and so $\pi_1^+ \geq 2+2N$. Therefore t has at least $2+2N$ entries of 1, necessarily in its first row, and we see that boxes $(1,6)$ and $(1,7)$ of t both contain 1. Removing this $\boxed{1|1}$ and deleting $\boxed{\mathbf{1}|\mathbf{2}}$ from positions $(3,1)$ and $(3,2)$ and then shifting boxes left or up (as seen when $N = 2$) defines a bijection $\mathrm{SSYT}(\sigma)_{(\pi^-,\pi^+)} \to \mathrm{SSYT}(f^{-1}(\sigma))_{(f^{-1}(\pi)^-, f^{-1}(\pi)^+)}$.

	(10,2,2,2)	(9,3,2,2)	(8,4,2,2)	(8,3,3,2)	(7,5,2,2)	(7,4,3,2)	(6,6,2,2)	(6,5,3,2)	(6,4,4,2)
⟨(4,4),(8)⟩	1	·	·	·	·	·	·	·	·
⟨(4,4),(7,1)⟩	1	1	·	·	·	·	·	·	·
⟨(4,4),(6,2)⟩	1	1	1	·	·	·	·	·	·
⟨(4,4),(6,1,1)⟩	1	2	1	1	·	·	·	·	·
⟨(4,4),(5,3)⟩	1	1	1	0	1	·	·	·	·
⟨(4,4),(5,2,1)⟩	1	2	2	1	1	1	·	·	·
⟨(4,4),(4,4)⟩	1	1	1	0	1	0	1	·	·
⟨(4,4),(4,3,1)⟩	1	2	2	1	2	1	1	1	·
⟨(4,4),(4,2,2)⟩	1	2	3	1	**2**	2	1	1	1

FIGURE 8.3. The stable transition matrix $K(1)$ in Example 8.2 with entries $K(1)_{\pi\sigma} = |\mathrm{SSYT}(\sigma)_{(\pi^-,\pi^+)}|$. Columns are labelled by the partition σ, rows by the 2-decomposition of π. We use $\cdot$ to denote a zero entry implied by Lemma 6.12. The entry highlighted in bold counting $\mathrm{SSYT}\big((7,5,2,2)\big)_{((4,4),(4,2,2))}$ is used in Example 8.1.

Example 8.2. The stable transition matrix $K(1)$ is shown in Figure 8.3 below. It was computed using the MAGMA code available as part of the arXiv submission of this paper using `TwistedIntervalMatrix(2, [6,4,4,2] : q := [10,2,2,2]);`. The entry relevant to Example 8.1 is highlighted in bold in the bottom row. The rows (recording the signed weight defining the relevant product $e_{\alpha^-}h_{\alpha^+}$) and columns (recording the relevant Schur function, or shape of the semistandard signed tableau) are ordered by the total order $\leq$ (see Definition 6.14) refining the 2-twisted dominance order $\trianglelefteq$ (see Definition 6.6). The relevant 2-decompositions (see Definition 6.1) are shown on the rows. As remarked at the end of §6.3 it is instructive to check that the entries of 0 correspond to pairs of partitions incomparable in the 2-twisted dominance order.

8.2. **Cut up-sets and the plethysm** $\langle s_{(3)+(M)} \circ s_{(4)}, s_{(4,4,4)\oplus M((1^2),(2)}\rangle$**.** The special case of Theorem 1.2 for the strongly maximal signed weights $\big((1^d),(m-d)\big)$ seen in Example 4.18(i) asserts that, if d is even, then the plethysm coefficients $\langle s_{\nu+(M)} \circ s_{(m)}, s_{\lambda\oplus M((1^d),(m-d))}\rangle$ are ultimately constant. To prove this using the Signed Weight Lemma, we need a stable partition system $(\mathcal{P}^{(M)})_{M\in\mathbb{N}_0}$ such that $\lambda \oplus M\big((1^d),(m-d)\big) \in \mathcal{P}^{(M)}$ for each $M \in \mathbb{N}_0$. As we saw in the overview in §2, we cannot expect to define $\mathcal{P}^{(M)}$ to be the up-set $\big(\lambda \oplus M(\kappa^-,\kappa^+)\big)^{\trianglelefteq}$, where $\trianglelefteq$ is the d-twisted dominance order, because typically the sizes of the up-sets grow, ruling out any bijection between them. In this subsection we shall see this problem in the particular case of the plethysm coefficients

$$\langle s_{(3)+(M)} \circ s_{(4)}, s_{(4,4,4)\oplus M((1^2),(2))}\rangle \tag{8.3}$$

and resolve it using the stable partition system constructed in §8.1. Then in §8.3 we use the 3-subsystem of this partition system to prove a different stability result.

Up-sets are not stable. We take $\ell^- = 2$ in the Signed Weight Lemma. The 2-decomposition of a partition π (see Definition 6.1) is defined by $\pi^- = (\pi'_1, \pi'_2)$ and $\pi^+ = (\pi_1 - 2, \pi_2 - 2, \ldots, \pi_r - 2)$, where r is maximal such that $\pi_r > 2$. For example, if $\pi = (4 + 2M, 4, 4, 2^M)$ then $\pi^- = (M + 3, M + 3)$ and $\pi^+ = (2 + 2M, 2, 2)$. By Lemma 6.12, if s_σ is a summand of $e_{\pi^-} h_{\pi^+}$ then $\sigma \trianglerighteq \pi$. Therefore, taking $g_\pi = e_{\pi^-} h_{\pi^+}$, the up-set of $(4 + 2M, 4, 4, 2^M)$ in the 2-twisted dominance order satisfies condition (i) in the Signed Weight Lemma. We cannot take the up-sets $(4 + 2M, 4, 4, 2^M)^{\trianglelefteq}$ as our partition system because, they are not stable. Indeed, since $\pi \leftrightarrow \bigl\langle (3{+}M, 3{+}M), (2{+}2M, 2, 2)\bigr\rangle$ and $(2^{3+b+M}, 1^{6-2b+2M}) \leftrightarrow \bigl\langle (9 - b + 3M, 3 + b + M), \varnothing \bigr\rangle$ we have

$$\pi \trianglelefteq (2^{3+b+M}, 1^{9-2b+2M})$$

for all $b \leq 3 + M$ and hence $\bigl|(4 + 2M, 4, 4, 2^M)^{\trianglelefteq}\bigr| \geq 3 + M$ and the sizes of the up-sets tend to infinity with M. This behaviour, that some ‘cut’ is necessary before a sequence of up-sets becomes stable, is typical.

Cut up-sets are stable. To get around the problem we use that condition (i) in the Signed Weight Lemma (Lemma 7.3) does not require that $\mathrm{supp}(g_\pi) \subseteq \mathcal{P}^{(M)}$ for all $\pi \in \mathcal{P}^{(M)}$ but instead, since $\nu = (3)$ and $\mu/\mu_\star = (4)$, only the weaker condition that $\mathrm{supp}(g_\pi) \cap \mathrm{supp}(s_{(3+M)} \circ s_{(4)}) \subseteq \mathcal{P}^{(M)}$ for all $\pi \in \mathcal{P}^{(M)}$. Since the support of the plethysm $s_{(3+M)} \circ s_{(4)}$ is contained in the support of $s_{(4)} \times \overset{3+M}{\cdots} \times s_{(4)}$, each partition in $\mathrm{supp}(s_{(3+M)} \circ s_{(4)})$ has at most $3 + M$ parts. We therefore only need to consider partitions such as $(4{+}2M, 4, 4, 2^M)$ for which $\ell(\sigma) \leq 3 + M$. This motivates the definition

$$\mathcal{P}^{(M)} = (4 + 2M, 4, 4, 2^M)^{\trianglelefteq} \cap \{\sigma \in \mathrm{Par}(12 + 4M) : \ell(\sigma) \leq 3 + M\}$$

already given in (8.1) in an obviously equivalent form. We saw in §8.1 that $(\mathcal{P}^{(M)})_{M \in \mathbb{N}_0}$ is a stable partition system for $M \geq 1$ with respect to $F : \mathcal{P}^{(M)} \to \mathcal{P}^{(M+1)}$ defined by $F(\lambda) = \lambda \oplus \bigl((1, 1), (2)\bigr) = \lambda + (2) \sqcup (2)$ and the symmetric functions g_π.

Proof that $\langle s_{(3)+(M)} \circ s_{(4)}, s_{(4,4,4) \oplus M((1^2),(2))} \rangle$ is ultimately constant. We shall check conditions (i) and (ii) in the Signed Weight Lemma (Lemma 7.3). Let $\pi \in \mathcal{P}^{(M)}$. As we saw in §8.1 we have $\pi^- = (3 + M, 3 + M)$. Hence

$$\begin{aligned} &\mathrm{supp}(g_\pi) \cap \mathrm{supp}(s_{(3+M)} \circ s_{(4)}) \\ &\quad\subseteq \pi^{\trianglelefteq} \cap \{\sigma \in \mathrm{Par}(12 + 4M) : \ell(\sigma) \leq 3 + M\} \\ &\quad\subseteq \{\sigma \in (4 + 2M, 4, 4, 2^M)^{\trianglelefteq} : \ell(\sigma) \leq 3 + M\} \\ &\quad= \mathcal{P}^{(M)} \end{aligned}$$

where the second line uses Lemma 6.12 on $\mathrm{supp}(g_\pi)$ and the length bound in the previous paragraph on partitions in $\mathrm{supp}(s_{(3+M)} \circ s_{(4)})$, and the third line follows from $\pi \trianglerighteq (4 + 2M, 4, 4, 2^M)$. Hence (i) holds for all $M \in \mathbb{N}_0$.

Now fix $M \in \mathbb{N}_0$ with $M \geq 1$ (so meeting the stability bound) and $\pi \in \mathcal{P}^{(M)}$. For (ii), it suffices to define a bijection

$$\mathcal{H} : \mathrm{PSSYT}\big((3+M),(4)\big)_{((3+M,3+M),\pi^+)} \rightarrow \mathrm{PSSYT}\big((3+M+1),(4)\big)_{((3+M+1,3+M+1),\pi^+ + (2))}.$$

Let T be in the right-hand side. Observe that T has $3+M+1$ integer entries of -1. necessarily lying in distinct (4)-tableau entries. A similar argument considering -2 now shows that each inner (4)-tableau in T is of the form $\boxed{\mathbf{1}}\boxed{\mathbf{2}}\boxed{x}\boxed{y}$ where $1 \leq x \leq y$. Since $\pi \in \mathcal{P}^{(M)}$ we have $\pi \trianglerighteq (4+2M,4,4,2^M)$ and hence $\pi^+ + (2) \trianglerighteq (2+2M+2,2,2)$. Therefore $a(\pi^+ + (2)) \geq 4+2M$ and, of the $8+2M$ positions in the (4)-tableau entries of T containing a positive entry, all but four positions contain 1. In particular, since $M \geq 1$, the leftmost (4)-tableau in T is $\boxed{\mathbf{1}}\boxed{\mathbf{2}}\boxed{1}\boxed{1}$. Hence we may define $\mathcal{H}$ by inserting this inner (4)-tableau as a new leftmost inner tableau in a given plethystic semistandard signed tableau in $\mathrm{PSSYT}\big((3+M),(4)\big)_{((3+M,3+M),\pi^+)}$.

We have now checked (i) and (ii) in the Signed Weight (Lemma 7.3) and conclude that $\langle s_{(3)+(M)} \circ s_{(4)}, s_{(4,4,4)\oplus M((1,1),(2))}\rangle$ is constant for $M \geq 1$. $\square$

Computation shows that the stable multiplicity is in fact 1. The stability of this plethysm is a special case of Theorem 1.2 and the map $\mathcal{H}$ is as in the proof of condition (ii) of the Signed Weight Lemma in the proof of Theorem 14.7.

Example 8.3. Applying the ω-involution to the result just proved, we obtain that $\langle s_{(3+M)} \circ s_{(1^4)}, s_{(3,3,3,3)\oplus M((2),(1,1))}\rangle$ is constant for $M \geq 1$. This is not an instance of Theorem 1.2 since, according to Definition 4.10, the singleton strongly maximal tableau families of shape (1^4) have as their unique elements the tableaux shown in the margin of signed weights $\big((4),\varnothing\big)$ and $\big(\varnothing,(1^4)\big)$ respectively. Therefore $\big((2),(1,1)\big)$ is not the signed weight of a strongly maximal tableau family of shape (1^4) and size 1. This illustrates Remark 4.16.

$$\begin{array}{|c|}\hline \mathbf{1} \\ \hline \mathbf{1} \\ \hline \mathbf{1} \\ \hline \mathbf{1} \\ \hline \end{array} \qquad \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline 4 \\ \hline \end{array}$$

8.3. **Re-use of a stable partition system: example of Theorem 1.1.** One reason for defining stable partition systems in Definition 7.1 as objects of interest in their own right, rather than including conditions (a) and (b) as hypotheses in the Signed Weight Lemma (Lemma 7.3) is that the same stable partition system can be used to prove the stability of multiple plethysms. Here we use the Signed Weight Lemma to prove the special case of Theorem 1.1 that $\langle s_{(3)} \circ s_{(4+2M,2^M)}, s_{(4+6M,4,4,2^{3M})}\rangle$ is ultimately constant, using the stable partition system in our running example.

Proof that $\langle s_{(3)} \circ s_{(4+2M,2^M)}, s_{(4+6M,4,4,2^{3M})}\rangle$ is ultimately constant. We consider the 3-subsystem (see Definition 7.1) of the stable partition system $(\mathcal{P}^{(M)})_{M\in\mathbb{N}_0}$ from §8.1. Thus we set $\mathcal{Q}^{(M)} = \mathcal{P}^{(3M)}$, so that

$$\mathcal{Q}^{(M)} = (4+6M,4,4,2^{3M})^{\trianglelefteq} \cap \{\pi \in \mathrm{Par}(12+12M) : \ell(\pi) \leq 3+3M\}.$$

For condition (i) in the Signed Weight Lemma (Lemma 7.3), observe that

$$\mathrm{supp}(s_{(3)} \circ s_{(4+2M,2^M)}) \subseteq \mathrm{supp}(s_{(4+2M,2^M)} \times s_{(4+2M,2^M)} \times s_{(4+2M,2^M)})$$

and so, by the Littlewood–Richardson rule, every partition in the right-hand side has at most $3(M+1)$ parts. Hence (i) holds by the same argument used in §8.2. For (ii), again a similar argument works. Fix $M \in \mathbb{N}_0$ with $M \geq 1$ (so again the stability bound holds), let $\pi \in \mathcal{Q}^{(M)}$ and let

$$T \in \mathrm{PSSYT}\big((3), (4+2(M+1), 2^{M+1})\big)_{((3+3(M+1), 3+3(M+1)), \pi^+ + (6))}.$$

Since T has $3M+6$ integer entries of -1 and $3M+6$ integer entries of -2, each $\big(4+2(M+1), 2^{M+1}\big)$-tableau in T has first column entries all -1 and second column entries all -2. Since $\pi \in \mathcal{Q}^{(M)}$ we have $\pi^+ \trianglerighteq (2 + 6M, 2, 2)$. Therefore $a(\pi^+ + (6)) \geq 6M + 8$ and, of the $6M+12$ positive entries in the three $\big(4+2(M+1), 2^{M+1}\big)$-tableaux entries of T, all but four are equal to 1. Hence removing $\boxed{\mathbf{1}\,|\,\mathbf{2}}$ from positions $(2,1)$ and $(2,2)$ in each inner $\big(4+2(M+1), 2^{M+1}\big)$-tableau and $\boxed{1\,|\,1}$ from positions $(1,3)$ and $(1,4)$ in each inner $(4+2(M+1), 2^{M+1})$-tableau, shifting the remaining entries one position up or left as appropriate, defines a bijection proving (ii). Therefore, by the Signed Weight Lemma 7.3, the plethysm multiplicity $\langle s_{(3)} \circ s_{(4+2M, 2^M)}, s_{(4+6M,4,4,2^{3M})}\rangle$ is constant for $M \geq 1$. □

8.4. **Stable partition systems as intervals.** A special feature of the stable partition system $\mathcal{P}^{(M)}$ in our running example is that all the partitions $\pi \in \mathcal{P}^{(M)}$ have the same negative part in their 2-decomposition, namely $(3+M, 3+M)$. All partitions in $\mathcal{P}^{(M)}$ have size $12+4M$. The greatest partition in the 2-twisted dominance order (see Definition 6.6) of $12+4M$ with these first two columns is $(8+2M, 2, 2, 2^M)$. By the definition in (8.1), the least element of $\mathcal{P}^{(M)}$ is $(4+2M, 4, 4, 2^M)$. Therefore for each $M \in \mathbb{N}_0$ we have

$$\mathcal{P}^{(M)} = \big[(4+2M, 4, 4, 2^M), (8+2M, 2, 2, 2^M)\big]_{\trianglelefteq}$$

and each $\mathcal{P}^{(M)}$ is an interval for the 2-twisted dominance order, but of the special type where all partitions in the interval have the same negative part. This was a deliberate choice in order to give a system that was not immediately stable, but still of manageable size and useful for proving stability results. We revisit this example in Example 10.9, showing that the upper bound in each interval is the partition given by Proposition 10.7.

To give a more typical example we suppose that instead of the plethysm coefficients $\langle s_{(3)+(M)} \circ s_{(4)}, s_{(4,4,4)\oplus M((1^2),(2)}\rangle$ in (8.3), we want to prove that

$$\big\langle s_{(4)+(M)} \circ s_{(4)}, s_{(6,6,4)\oplus M((1^2),(2))}\big\rangle$$

is ultimately constant. Since the Schur functions constituents of $s_{(4+M)} \circ s_{(4)}$ have at most $4+M$ parts we must relax the length bound in $\mathcal{P}^{(M)}$, and so we now define

$$\mathcal{R}^{(M)} = \big\{\sigma \in \mathrm{Par}(16+4M) : \sigma \trianglerighteq (6+2M, 6, 4, 2^M), \ell(\sigma) \leq 4+M\big\}$$

still working with the 2-twisted dominance order. Observe that $\mathcal{R}^{(0)}$ contains $(6,6,4)$, $(7,7,1,1)$, $(6,5,4,1)$, $(5,5,4,2)$ with increasing negative parts $(3,3)$, $(4,2)$, $(4,3)$ and $(4,4)$ respectively. To show that $(\mathcal{R}^{(M)})_{M\in\mathbb{N}_0}$ is stable we reinterpret each set $\mathcal{R}^{(M)}$ as an interval for the 2-twisted dominance order using the idea seen in Example 6.15. Observe that $\ell(\sigma) \leq 4+M$ if and only if

$\sigma^- \unlhd (4{+}M, 4{+}M)$. Since $(10{+}2M, 2, 2, 2, 2^M) \leftrightarrow \big\langle (4{+}M, 4{+}M), (8{+}2M)\big\rangle$ is the greatest partition of $16{+}4M$ in the 2-twisted dominance order satisfying this condition, we have

$$\mathcal{R}^{(M)} = \big[(6 + 2M, 6, 4, 2^M), (10 + 2M, 2, 2, 2, 2^M)\big]_{\unlhd}.$$

The stability of $(\mathcal{R}^{(M)})_{M\in\mathbb{N}_0}$ is then a special case of Corollary 9.20. The four bounds in this corollary are $M \geq -2$, $M \geq 1$, $M \geq 2$ and $M \geq 1$, respectively, so the stability bound is $M \geq 2$. By this corollary, this bound is a sufficient condition for $F : \mathcal{R}^{(M)} \to \mathcal{R}^{(M+1)}$ to be a bijection; computation using the Magma code mentioned after Definition 9.14 shows that this bound is also necessary: $|\mathcal{R}^{(M)}| = 40, 57, 60, 60$ for $0 \leq M \leq 3$.

9. Stable partition systems defined by twisted intervals

In this section we prove the technical result, Corollary 9.20, that suitable sequences of intervals in the ℓ^--twisted dominance order define stable partition systems. These are the stable partition systems we use in the Signed Weight Lemma (Lemma 7.3) to prove Theorems 1.1 and 1.2.

9.1. Unsigned intervals.

Recall from §3 that we write $\blacktriangleleft$ for the dominance order extended to partitions possibly of different sizes. Given partitions γ and δ each with at most p parts, we define the *unsigned interval* $[\gamma, \delta]^{(p)}_{\blacktriangleleft}$ by

$$[\gamma, \delta]^{(p)}_{\blacktriangleleft} = \{\sigma \in \mathrm{Par} : \gamma \blacktriangleleft \sigma \blacktriangleleft \delta \text{ and } \ell(\sigma) \leq p\}.$$

Note that unless $|\gamma| \leq |\delta|$ the interval is empty.

Remark 9.1. If $|\gamma| = |\delta|$ and $\ell(\gamma) \leq p$ then $[\gamma, \delta]^{(p)}_{\blacktriangleleft} = \{\sigma \in \mathrm{Par} : \gamma \unlhd \sigma \unlhd \delta\}$ since any partition σ such that $\gamma \unlhd \sigma$ satisfies $\ell(\sigma) \leq \ell(\gamma)$; thus in this case we have $[\gamma, \delta]^{(p)}_{\blacktriangleleft} = [\gamma, \delta]_{\unlhd}$ and there is no ambiguity in using this simpler notation.

In our applications, whenever $|\gamma| < |\delta|$, we will take $p = \ell^-$ where ℓ^- is the length of the negative part of the relevant signed weight, and so the partitions in the unsigned interval $[\gamma, \delta]^{(p)}_{\blacktriangleleft}$ all have at most ℓ^- parts, as in the ℓ^--decomposition (see Definition 6.1).

Definition 9.2. Given partitions λ and ω with $\lambda \blacktriangleleft \omega$ and a non-empty partition κ, each having at most p parts, let $\ell = \ell(\kappa)$ and set

$$L_k = \frac{2\sum_{i=1}^{k-1} \omega_i + \omega_k + \omega_{k+1} - 2\sum_{i=1}^{k} \lambda_i}{\kappa_k - \kappa_{k+1}}$$

for k such that $1 \leq k \leq \ell$ and $\kappa_k > \kappa_{k+1}$. Set $L_k = 0$ if $\kappa_k = \kappa_{k+1}$. Define $\mathrm{L}\big([\lambda, \omega]^{(p)}_{\blacktriangleleft}, \kappa\big)$ to be the maximum of $L_1, \ldots, L_\ell$ if $p > \ell$ and the maximum of $L_1, \ldots, L_{\ell-1}$ and $(|\omega| - |\lambda| - \omega_\ell)/\kappa_\ell$ if $p = \ell$. Set $\mathrm{L}\big([\lambda, \omega]^{(p)}_{\blacktriangleleft}, \varnothing\big) = 0$.

We remark that if $\ell(\lambda) \leq \ell$ and $\ell(\omega) \leq \ell$ then $L_\ell = (2|\omega| - 2|\lambda| - \omega_\ell)/\kappa_\ell$. Thus the bound in Definition 9.2 may in this case be strictly less than the maximum of $L_1, \ldots, L_\ell$.

Let κ be a partition with $\ell(\kappa) \le p$. Since $\alpha \trianglelefteq \beta$ implies $\alpha + \kappa \trianglelefteq \beta + \kappa$ for any partitions α, β, adding κ defines an injective map from $[\gamma, \delta]^{(p)}_{\trianglelefteq}$ to $[\gamma + \kappa, \delta + \kappa]^{(p)}_{\trianglelefteq}$.

Proposition 9.3. *Let λ and ω be partitions and let κ be a non-empty partition, each having at most p parts. Let $F : \mathrm{Par} \to \mathrm{Par}$ be defined by $F(\sigma) = \sigma + \kappa$. Let $M \in \mathbb{N}_0$. The injective map*

$$F : [\lambda + M\kappa, \omega + M\kappa]^{(p)}_{\trianglelefteq} \longrightarrow [\lambda + (M+1)\kappa, \omega + (M+1)\kappa]^{(p)}_{\trianglelefteq}$$

is bijective provided $M \ge \mathrm{L}\bigl([\lambda, \omega]^{(p)}_{\trianglelefteq}, \kappa\bigr)$.

Proof. Let $\ell = \ell(\kappa)$ and let $N = M + 1$. Let $\tau \in [\lambda + N\kappa, \omega + N\kappa]^{(p)}_{\trianglelefteq}$. Observe that τ is of the form $\sigma + \kappa$ for a partition σ if and only if all ℓ inequalities in the chain

$$\tau_1 - \kappa_1 \ge \tau_2 - \kappa_2 \ge \ldots \ge \tau_\ell - \kappa_\ell \ge \tau_{\ell+1} \tag{9.1}$$

hold. (For instance if $\tau_k < \kappa_k$ for some k then since $\tau_k - \kappa_k < 0 \le \tau_{\ell+1}$, at least one inequality fails to hold.) Fix $k \le \ell$. Using the hypotheses $\tau \trianglerighteq \lambda + N\kappa$ and $\tau \trianglelefteq \omega + N\kappa$ we have

$$\sum_{i=1}^{k-1} \tau_i \le \sum_{i=1}^{k-1} \omega_i + N \sum_{i=1}^{k-1} \kappa_i, \tag{9.2}$$

$$\sum_{i=1}^{k} \tau_i \ge \sum_{i=1}^{k} \lambda_i + N \sum_{i=1}^{k} \kappa_i, \tag{9.3}$$

$$\sum_{i=1}^{k+1} \tau_i \le \sum_{i=1}^{k+1} \omega_i + N \sum_{i=1}^{k+1} \kappa_i. \tag{9.4}$$

Subtracting (9.2) from (9.3) we get $\tau_k \ge -\sum_{i=1}^{k-1} \omega_k + \sum_{i=1}^{k} \lambda_i + N\kappa_k$ and subtracting (9.3) from (9.4) we get $\tau_{k+1} \le \sum_{i=1}^{k+1} \omega_k - \sum_{i=1}^{k} \lambda_i + N\kappa_{k+1}$. Subtracting these two equations in turn, to form the linear combination $-(9.4) + 2(9.3) - (9.2)$, we get

$$\tau_k - \tau_{k+1} \ge -2 \sum_{i=1}^{k-1} \omega_i - \omega_k - \omega_{k+1} + 2 \sum_{i=1}^{k} \lambda_i + N(\kappa_k - \kappa_{k+1}). \tag{9.5}$$

Recalling that $M = N - 1$, we deduce that

$$(\tau_k - \kappa_k) - (\tau_{k+1} - \kappa_{k+1}) \ge B_k + M(\kappa_k - \kappa_{k+1}) \tag{9.6}$$

where $B_k = -2\sum_{i=1}^{k-1} \omega_i - \omega_k - \omega_{k+1} + 2\sum_{i=1}^{k} \lambda_i$. Note that if $\kappa_k = \kappa_{k+1}$, the inequality $\tau_k - \kappa_k \ge \tau_{k+1} - \kappa_{k+1}$ holds simply because τ is a partition. Therefore by taking $M \ge -B_k/(\kappa_k - \kappa_{k+1})$ for each k such that $\kappa_k > \kappa_{k+1}$, we deduce from (9.6) that every inequality in the chain (9.1) holds. Hence, provided $M \ge L_1, \ldots, L_\ell$, we may define $\sigma = \tau - \kappa$, knowing that σ is a well-defined partition.

If $p = \ell$ then rather than $M \ge L_\ell$, we have only the weaker hypothesis that $M \ge (|\omega| - |\lambda| - \omega_\ell)/\kappa_\ell$. However, in this case $\ell(\lambda) \le \ell$, $\ell(\tau) \le \ell$ and

$\ell(\omega) \le \ell$ and

$$\tau_\ell = \sum_{i=1}^{\ell} \tau_i - \sum_{i=1}^{\ell-1} \tau_{i-1} \ge \Big(\sum_{i=1}^{\ell} \lambda_i + \sum_{i=1}^{\ell} N\kappa_i\Big) - \Big(\sum_{i=1}^{\ell-1} \omega_i + \sum_{i=1}^{\ell-1} N\kappa_i\Big)$$
$$= \sum_{i=1}^{\ell} \lambda_i - \sum_{i=1}^{\ell-1} \omega_i + N\kappa_\ell = |\lambda| - |\omega| + \omega_\ell + N\kappa_\ell.$$

Hence $\tau_\ell \ge \kappa_\ell$, as we require, provided $(N-1)\kappa_\ell \ge |\omega| - |\lambda| - \omega_\ell$. Therefore in the case $p = \ell$ we may replace L_ℓ with the weaker bound $(|\omega| - |\lambda| - \omega_\ell)/\kappa_\ell$, and again σ is a well-defined partition.

It remains to show that $\sigma \in [\lambda + M\kappa, \omega + M\kappa]^{(p)}_{\underline{\blacktriangleleft}}$. Since $\tau \mathrel{\underline{\blacktriangleright}} \lambda + (M+1)\kappa$ we have $\sum_{i=1}^{k} \tau_i \ge \sum_{i=1}^{k} \lambda_i + (M+1) \sum_{i=1}^{k} \kappa_i$ for each $k \in \mathbb{N}$. Therefore $\sum_{i=1}^{k} \sigma_i \ge \sum_{i=1}^{k} \lambda_i + M \sum_{i=1}^{k} \kappa_i$ for each $k \in \mathbb{N}$, and hence $\sigma \mathrel{\underline{\blacktriangleright}} \lambda + M\kappa$. Very similarly one shows that $\sigma \mathrel{\underline{\blacktriangleleft}} \omega + M\kappa$. Finally since $\tau \in [\lambda + N\kappa, \omega + N\kappa]^{(p)}_{\underline{\blacktriangleleft}}$ we have $\ell(\tau) \le p$, and since $\ell(\kappa) = \ell \le p$, it follows that $\ell(\sigma) \le p$. Therefore $\sigma \in [\lambda + M\kappa, \omega + M\kappa]^{(p)}_{\underline{\blacktriangleleft}}$ is a preimage of τ under F and since F is injective, it follows that F is bijective for $M \ge \mathrm{L}\big([\lambda, \omega]^{(p)}_{\underline{\blacktriangleleft}}, \kappa\big)$. $\square$

We give one of the smallest examples in which the bound in Definition 9.2 and Proposition 9.3 is 2: see Examples 9.13 and 9.15 for cases where two parts of κ agree.

Example 9.4. We take $\lambda = (1,1,1)$, $\omega = (3)$ and $\kappa = (3,2,1)$. Routine calculations show that the unsigned intervals $[(1,1,1),(3)]_{\unlhd}$, $[(4,3,2),(6,2,1)]_{\unlhd}$, $[(7,5,3),(9,4,2)]_{\unlhd}$ and $[(10,7,4),(12,6,3)]_{\unlhd}$ are as shown below

$$\left\{\begin{matrix} (3) \\ (2,1) \\ (1,1,1) \end{matrix}\right\} \hookrightarrow \left\{\begin{matrix} (6,2,1) \\ (5,3,1) \\ (\mathbf{5},\mathbf{2},\mathbf{2}) \\ (\mathbf{4},\mathbf{4},\mathbf{1}) \\ (4,3,2) \end{matrix}\right\} \hookrightarrow \left\{\begin{matrix} (9,4,2) \\ (\mathbf{9},\mathbf{3},\mathbf{3}) \\ (8,5,2) \\ (8,4,3) \\ (7,6,2) \\ (7,5,3) \end{matrix}\right\} \hookrightarrow \left\{\begin{matrix} (12,6,3) \\ (12,5,4) \\ (11,7,3) \\ (11,6,4) \\ (10,8,3) \\ (10,7,4) \end{matrix}\right\}.$$

The elements not in the image of the map $\sigma \stackrel{F}{\longmapsto} \sigma + (3,2,1)$ are highlighted. Setting $\mathcal{P}^{(M)} = [(1,1,1) + M(3,2,1), (3) + M(3,2,1)]_{\unlhd}$ we see that $F : \mathcal{P}^{(2)} \to \mathcal{P}^{(3)}$ is a bijection. Correspondingly, by Proposition 9.3, $F : \mathcal{P}^{(M)} \to \mathcal{P}^{(M+1)}$ is bijective provided $M \ge \mathrm{L}\big([(1,1,1),(3)]_{\unlhd}, (3,2,1)\big)$ and the right-hand side is the maximum of $\max\{\frac{3-2}{3-2}, \frac{6-4}{2-1}\} = \max\{1,2\} = 2$ and $\frac{3-3-0}{1-0} = 0$.

9.2. **Twisted intervals.** We now extend Proposition 9.3 to the twisted case. Recall from §6.5 that for fixed $\ell^- \in \mathbb{N}_0$, and partitions γ, δ of the same size we defined the twisted interval $[\gamma, \delta]_{\unlhd} = \{\sigma \in \mathrm{Par}(p) : \gamma \unlhd \sigma \unlhd \delta\}$, where $\unlhd$ is the ℓ^--twisted dominance order. It is obvious that addition of partitions preserves the dominance order. By conjugating partitions, the same result holds for joining. Despite this, addition *does not* preserve the ℓ^--twisted dominance order. For instance, taking $\ell^- = 1$ we have

$$\big\langle (1),(1) \big\rangle \leftrightarrow (2) \unlhd (1,1) \leftrightarrow \big\langle (2), \varnothing \big\rangle,$$

whereas after adding $(1,1)$,

$$\big\langle (2),(2)\big\rangle \leftrightarrow (3,1) \trianglerighteq (2,2) \leftrightarrow \big\langle (2),(1,1)\big\rangle.$$

The problem does not arise for addition of δ when the partitions involved are $\big(\ell^-, \ell(\delta)\big)$-large, in the sense of Definition 3.1. Moreover, joining is better behaved. We establish this in a series of easy lemmas.

Lemma 9.5. *Fix $\ell^- \in \mathbb{N}_0$. Let α, γ and δ be partitions.*
(i) *If α is $\big(\ell^-, \ell(\delta)\big)$-large then $(\alpha+\delta)^- = \alpha^-$ and $(\alpha+\delta)^+ = \alpha^+ + \delta$.*
(ii) *If $\ell(\gamma) \leq \ell^-$ then $(\alpha \sqcup \gamma')^- = \alpha^- + \gamma^-$ and $(\alpha \sqcup \gamma')^+ = \alpha^+$.*

Proof. The most transparent proof uses Young diagrams. By hypothesis $[\alpha]$ contains the boxes (i,j) for $1 \leq i \leq \ell(\delta)$ and $1 \leq j \leq \ell^-$. Hence addition of δ creates no new boxes in the first ℓ^- columns of α. Similarly joining γ' creates no new boxes outside the first ℓ^- columns of α. □

Lemma 9.6. *Let κ^- and κ^+ be partitions. If α is a $\big(\ell(\kappa^-), \ell(\kappa^+)\big)$-large partition then in the $\ell(\kappa^-)$-decomposition of $\alpha \oplus (\kappa^-, \kappa^+)$ we have $\big(\alpha \oplus (\kappa^-,\kappa^+)\big)^-$ $= \alpha^- + \kappa^-$ and $\big(\alpha \oplus (\kappa^-,\kappa^+)\big)^+ = \alpha^+ + \kappa^+$. Moreover, adding and joining to α are commuting operations.*

Proof. This is immediate from Lemma 9.5. □

In particular, if $K \in \mathbb{N}$ and α is a $\big(\ell(\kappa^-), \ell(\kappa^+)\big)$-large partition then $\alpha \oplus (K-1)(\alpha^-, \alpha^+)$ is a partition having ℓ^--decomposition $K\big\langle \alpha^-, \alpha^+\big\rangle$. We use this remark in the proof of Lemma 13.23.

Lemma 9.7 (Twisted dominance order on large partitions is preserved by adjoining)**.** *Let κ^-, κ^+ be partitions. Set $\ell^- = \ell(\kappa^-)$. Suppose that α and β are $\big(\ell(\kappa^-), \ell(\kappa^+)\big)$ large. Then, working in the ℓ^--twisted dominance order, $\alpha \trianglelefteq \beta$ if and only if $\alpha \oplus (\kappa^-,\kappa^+) \trianglelefteq \beta \oplus (\kappa^-,\kappa^+)$.*

Proof. By Lemma 9.6 we have

$$\big(\lambda \oplus (\kappa^-,\kappa^+)\big)^- = \lambda^- + \kappa^-, \tag{9.7}$$

$$\big(\lambda \oplus (\kappa^-,\kappa^+)\big)^+ = \lambda^+ + \kappa^+. \tag{9.8}$$

Therefore it is equivalent to show that $\langle \alpha^-, \alpha^+\rangle \trianglelefteq \langle \beta^-, \beta^+\rangle$ if an only if $\big\langle \alpha^- + \kappa^-, \alpha^+ + \kappa^+\big\rangle \trianglelefteq \big\langle \beta^- + \kappa^-, \beta^+ + \kappa^+\big\rangle$, which is obvious. □

Lemma 9.8. *Fix $\ell^- \in \mathbb{N}_0$ and let $\ell^+ \in \mathbb{N}_0$. Let ω be a $(\ell^- + 1, \ell^+)$-large partition. If $\pi \trianglelefteq \omega$ then π is $(\ell^- + 1, \ell^+)$-large.*

Proof. By Remark 6.2, a partition α is $(\ell^- + 1, \ell^+)$-large if and only if $\ell(\alpha^+) \geq \ell^+$. Therefore $\ell(\omega^+) \geq \ell^+$ and since $\pi \trianglelefteq \omega$, Lemma 6.10 implies that $\ell(\pi^+) \geq \ell^+$. □

Lemma 9.9. *Fix $\ell^- \in \mathbb{N}_0$ and let $\ell^+ \in \mathbb{N}_0$. Let λ and ω be partitions such that ω is $(\ell^- + 1, \ell^+)$-large. If $\pi \in [\lambda, \omega]_{\trianglelefteq}$ then π is (ℓ^-, ℓ^+)-large. In particular λ is (ℓ^-, ℓ^+)-large.*

Proof. This is immediate from Lemma 9.8 since, as used in Remark 6.2, if a partition is $(\ell^- + 1, \ell^+)$-large then it is (ℓ^-, ℓ^+)-large. □

The hypothesis in the previous lemma cannot be weakened to the apparently more natural condition that ω is (ℓ^-,ℓ^+)-large. For example, both $(3,2)$ and $(2,2,1)$ are $(2,2)$-large, but since

$$(3,2)\leftrightarrow\big\langle(2,2),(1)\big\rangle,\ (3,1,1)\leftrightarrow\big\langle(3,1),(1)\big\rangle,\ (2,2,1)\leftrightarrow\big\langle(3,2),(1)\big\rangle,$$

the twisted interval $[(3,2),(2,2,1)]_\unlhd$ for the 2-twisted dominance order contains $(3,1,1)$ which is not $(2,2)$-large. See Remark 6.2 for one sign that the hypothesis in Lemma 9.9 is in fact the correct one. By Remark 3.2, any partition can be made $\big(\ell(\kappa^-)+1,\ell(\kappa^+)\big)$-large by sufficiently many applications of the adjoining map $\lambda\mapsto\lambda\oplus(\kappa^-,\kappa^+)$ so, as usual, any 'largeness' assumption are made without loss of generality.

The L bounds in the following proposition are defined in Definition 9.2. Remark 9.1 explains the different notations for intervals in the dominance order in the first two bounds in the lemma.

Proposition 9.10 (Partition Stability)**.** *Let κ^- and κ^+ be partitions. Fix $\ell^-=\ell(\kappa^-)$ and $\ell^+=\ell(\kappa^+)$. Let ω be a $\big(\ell^-+1,\ell^+\big)$-large partition and let $\lambda\unlhd\omega$ in the ℓ^--twisted dominance order. For each $M\in\mathbb{N}_0$, there is an injective map of intervals for the ℓ^--twisted dominance order*

$$\begin{aligned}F:\ &\big[\lambda\oplus M(\kappa^-,\kappa^+),\omega\oplus M(\kappa^-,\kappa^+)\big]_\unlhd\\ &\quad\hookrightarrow\big[\lambda\oplus(M+1)(\kappa^-,\kappa^+),\omega\oplus(M+1)(\kappa^-,\kappa^+)\big]_\unlhd\end{aligned}$$

defined, using the ℓ^--decomposition, by $F(\sigma)=\sigma\oplus(\kappa^-,\kappa^+)$. This map is bijective provided $M\geq L$ where L is the maximum of

- $\mathrm{L}\big([\lambda^-,\omega^-]^{(\ell^-)}_{\blacktriangleleft},\kappa^-\big)$,
- $\mathrm{L}\big([\lambda^+,\omega^++(|\lambda^+|-|\omega^+|)]_\unlhd,\kappa^+\big)$,
- $\big(\omega^+_1+\omega^+_2-2\lambda^+_1+2|\lambda^+|-2|\omega^+|\big)/(\kappa^+_1-\kappa^+_2)$,
- $\big(\max(\ell(\lambda^+),\ell^+)+|\omega^-|-|\lambda^-|-\omega^-_{\ell^-}\big)/\kappa^-_{\ell^-}$

where the third is omitted if $\kappa^+_1=\kappa^+_2$ and the fourth is omitted if $\kappa^-=\varnothing$.

Proof. By hypothesis the partitions $\omega\oplus M(\kappa^-,\kappa^+)$ and $\omega\oplus(M+1)(\kappa^-,\kappa^+)$ are $\big(\ell^-+1,\ell^+\big)$-large. Hence, by Lemma 9.9, every partition in each twisted interval is $\big(\ell^-,\ell^+\big)$-large. By the 'only if' direction of Lemma 9.7 it now follows that the map F on these twisted intervals preserves the ℓ^--twisted dominance order. Hence the image of the left-hand twisted interval under F is contained in the right-hand twisted interval. Set $N=M+1$ and suppose that M satisfies the inequalities in the proposition. By Lemma 9.6 and Lemma 6.7 we have $\tau\in\big[\lambda\oplus N(\kappa^-,\kappa^+),\omega\oplus N(\kappa^-,\kappa^+)\big]_\unlhd$ if and only if

(a) $\lambda^-+N\kappa^-\blacktriangleleft\tau^-\blacktriangleleft\omega^-+N\kappa^-$;

(b)(i) $\lambda^++N\kappa^+\unlhd\tau^++\big(|\lambda^+|+N|\kappa^+|-|\tau^+|\big)$ and $|\tau^+|\leq|\lambda^+|+N|\kappa^+|$;

(b)(ii) $\tau^+\unlhd\omega^++N\kappa^++\big(|\tau^+|-|\omega^+|-N|\kappa^+|\big)$ and $|\omega^+|+N|\kappa^+|\leq|\tau^+|$.

It is easily seen that (b)(i) and (b)(ii) are equivalent to the two conditions $\lambda^++N\kappa^+\unlhd\tau^++\big(|\lambda^+|+N|\kappa^+|-|\tau^+|\big)\unlhd\omega^++N\kappa^++\big(|\lambda^+|-|\omega^+|\big)$ and

$$|\omega^+|+N|\kappa^+|\leq|\tau^+|\leq|\lambda^+|+N|\kappa^+|. \tag{9.9}$$

Note that by definition of the ℓ^--decomposition (see Definition 6.1), λ^- and ω^- have at most ℓ^- parts, where $\ell^-=\ell(\kappa^-)$. Thus (a), (b)(i) and (b)(ii) hold if and only if (9.9) holds and

$$\tau^- \in [\lambda^- + N\kappa^-, \omega^- + N\kappa^-]^{(\ell^-)}_{\blacktriangleleft}$$

and

$$\tau^+ + \bigl(|\lambda^+| + N|\kappa^+| - |\tau^+|\bigr) \in \bigl[\lambda^+ + N\kappa^+, \omega^+ + N\kappa^+ + (|\lambda^+| - |\omega^+|)\bigr]_{\trianglelefteq}.$$

By Proposition 9.3, the map

$$[\lambda^- + (N-1)\kappa^-, \omega^- + (N-1)\kappa^-]^{(\ell^-)}_{\blacktriangleleft} \to [\lambda^- + N\kappa^-, \omega^- + N\kappa^-]^{(\ell^-)}_{\blacktriangleleft}$$

defined by adding κ^- is bijective if $N - 1 \geq \mathrm{L}\bigl([\lambda^-, \omega^-]^{(\ell^-)}_{\blacktriangleleft}, \kappa^-\bigr)$, as we have assumed. Similarly, the map

$$\begin{aligned}[\lambda^+ + (N-1)\kappa^+, &\omega^+ + (N-1)\kappa^+ + (|\lambda^+| - |\omega^+|)]_{\trianglelefteq} \\ &\to [\lambda^+ + N\kappa^+, \omega^+ + N\kappa^+ + (|\lambda^+| - |\omega^+|)]_{\trianglelefteq}\end{aligned}$$

defined by adding κ^+ is bijective if $N-1 \geq \mathrm{L}\bigl([\lambda^+, \omega^+ + (|\lambda^+| - |\omega^+|)]_{\trianglelefteq}, \kappa^+\bigr)$, again as we have assumed. Hence there exist unique partitions

$$\sigma^- \in [\lambda^- + (N-1)\kappa^-, \omega^- + (N-1)\kappa^-]^{(\ell^-)}_{\blacktriangleleft} \tag{9.10}$$

such that $\tau^- = \sigma^- + \kappa^-$ and

$$\vartheta \in [\lambda^+ + (N-1)\kappa^+, \omega^+ + (N-1)\kappa^+ + (|\lambda^+| - |\omega^+|)\bigr]_{\trianglelefteq} \tag{9.11}$$

such that

$$\tau^+ + (|\lambda^+| + N|\kappa^+| - |\tau^+|) = \vartheta + \kappa^+. \tag{9.12}$$

The unique integer sequence σ^+ such that $\sigma^+ + \kappa^+ = \tau^+$ is

$$\sigma^+ = \vartheta - (|\lambda^+| + N|\kappa^+| - |\tau^+|). \tag{9.13}$$

We shall show that σ^+ is a partition, provided N is sufficiently large. Suppose first of all that $\kappa^+_1 = \kappa^+_2$. Then by (9.12), $\vartheta_1 - \vartheta_2 = \tau^+_1 + (|\lambda^+| + N|\kappa^+| - |\tau^+|) - \tau^+_2 \geq |\lambda^+| + N|\kappa^+| - |\tau^+|$ and hence by (9.13), $\sigma^+_1 - \sigma^+_2 \geq 0$, with no condition on N. Now suppose that $\kappa^+_1 > \kappa^+_2$. By (9.11), we have

$$\begin{aligned}\vartheta_1 - \vartheta_2 &= 2\vartheta_1 - (\vartheta_1 + \vartheta_2) \\ &\geq 2\bigl(\lambda^+_1 + (N-1)\kappa^+_1\bigr) \\ &\qquad - \bigl(\omega^+_1 + \omega^+_2 + (N-1)(\kappa^+_1 + \kappa^+_2) + (|\lambda^+| - |\omega^+|)\bigr) \\ &= 2\lambda^+_1 - \omega^+_1 - \omega^+_2 + (N-1)(\kappa^+_1 - \kappa^+_2) - |\lambda^+| + |\omega^+|\end{aligned}$$

By (9.13),

$$\begin{aligned}\sigma^+_1 - \sigma^+_2 &= \vartheta_1 - \vartheta_2 - |\lambda^+| - N|\kappa^+| + |\tau^+| \\ &\geq \vartheta_1 - \vartheta_2 - |\lambda^+| + |\omega^+| \\ &\geq 2\lambda^+_1 - \omega^+_1 - \omega^+_2 + (N-1)(\kappa^+_1 - \kappa^+_2) - 2|\lambda^+| + 2|\omega^+|\end{aligned}$$

where the middle line follows from the first inequality in (9.9) that $|\omega^+| + N|\kappa^+| \leq |\tau^+|$ and the third line by substituting the expression for $\vartheta_1 - \vartheta_2$ just found. Hence it suffices if

$$N - 1 \geq \frac{\omega^+_1 + \omega^+_2 - 2\lambda^+_1 + 2|\lambda^+| - 2|\omega^+|}{\kappa^+_1 - \kappa^+_2}$$

which, setting $M = N - 1$, is the third condition.

We have now defined partitions σ^- and σ^+ such that, provided $\langle \sigma^-, \sigma^+ \rangle$ is a well-defined ℓ^--decomposition, the partition σ defined by $\sigma \leftrightarrow \langle \sigma^-, \sigma^+ \rangle$ satisfies $\sigma \oplus (\kappa^-, \kappa^+) = \tau$. If $\ell^- = 0$ (or equivalently, $\kappa^- = \varnothing$) this is immediate, and similarly it is immediate if $\kappa^+ = \varnothing$. We may therefore assume that $\ell^- \geq 1$ and $\kappa^+ \neq \varnothing$. We then require $\sigma^-_{\ell^-} \geq \ell(\sigma^+)$. By (9.10) we have $\sigma^- \blacktriangleleft \omega^- + (N-1)\kappa^-$. Define an integer sequence ψ by $\psi_j = \sigma^-_j$ for $1 \leq j < \ell^-$ and $\psi_{\ell^-} = \sigma^-_{\ell^-} + |\omega^-| + (N-1)|\kappa^-| - |\sigma^-|$. Since $\sigma^- \blacktriangleleft \omega^- + (N-1)\kappa^-$ by (9.10), σ^- is a weight, having non-negative entries. Moreover, after this equalization of sizes, we have $\psi \trianglelefteq \omega^- + (N-1)\kappa^-$. Since each side has at most ℓ^- parts, it follows from the dominance order that $\psi_{\ell^-} \geq \omega^-_{\ell^-} + (N-1)\kappa^-_{\ell^-}$. Now using that $|\omega^-| + (N-1)|\kappa^-| - |\sigma^-| = |\omega^-| + N|\kappa^-| - |\tau^-| \leq \big(|\omega^-| + N|\kappa^-|\big) - \big(|\lambda^-| + N|\kappa^-|\big) = |\omega^-| - |\lambda^-|$ we obtain

$$\sigma^-_{\ell^-} \geq \omega^-_{\ell^-} + (N-1)\kappa^-_{\ell^-} - (|\omega^-| - |\lambda^-|).$$

By (9.11) we have $\vartheta \trianglerighteq \lambda^+ + (N-1)\kappa^+$, and since $\ell(\kappa^+) = \ell^+$ we have

$$\ell(\sigma^+) = \ell(\vartheta) \leq \max(\ell(\lambda^+), \ell^+).$$

Therefore, comparing the two previous displayed equations, a sufficient condition for σ to be well-defined is

$$\omega^-_{\ell^-} + (N-1)\kappa^-_{\ell^-} - (|\omega^-| - |\lambda^-|) \geq \max(\ell(\lambda^+), \ell^+).$$

Rearranging and, as before, setting $M = N - 1$, this becomes the fourth condition. □

This shows that twisted intervals for the ℓ^--twisted dominance order, defined for suitable large partitions, satisfy condition (i) in the definition of a stable partition system (Definition 7.1) for the map F in Proposition 9.10 (Partition Stability). We remark that the example in §8.4 shows one case where the third bound, required in the middle part of the proof, is the only bound that is tight and Example 9.13 below shows that the most technical fourth bound may also be the only bound that is tight.

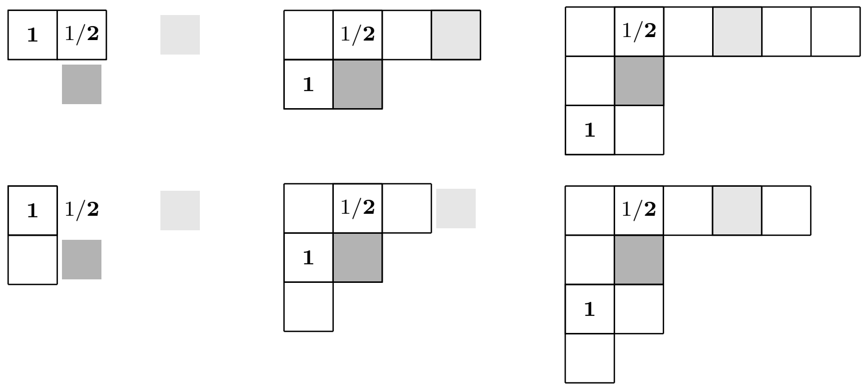
9.3. **Positions for tableau stability.** We must now verify condition (b) in the definition of a stable partition system (Definition 7.1). The critical positions in tableaux are defined below. In this section, only the case where $\mu_\star = \varnothing$ is needed: the definition is used in full generality in §11.5 below.

Definition 9.11. Let κ^- and κ^+ be partitions. Fix $\ell^- = \ell(\kappa^-)$ and let $\ell^+ = \ell(\kappa^+)$. Let $\mu/\mu_\star$ be a skew partition. For $1 \leq r^- \leq \ell^-$ and $1 \leq r^+ \leq \ell^+$,

(a) the r^--*top position* of $\mu/\mu_\star$ is $\big(\max(\ell(\mu_\star), \ell(\mu^+), \ell^+, \mu^-_{r^-+1}), r^-\big)$,
(b) the r^--*bottom position* of $\mu/\mu_\star$ is $(k + \kappa^-_{r^-} - \kappa^-_{r^-+1}, r^-)$ where (k, r^-) is the r^--top position of $\mu/\mu_\star$.
(c) the r^+-*left position* of $\mu/\mu_\star$ is $\big(r^+, \ell^- + \max(a(\mu_\star), \mu^+_{r^++1})\big)$,
(d) the r^+-*right position* of $\mu/\mu_\star$ is $(r^+, k + \kappa^+_{r^+} - \kappa^+_{r^++1})$ where (r^+, k) is the r^+-left position of $\mu/\mu_\star$.

Note that if $\mu_{r^+} < \ell^-$ then $\mu^+_{r^++1} = 0$ and so the r^+-left position of $\mu/\mu_\star$ is $(r^+, \ell^- + a(\mu_\star))$ and is not contained in $[\mu]$. Similarly, if $\mu_\star = \varnothing$ and $\mu^+ = \varnothing$ and $\kappa^+ = \varnothing$ then since μ^- has at most ℓ^- parts, the ℓ^--top position is $(0, \ell^-)$. We therefore refer to 'positions' rather than 'boxes'.

Example 9.12. Take $\kappa^- = (1,1)$ and $\kappa^+ = (2)$. The map $F : \mathrm{Par} \rightarrow \mathrm{Par}$ in Proposition 9.10 is defined by $F(\sigma) = \sigma \oplus \big((1,1),(2)\big) = \sigma + (2) \sqcup (2)$. The numbers in the diagrams below show the 1-top, 2-top and 1-left positions in the partitions obtained from (2) and $(1,1)$ by adjoining according to F. Following our usual convention, top positions, relevant to the insertion of negative entries, are marked by bold numbers. For instance the 2-top position is $(1,2)$ in every partition. The 2-bottom and 1-right positions are indicated by shading;



Since $\kappa^-_1 = \kappa^-_2$ the 1-top and 1-bottom positions coincide in every case. (We shall see in Definition 9.14 that this makes them irrelevant to our application.) Since $\kappa^-_2 - \kappa^-_3 = 1 - 0 = 1$, the 2-bottom position is always one position below the 2-top position and since $\kappa^+_1 - \kappa^+_2 = 2 - 0 = 2$, the 1-right position is always two positions right of the 1-left position.

For a further example in the general skew case, also showing the behaviour when $\ell(\mu^+) > \ell(\kappa^+)$, see Example 11.5.

9.4. **The $\mathcal{F}$ insertion map on tableaux.** We now show how these positions can be used to define a bijection between semistandard signed tableaux. We admit the following results are technical, and so we give two substantial examples. See also Example 7.2 and §8.3 for two earlier bijections; both can now be seen to be instances of $\mathcal{F}$.

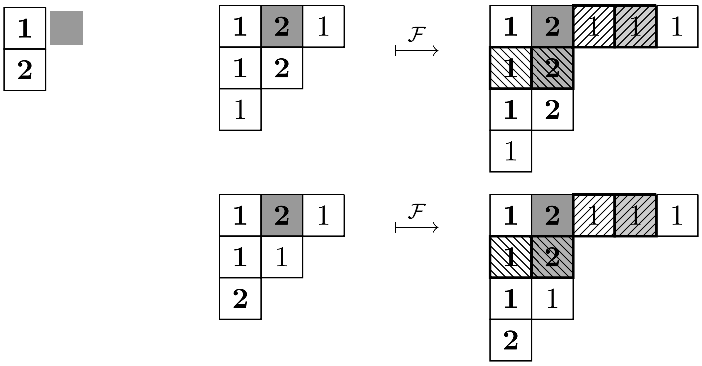
Example 9.13. Consider the twisted intervals

$$\mathcal{P}^{(M)} = \big[(4,2) + (2M) \sqcup (2^M), (3,2,1) + (2M) \sqcup (2^M)\big]_\unlhd.$$

for the 2-twisted dominance order. By Proposition 9.10 (Partition Stability), the map $F : \mathcal{P}^{(M)} \rightarrow \mathcal{P}^{(M+1)}$ defined by $\lambda \mapsto \lambda \oplus \big((1,1),(2)\big)$ is bijective for $M \geq 0$. (Note that $(3,2,1)$ is $(3,1)$-large; the four bounds on M are respectively $M \geq -1$, $M \geq -1$, $M \geq -\frac{1}{2}$ and $M \geq 0$.) This gives a bijection between the row and column labels of the matrices $K(M)$ in condition (b) of a stable partition system (Definition 7.1), as indicated below. We include the set $\mathcal{P}^{(-1)} = [(2),(1,1)]_\unlhd$ below: even though (2) is not $(3,1)$-large, the proof of Proposition 9.10 still applies; the bounds on M are now $M \geq 0$, $M \geq 0$, $M \geq 0$ and $M \geq 1$, so the necessary restriction on M comes from the technical final paragraph of the proof.

$$
\begin{array}{r} \\ \langle(2),\varnothing\rangle \\ \langle(1,1),\varnothing\rangle \end{array}
\begin{array}{c} \begin{array}{cc} (1,1) & (2) \end{array} \\ \begin{pmatrix} 1 & \cdot \\ 1 & 1 \end{pmatrix} \end{array}
\qquad
\begin{array}{r} \\ \langle(3,2),(1)\rangle \\ \langle(3,1),(2)\rangle \\ \langle(2,2),(2)\rangle \end{array}
\begin{array}{c} \begin{array}{ccc} (3,2,1) & (4,1,1) & (4,2) \end{array} \\ \begin{pmatrix} 1 & \cdot & \cdot \\ 1 & 1 & \cdot \\ 2 & 1 & 1 \end{pmatrix} \end{array}
\qquad
\begin{array}{r} \\ \langle(4,3),(3)\rangle \\ \langle(4,2),(4)\rangle \\ \langle(3,3),(4)\rangle \end{array}
\begin{array}{c} \begin{array}{ccc} (5,2,2,1) & (6,2,1,1) & (6,2,2) \end{array} \\ \begin{pmatrix} 1 & \cdot & \cdot \\ 1 & 1 & \cdot \\ 2 & 1 & 1 \end{pmatrix} \end{array}
$$

The tableaux enumerated by the bottom left matrix entries of 1, 2 and 2 are shown below.



We saw in Example 9.12 that the 2-top and 1-left positions of $(3,2,1)$ are both $(1,2)$; these positions are shaded dark grey in all tableaux. Insertion of $\boxed{\mathbf{1}}\boxed{\mathbf{2}}$ in the two positions $(2,1)$, $(2,2)$ below the 2-top position, moving each box in columns 1 and 2 one row down, gives a semistandard tableau. Similarly insertion of $\boxed{1}\boxed{1}$ into the positions $(1,3)$, $(1,4)$, right of the 1-left position, moving each box in row 1 two columns right, again gives a semistandard tableau. These operations commute. The inverse map is defined by deleting $\boxed{\mathbf{1}}\boxed{\mathbf{2}}$ and $\boxed{1}\boxed{1}$ from the 2-bottom and 1-right positions in the $(5,2,2,1)$-tableau; these positions are again shaded and the newly inserted boxes which should be deleted are hatched. We therefore have a bijection

$$\mathrm{SSYT}\big((3,2,1)\big)_{((2,2),(2))} \xrightarrow{\mathcal{F}} \mathrm{SSYT}\big((5,2,2,1)\big)_{((3,3),(4))}.$$

This bijection establishes, via Lemma 5.3 (Twisted Kostka Numbers), that the bottom-left entries $\langle e_{(2,2)}h_{(2)}, s_{(3,2,1)}\rangle$ and $\langle e_{(3,3)}h_{(4)}, s_{(5,2,2,1)}\rangle$ of the final two matrices above are equal. This bijection is generalized in Definition 9.14: in general $\kappa^-_r - \kappa^-_{r+1}$ rows of length r and $\kappa^+_r - \kappa^+_{r+1}$ columns of length r are inserted/deleted. This feature may be seen in this example: for instance, since $\kappa^-_1 = \kappa^-_2$, there was no need to consider the 1-top and 1-bottom positions.

Generalizing this example, we now define the insertion map $\mathcal{F}$ in the general skew case; this generality is needed later in the proof of Proposition 11.13. Note that when $\sigma_\star = \varnothing$ then the only hypothesis needed is that σ is $\big(\ell(\kappa^-), \ell(\kappa^+)\big)$-large. Recall from Definition 3.3 that $\mathrm{YT}\big(\sigma/\sigma_\star\big)$ is the set of signed tableaux of shape $\sigma/\sigma_\star$; note the tableaux in $\mathrm{YT}\big(\sigma/\sigma_\star\big)$ are not necessarily semistandard.

Definition 9.14. Let κ^- and κ^+ be partitions. Let $\sigma/\sigma_\star$ be a $\big(\ell(\kappa^-) + a(\sigma_\star), \ell(\kappa^+)\big)$-large and $\big(\ell(\kappa^-), \ell(\mu_\star)\big)$-large skew partition. Define

$$\mathcal{F} : \mathrm{SSYT}(\sigma/\sigma_\star) \to \mathrm{YT}\big(\sigma/\sigma_\star \oplus (\kappa^-, \kappa^+)\big)$$

by performing (1) then (2) below:

1. starting with $r^- = 1$ and finishing with $r^- = \ell(\kappa^-)$, insert $\kappa^-_{r^-} - \kappa^-_{r^-+1}$ new rows each with entries $-1, \ldots, -r^-$, each with their rightmost box immediately below the r^--top position of σ;
2. starting with $r^+ = 1$ and finishing with $r^+ = \ell(\kappa^+)$, insert $\kappa^+_{r^+} - \kappa^+_{r^++1}$ new columns each with entries $1, \ldots, r^+$, each with their bottom box immediate right of the r^+-left position of σ.

If $\kappa^-_{r^-} = \kappa^-_{r^-+1}$ or $\kappa^+_{r^+} = \kappa^+_{r^++1}$ then there is nothing to do in that step.

The partitions κ^- and κ^+ will always be clear from context. It has to be checked that $\mathcal{F}$ is well-defined (meaning that the insertions give a tableau of skew partition shape), but as we shall see in Lemma 9.18, this is not hard to prove. Our aim, achieved in Proposition 9.19, is to show that

$$\mathcal{F} : \mathrm{SSYT}\big(\sigma\big)_{(\pi^-,\pi^+)} \to \mathrm{SSYT}\big(\sigma \oplus (\kappa^-, \kappa^+)\big)_{(\pi^-+\kappa^-,\pi^++\kappa^+)}.$$

is a well-defined bijection for σ and π suitable elements of a twisted interval for the $\ell(\kappa^-)$-twisted dominance order. Example 9.13 shows the special case where $\sigma = (3, 2, 1)$ and $\pi = \big\langle (2, 2), (2) \big\rangle \leftrightarrow (4, 2)$.

To help guide the reader through the remaining technicalities we give a further 'unsigned' example below. This example, like many others in this paper, was created with the help of the Magma code mentioned in the introduction using `TwistedIntervalInjectionM( [], [3,3,1], [2,1,1] : q := [4], NSteps := 2)`; varying the parameters to `[1,1]`, `[2]`, `[4,2]`, `q := [3,2,1]` gives the bijection in Example 9.13.

Example 9.15. Take $\kappa^- = \varnothing$, $\kappa^+ = (3, 3, 1)$ so $\ell^- = 0$ and $\ell^+ = 3$. By Definition 9.2, $\mathrm{L}\big([(2, 1, 1), (4)]_\trianglelefteq, (3, 3, 1)\big) = 1$; the only strictly positive quantity comes from the case $k = 2$. (Note that we disregard the case $k = 1$ since $\kappa_1 = \kappa_2$.) Therefore, by Proposition 9.3, the $\mathcal{F}$ map adding $(3, 3, 1)$ is an injection

$$[(2, 1, 1), (4)]_\trianglelefteq \xrightarrow{+(3,3,1)} [(5, 4, 2), (7, 3, 1)]_\trianglelefteq \xrightarrow{+(3,3,1)} [(8, 7, 3), (10, 6, 2)]_\trianglelefteq$$

and the second map is a bijection. (We remark that Proposition 9.10 could also be used; the intervals are then interpreted for the 0-twisted dominance order, which by Remark 6.8 is the usual dominance order on partitions, and the additional bounds are, as expected, irrelevant.) Figure 9.1 shows the Kostka matrices $\langle h_\pi, s_\sigma \rangle$ for π, σ in each interval; and several features of the $\mathcal{F}$ bijection

$$\mathrm{SSYT}\big((7, 3, 1)\big)_{(\varnothing,(5,4,2))} \xrightarrow{\mathcal{F}} \mathrm{SSYT}\big((10, 6, 2)\big)_{(\varnothing,(8,7,3))}$$

establishing the equality $\langle h_{(5,4,2)}, s_{(7,3,1)}\rangle = \langle h_{(8,7,3)}, s_{(10,6,2)}\rangle$ of the bottom-left matrix entries of 2.

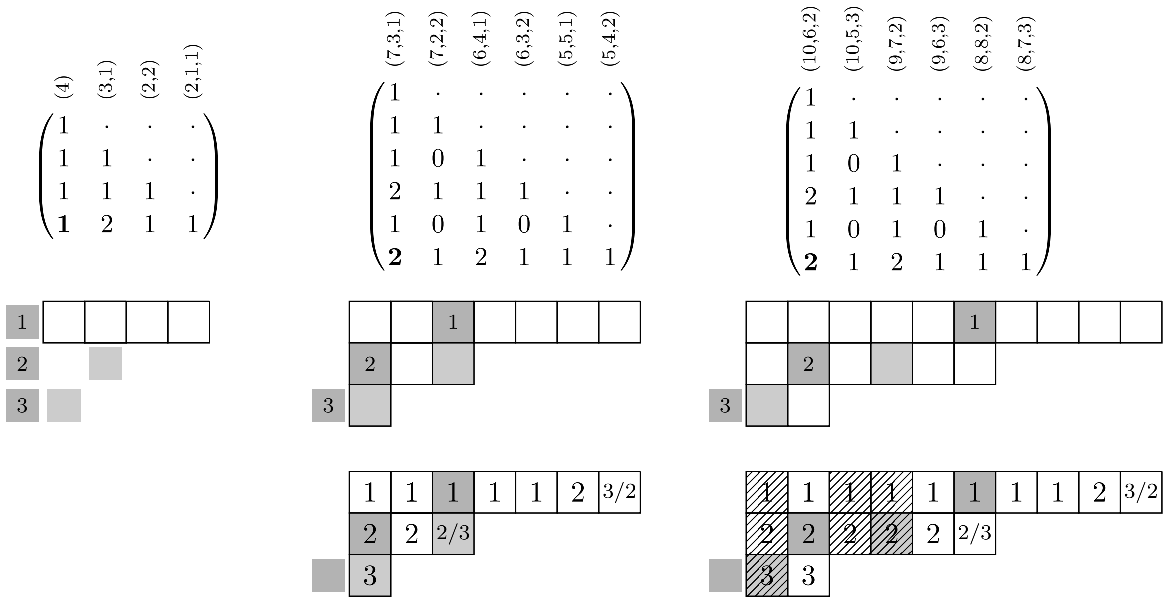


FIGURE 9.1. Kostka matrices for the intervals $[(2,1,1),(4)]_{\trianglelefteq}$, $[(5,4,2),(7,3,1)]_{\trianglelefteq}$, $[(8,7,3),(10,6,2)]_{\trianglelefteq}$ showing the bijection F (see Proposition 9.10) between the two larger intervals defined by adding $(3,3,1)$. Observe that while $(7,3,1) = (4)+(3,3,1)$ and $(5,4,2) = (2,1,1)+(3,3,1)$, we have $|\operatorname{SSYT}\big((4)\big)_{(\varnothing,(2,2,1))}| = 1$ but $|\operatorname{SSYT}\big((7,3,1)\big)_{(\varnothing,(5,4,2))}| = 2$ so in the step from the first interval to the second we do not have tableau stability, in the sense of Proposition 9.19, *even if we consider only those partitions in the image of the addition map.* Below the matrices we show the 1-, 2- and 3-left and 1-, 2- and 3-right position for the partitions $(4), (7,3,1), (10,6,2)$; note the 1-left and 1-right positions coincide. At the bottom we show the bijection $\mathcal{F} : \operatorname{SSYT}\big((7,3,1)\big)_{(\varnothing,(5,4,2))} \xrightarrow{\mathcal{F}} \operatorname{SSYT}\big((10,6,2)\big)_{(\varnothing,(8,7,3))}$ defined by inserting two columns of length 2 immediately right of the 2-left position $(2,1)$ and a single column of height 3 immediately right of the 3-left position $(3,0)$, using 2/3 and 3/2 to indicate the two boxes that have a choice of entry. The shading and hatching conventions are as in Example 9.13. This gives a bijective proof of the equality of the bottom left entries of 2 in the two larger matrices marked in bold.

9.5. **Technical lemmas on positions.** In the following lemma we use the bounds $\mathrm{L}\big([\lambda,\omega]^{(p)}_{\blacktriangleleft},\kappa\big)$ and $\mathrm{L}\big(\big[\lambda^+,\omega^+ + (|\lambda^+| - |\omega^+|)\big],\kappa^+\big)_{\trianglelefteq}$, defined in Definition 9.2. (See Remark 9.1 for the difference in notation.) We remark that the bounds in the following lemma are the first, second and fourth from Proposition 9.10 (Partition Stability), so whenever the conditions for this proposition hold, so do the conditions for this lemma.

Lemma 9.16. *Let κ^- and κ^+ be partitions. Fix $\ell^- = \ell(\kappa^-)$ and $\ell^+ = \ell(\kappa^+)$. Let λ and ω be (ℓ^-, ℓ^+)-large partitions and let $\lambda \trianglelefteq \omega$ in the ℓ^--twisted dominance order. Let L be the maximum of the twisted interval bounds*
- $\mathrm{L}\big([\lambda^-, \omega^-]^{(\ell^-)}_{\blacktriangleleft}, \kappa^-\big)$,
- $\mathrm{L}\big(\big[\lambda^+, \omega^+ + (|\lambda^+| - |\omega^+|)\big]_{\trianglelefteq}, \kappa^+\big)$
- $\big(\max(\ell(\lambda^+), \ell^+) + |\omega^-| - |\overline{\lambda}^-| - \omega^-_{\ell^-}\big)/\kappa^-_{\ell^-}$

omitting the third if $\kappa^- = \varnothing$. Let σ and π be partitions in the interval

$$\big[\lambda \oplus M(\kappa^-, \kappa^+), \omega \oplus M(\kappa^-, \kappa^+)\big]_{\trianglelefteq}$$

for the ℓ^--twisted dominance order such that σ is (ℓ^-, ℓ^+)-large. Let $t \in \mathrm{SSYT}(\sigma)_{(\pi^-, \pi^+)}$. If $M - 1 \geq L$ then
- (i) *the r^--bottom position of t contains $-r^-$ if $r^- < \ell^-$ and $\kappa^-_{r^-} > \kappa^-_{r^-+1}$;*
- (ii) *if $\kappa^- \neq \varnothing$ then the ℓ^--bottom position of t contains $-\ell^-$;*
- (iii) *if $\kappa^- \neq \varnothing$ and $\kappa^+ \neq \varnothing$ then the box (ℓ^+, ℓ^-) of t contains $-\ell^-$;*
- (iv) *the r^+-right position of t contains r^+ if $r^+ < \ell^+$ and $\kappa^+_{r^+} > \kappa^+_{r^++1}$.*
- (v) *the ℓ^+-right position of t contains ℓ^+.*

Moreover if $M \geq L$ then the same results hold replacing 'bottom' with 'top' and 'right' with 'left', except that
- (ii) *if $\sigma^+ = \varnothing$ and $\kappa^+ = \varnothing$ then the ℓ^--top position is $(0, \ell^-)$;*
- (iv) *and* (v) *if $\sigma_{r^++1} \leq \ell^-$, and so the r^+-left position is (r^+, ℓ^-), then it contains a negative entry.*

Proof. First note that, by Lemma 9.6, we have $\big(\lambda \oplus M(\kappa^-, \kappa^+)\big)^- = \lambda^- + M\kappa^-$ and three further analogous equations replacing $-$ with $+$ or λ with ω. We also record a key observation on where negative entries lie in t:

($-$) The negative entries of t lie in the boxes in $[\alpha]$ where α is a subpartition of σ such that $a(\alpha) \leq \ell^-$ and $|\alpha| = |\pi^-|$.

For (i), there is nothing to prove if $\kappa^- = \varnothing$. Let $r^- < \ell^-$. Since σ is (ℓ^-, ℓ^+)-large, we have $\sigma^-_{r^-+1} \geq \ell^+$. Hence the r^--top position of t is $(\sigma^-_{r^-+1}, r^-)$. Suppose for a contradiction that this position has either a positive entry, or some $-s$ with $-s > -r^-$ in the order in Definition 3.7, meaning that $s > r$. In either case, ($-$) implies that the total number of entries of t in the set $\{-1, \ldots, -r^-\}$ is at most $\sigma^-_1 + \cdots + \sigma^-_{r^--1} + \sigma^-_{r^-+1} - 1$. (At this point we suggest that the reader refers to Figure 9.2 to see ($-$) graphically: it is also helpful to note that $\sigma^-_j = \sigma'_j$ for $1 \leq j \leq \ell^-$. See Figure 6.1 for a reminder of this notation.) On the other hand, t has exactly $\pi^-_1 + \cdots + \pi^-_{r^--1} + \pi^-_{r^-}$ such entries. Hence

$$\sum_{j=1}^{r^--1} \sigma^-_j + \sigma^-_{r^-+1} > \sum_{j=1}^{r^-} \pi^-_j. \tag{9.14}$$

Using $\pi \trianglerighteq \lambda \oplus M(\kappa^-, \kappa^+)$ and so, by Lemma 6.7(a), $\pi^- \blacktriangleright\!\!\!\!\underline{\ \ } \lambda^- + M\kappa^-$ we have $\sum_{j=1}^{r^-} \pi^-_j \geq \sum_{j=1}^{r^-} \lambda^-_j + M \sum_{j=1}^{r^-} \kappa^-_j$. Hence

$$\sum_{j=1}^{r^--1} \sigma^-_j + \sigma^-_{r^-+1} > \sum_{j=1}^{r^-} \lambda^-_j + M \sum_{j=1}^{r^-} \kappa^-_j. \tag{9.15}$$

Since $\sigma \in [\lambda \oplus M(\kappa^-, \kappa^+), \omega \oplus M(\kappa^-, \kappa^+)]_{\trianglelefteq}$ we have $\sigma \trianglelefteq \omega \oplus M(\kappa^-, \kappa^+)$ and so by Lemma 6.7(a), $\sigma^- \blacktriangleleft\!\!\!\!\underline{\ \ } \ \omega^- + M\kappa^-$, we also have for each k,

$$\sum_{j=1}^{k} \lambda_j^- + M \sum_{j=1}^{k} \kappa_j^- \leq \sum_{j=1}^{k} \sigma_j^- \leq \sum_{j=1}^{k} \omega_j^- + M \sum_{j=1}^{k} \kappa_j^-. \tag{9.16}$$

Taking $k = r^-$ in (9.15) and $k = r^- + 1$ in (9.16) the right-hand inequality and subtracting we get $\sigma_{r+1}^- \leq \sum_{j=1}^{r^-+1} \omega_j^- - \sum_{j=1}^{r^-} \lambda_j^- + M\kappa_{r^-+1}^-$. Hence by another use of the right-hand inequality in (9.16) taking $k = r^- - 1$,

$$\sum_{j=1}^{r^--1} \sigma_j^- + \sigma_{r^-+1}^- \leq 2 \sum_{j=1}^{r^--1} \omega_j^- + \omega_{r^-}^- + \omega_{r^-+1}^- - \sum_{j=1}^{r^-} \lambda_j^- + M \sum_{j=1}^{r^--1} \kappa_j^- + M\kappa_{r^-+1}^-.$$

Now (9.15) and the previous inequality imply

$$2 \sum_{j=1}^{r^--1} \omega_j^- + \omega_{r^-}^- + \omega_{r^-+1}^- - 2 \sum_{j=1}^{r^-} \lambda_j^- > M(\kappa_{r^-}^- - \kappa_{r^-+1}^-). \tag{9.17}$$

Taking $k = r^-$ in the definition of $\mathrm{L}\big([\lambda^-, \omega^-]^{(\ell^-)}_{\underline{\blacktriangleleft}}, \kappa^-\big)$ in Definition 9.2 we get $2 \sum_{j=1}^{r^--1} \omega_j^- + \omega_k^- + \omega_{k+1}^- - 2 \sum_{j=1}^{r^-} \lambda_j^- \leq M(\kappa_{r^-} - \kappa_{r^-+1})$. This contradicts (9.17). Hence, provided we have the first condition on M that $M \geq \mathrm{L}\big([\lambda^-, \omega^-]^{(\ell^-)}_{\underline{\blacktriangleleft}}, \kappa^-\big)$, (i) holds for top positions.

The r^--bottom position lies $\kappa_{r^-}^- - \kappa_{r^-+1}^-$ boxes below the r^--top position. Supposing similarly that it does not contain r^- we deduce that the total number of entries of t in the set $\{-1, \ldots, -r^-\}$ is at most $\kappa_{r^-}^- - \kappa_{r^-+1}^-$ plus the left-hand side of (9.14). Running the same argument, using the same inequalities (9.15) and (9.16), we obtain (9.17) with $\kappa_{r^-}^- - \kappa_{r^-+1}^-$ subtracted from the right hand side, which is therefore $(M-1)(\kappa_{r^-}^- - \kappa_{r^-+1}^-)$. We then get a contradiction as before from $M - 1 \geq \mathrm{L}\big([\lambda^-, \omega^-]^{(\ell^-)}_{\underline{\blacktriangleleft}}, \kappa^-\big)$.

For (ii), we may assume that $\kappa^- \neq \varnothing$; then by Definition 9.11, the ℓ^--*top position* of t is $\big(\max(\ell(\sigma^+), \ell^+), \ell^-\big)$. If $\sigma^+ = \varnothing$ then we are in the exceptional case at the end of the statement of the lemma; otherwise, since σ is (ℓ^-, ℓ^+)-large, this is a box of t. Suppose for a contradiction that this box does not contain $-\ell^-$. The analogue of (9.15) is

$$\sum_{j=1}^{\ell^--1} \sigma_j^- + \max(\ell(\sigma^+), \ell^+) > \sum_{j=1}^{\ell^-} \lambda_j^- + M \sum_{j=1}^{\ell^-} \kappa_j^- = |\lambda^-| + M|\kappa^-|.$$

By

$$\sum_{j=1}^{\ell^--1} \sigma_j^- \leq \sum_{j=1}^{\ell^--1} \omega_j^- + M \sum_{j=1}^{\ell^--1} \kappa_j^- = |\omega^-| - \omega_{\ell^-}^- + M|\kappa^-| - M\kappa_{\ell^-}^- \tag{9.18}$$

obtained from the upper bound in (9.16) we deduce $|\lambda^-| - \max(\ell(\sigma^+), \ell^+) < |\omega^-| - \omega_{\ell^-}^- - M\kappa_{\ell^-}^-$. By Lemma 6.10, since $\sigma \trianglerighteq \lambda$, we have $\ell(\sigma^+) \leq \ell(\lambda^+)$. Therefore

$$M\kappa_{\ell^-}^- < |\omega^-| - |\lambda^-| - \omega_{\ell^-}^- + \max(\ell(\lambda^+), \ell^+). \tag{9.19}$$

This contradicts the third bound in the statement of this lemma, namely $M \geq \big(\max(\ell(\lambda^+), \ell^+) + |\omega^-| - |\lambda^-| - \omega^-_{\ell^-}\big)/\kappa^-_{\ell^-}$. This proves (ii) for the top position. The modifications for the ℓ^--bottom position are precisely analogous to (i), leading to (9.18) with $\kappa^-_{\ell^-}$ subtracted from the right-hand side, and (9.19) with M replaced by $M-1$, as required.

Part (iii) follows from (ii) because the ℓ^--top position is (k, ℓ^-) where $k \geq \ell^+$, and since this position contains $-\ell^-$, so does position (ℓ^+, ℓ^-). This argument is indicated in the caption to Figure 9.2.

For (iv) and (v), we first note that if $\kappa^+ = \varnothing$ then there is nothing to prove. Suppose that $\kappa^+ \neq \varnothing$. By (iii) we have

(+) The positive entries of t in $\{1, 2, \ldots, \ell^+\}$ lie either in boxes in the first ℓ^- columns of t in rows strictly below row ℓ^+, or in boxes (i, j) with $i \leq \ell^+$ and $j > \ell^-$.

This restrictions from (−) and (+) are shown diagrammatically in Figure 9.2.

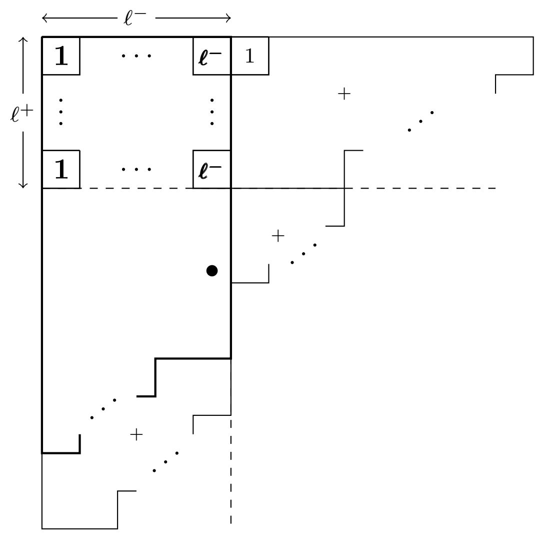


FIGURE 9.2. Entries in a tableau $t \in \mathrm{SSYT}(\sigma)_{(\pi^-, \pi^+)}$ when σ is (ℓ^-, ℓ^+)-large showing the conditions (−) and (+) in the proof of Lemma 9.16. The positive entries not in the first ℓ^+ rows lie in the regions marked +. Note that by (iii) in Lemma 9.16 the box in position (ℓ^-, ℓ^+) contains $-\ell^-$. We have shown the case where $\ell(\sigma^+) > \ell^+$, and so the ℓ^--top position is $\big(\ell(\sigma^+), \ell^-\big)$ marked $\bullet$. By part (ii) of the lemma, when M is sufficiently large, this position also contains $-\ell^-$, and so the first $\ell(\sigma^+)$ rows of t are equal in their first ℓ^- columns. The ℓ^--bottom position is $\kappa^-_{\ell^-}$ rows below the ℓ^--top position. Observe that since the ℓ^--top position is in row $\ell(\sigma^+)$, deleting a row of length ℓ^- strictly below the ℓ^--top position and weakly above the ℓ^--bottom position preserves partition shape: this is relevant to the bijection $\mathcal{F}$ defined in Definition 9.14.

By (+), there are exactly $|\sigma^-| - |\pi^-|$ positive entries in the first ℓ^- columns of t. Let $r^+ \leq \ell^+$ and suppose, as we may, that $\kappa_{r^+} > \kappa_{r^+ + 1}$. The r^+-left position of t is $(r^+, \ell^- + \sigma^+_{r^+ + 1})$. If $\sigma^+_{r^+ + 1} = 0$ then (iii) implies that this position contains a negative entry, as required in the exceptional cases for left-positions. We may therefore assume that $\sigma^+_{r^++1} > 0$, so the r^+-left position is not in the first ℓ^- columns of t. Suppose, for a contradiction, that this position does not contain r^+. The total number of entries of t lying in the set $\{1, \dots, r^+\}$ is then at most $|\sigma^-| - |\pi^-| + \sigma^+_1 + \cdots + \sigma^+_{r^+} + \sigma^+_{r^++1} - 1$. On the other hand t has exactly $\pi^+_1 + \cdots + \pi^+_{r^+-1} + \pi^+_{r^+}$ such entries. Hence

$$|\sigma^-| - |\pi^-| + \sum_{i=1}^{r^+-1} \sigma^+_i + \sigma^+_{r^++1} > \sum_{i=1}^{r^+} \pi^+_i. \tag{9.20}$$

Now using $\pi \trianglerighteq \lambda \oplus M(\kappa^-, \kappa^+)$ and so, by Lemma 6.7(b), $\pi^+ + \big((|\lambda^+| + M|\kappa^+|) - |\pi^+|\big) \trianglerighteq \lambda^+ + M\kappa^+$, we have

$$\sum_{i=1}^{r^+} \pi^+_i \geq \sum_{i=1}^{r^+} \lambda^+_i + M \sum_{i=1}^{r^+} \kappa^+_i - |\lambda^+| - M|\kappa^+| + |\pi^+|. \tag{9.21}$$

In exactly the same way, since $\sigma \trianglerighteq \lambda \oplus M(\kappa^-, \kappa^+)$, we have, for each $k \geq 1$,

$$\sum_{i=1}^{k} \sigma^+_i \geq \sum_{i=1}^{k} \lambda^+_i + M \sum_{i=1}^{k} \kappa^+_i - |\lambda^+| - M|\kappa^+| + |\sigma^+| \tag{9.22}$$

and using $\sigma \trianglelefteq \omega \oplus M(\kappa^-, \kappa^+)$ and so $\sigma^+ \trianglelefteq \omega^+ + \big(|\sigma^+| - |\omega^+| - M|\kappa^+|\big) + M\kappa^+$, we have

$$\sum_{i=1}^{k} \sigma^+_i \leq \sum_{i=1}^{k} \omega^+_i + M \sum_{i=1}^{k} \kappa^+_i + |\sigma^+| - |\omega^+| - M|\kappa^+| \tag{9.23}$$

for each k. Taking $k = r^+$ in (9.22) and $k = r^+ + 1$ in (9.23) and subtracting (as before for the negative case) we get

$$\sigma^+_{r^++1} \leq \sum_{i=1}^{r^++1} \omega^+_i - \sum_{i=1}^{r^+} \lambda^+_i + M\kappa_{r^++1} + |\lambda^+| - |\omega^+|.$$

Hence

$$\begin{aligned}
\sum_{i=1}^{r^+-1} \sigma_i + \sigma_{r^++1} \leq{} & \sum_{i=1}^{r^+-1} \omega^+_i + M \sum_{i=1}^{r^+-1} \kappa^+_i + |\sigma^+| - |\omega^+| - M|\kappa^+| \\
& + \sum_{i=1}^{r^++1} \omega^+_i - \sum_{i=1}^{r^+} \lambda^+_i + M\kappa^+_{r^++1} + |\lambda^+| - |\omega^+| \\
={} & 2\sum_{i=1}^{r^+-1} \omega^+_i + \omega^+_{r^+} + \omega^+_{r^++1} - \sum_{i=1}^{r^+} \lambda^+_i + M \sum_{i=1}^{r^+-1} \kappa^+_i + M\kappa^+_{r^++1} \\
& + |\sigma^+| + |\lambda^+| - 2|\omega^+| - M|\kappa^+|
\end{aligned}$$

and so, writing A for the right-hand side above, (9.20) and (9.21) imply

$$|\pi^-| - |\sigma^-| + \sum_{i=1}^{r^+} \lambda^+_i + M \sum_{i=1}^{r^+} \kappa^+_i - |\lambda^+| - M|\kappa^+| + |\pi^+| < A.$$

We now rearrange and simplify using $|\sigma^-| + |\sigma^+| = |\pi^-| + |\pi^+|$ to get

$$\begin{aligned} M(\kappa^+_{r^+} - \kappa^+_{r^++1}) &< 2\sum_{i=1}^{r^+-1} \omega^+_i + \omega^+_{r^+} + \omega^+_{r^++1} - 2\sum_{i=1}^{r^+} \lambda^+_i \\ &\qquad + |\sigma^+| + 2|\lambda^+| - 2|\omega^+| - |\pi^-| + |\sigma^-| - |\pi^+| \\ &= 2\sum_{i=1}^{r^+-1} \omega^+_i + \omega^+_{r^+} + \omega^+_{r^++1} - 2\sum_{i=1}^{r^+} \lambda^+_i + 2|\lambda^+| - 2|\omega^+|. \end{aligned} \tag{9.24}$$

But from the hypothesis $M \geq \mathrm{L}\big([\lambda^+, \omega^+ + (|\lambda^+| - |\omega^+|)]_{\unlhd}, \kappa^+\big)$ we have

$$M \geq \frac{2\sum_{i=1}^{r^+-1} \omega^+_i + \omega^+_{r^+} + \omega^+_{r^++1} - 2\sum_{i=1}^{r^+} \lambda^+_i + 2|\lambda^+| - 2|\omega^+|}{\kappa^+_{r^+} - \kappa^+_{r^++1}}. \tag{9.25}$$

Comparing with the previous inequality we get a contradiction. This proves (iv) and (v) for left positions. The modifications for (iv) and (v) for right positions are precisely analogous to the negative case; the relevant quantity to subtract from the left-hand side of (9.24) is $\kappa^+_{r^+} - \kappa^+_{r^++1}$, so we obtain (9.25) with M replaced by $M - 1$, as already seen twice before. This completes the proof. □

For later use in the proof of Lemma 11.7 we give the following lemma in the general skew case.

Lemma 9.17. *Let κ^- and κ^+ be partitions. Set $\ell^- = \ell(\kappa^-)$ and $\ell^+ = \ell(\kappa^+)$. Let $\sigma/\sigma_\star$ be a $\big(\ell^- + a(\sigma_\star), \ell^+\big)$-large and $\big(\ell^-, \ell(\sigma_\star)\big)$-large skew partition. The r^--top position of $\sigma/\sigma_\star$ is*

$$\begin{cases} \big(\sigma'_{r^-+1}, r^-\big) & \text{if } r^- < \ell^- \\ \big(\max(\ell(\sigma_\star), \ell^+, \ell(\sigma^+)), \ell^-\big) & \text{if } r^- = \ell^- \end{cases}$$

and the r^+-left position of σ is

$$\begin{cases} \big(r^+, \sigma_{r^++1}\big) & \text{if } r^+ < \ell^+ \\ \big(\ell^+, \max(\ell^- + a(\sigma_\star), \sigma_{\ell^++1})\big) & \text{if } r^+ = \ell^+ \end{cases}$$

Either $\sigma_\star = \varnothing$ and $\sigma^+ = \varnothing$ and $\ell^+ = 0$ or all top positions are in row $\max(\ell(\sigma_\star), \ell(\sigma^+), \ell^+)$ of σ or further below. All left positions are in column $\ell^- + a(\sigma_\star)$ of σ or further right.

Proof. Since $\sigma/\sigma_\star$ is $\big(\ell^-, \ell(\sigma_\star)\big)$-large, we have $\sigma^-_{\ell^-} \geq \ell(\sigma_\star)$. Since $\sigma/\sigma_\star$ is $\big(\ell^- + a(\sigma_\star), \ell^+\big)$-large we have $\sigma^-_{\ell^-} \geq \ell^+$ and $\ell^- + \sigma^+_{\ell^+} \geq \ell^- + a(\sigma_\star)$. Moreover, by Remark 6.2, we have $\sigma^-_{\ell^-} \geq \ell(\sigma^+)$. Summarising

$$\sigma^-_{\ell^-} \geq \max(\ell(\sigma_\star), \ell(\sigma^+), \ell^+), \tag{9.26}$$

$$\sigma^+_{\ell^+} \geq a(\sigma_\star). \tag{9.27}$$

By Definition 9.11, the r^--top position of $\sigma/\sigma_\star$ is

$$\big(\max(\ell(\sigma_\star), \ell(\sigma^+), \ell^+, \sigma^-_{r^-+1}), r^-\big)$$

for each $r^- \le \ell^-$. The claim on the ℓ^--top position is now immediate. If $r^- < \ell^-$, then by (9.26), the maximum defining the row is at least $\sigma^-_{\ell^-}$, and so the position is $(\sigma^-_{r^-+1}, r^-)$, as required. This also proves the claim on the rows of these positions. Again by Definition 9.11, the r^+-left position of $\sigma/\sigma_\star$ is

$$\big(r^+, \ell^- + \max(a(\sigma_\star), \sigma^+_{r^++1})\big)$$

for $r^+ \le \ell^+$. If $r^+ < \ell^+$, then by (9.27), the maximum is at least σ_{ℓ^+} and so the position is $\big(r^+, \ell^- + \sigma^+_{r^++1}\big)$, which is as required. If $\sigma^+_{\ell^++1} = 0$ then the column of the ℓ^+-left position is $\ell^- + a(\sigma_\star)$, as claimed, while if $\sigma^+_{\ell^++1} > 0$ then $\sigma_{\ell^++1} = \ell^- + \sigma^+_{\ell^++1}$ and so

$$\ell^- + \max(a(\sigma_\star), \sigma^+_{\ell^++1}) = \max(\ell^- + a(\sigma_\star), \sigma_{\ell^++1}\big)$$

as required. This also proves the claim on the columns of these positions. □

Recall from Definitions 3.3 and 3.7 that $\mathrm{YT}(\mu/\mu_\star)$ is the set of signed tableaux of shape $\mu/\mu_\star$ and $\mathrm{SSYT}^\pm(\mu/\mu_\star)$ is the subset of signed semistandard tableaux of shape $\mu/\mu_\star$.

Lemma 9.18. *Let κ^- and κ^+ be partitions. Let $\sigma/\sigma_\star$ be a $\big(\ell(\kappa^-) + a(\sigma_\star), \ell(\kappa^+)\big)$-large and $\big(\ell(\kappa^-), \ell(\sigma_\star)\big)$-large skew partition.*

(i) *The map $\mathcal{F} : \mathrm{SSYT}^\pm(\sigma/\sigma_\star) \to \mathrm{YT}\big(\sigma/\sigma_\star \oplus (\kappa^-, \kappa^+)\big)$ is well-defined.*

(ii) *If $t \in \mathrm{SSYT}^\pm(\sigma/\sigma_\star)$ has signed weight (π^-, π^+) then $\mathcal{F}(t)$ has signed weight $(\pi^- + \kappa^-, \pi^+ + \kappa^+)$.*

Proof. For (i) we must check that the insertions preserve partition shape. By Lemma 9.17, each r^--top position is immediately above a row of length at most r^- not meeting $[\sigma_\star]$. (Note particularly that this holds when $r^- = \ell^-$ because the ℓ^--top position lies in row $\max(\ell(\sigma_\star), \ell(\sigma^+), \ell^+)$, as remarked on in the caption to Figure 9.2.) The definition of $\mathcal{F}$ in Definition 9.14 specifies that insertions are performed working from bottom to top and top positions used for later insertions in (1) are not changed by earlier insertions in (1). Therefore the row insertions are well-defined. By the claim on the rows of the top positions in the lemma, the left positions are not changed by these insertions. Therefore the collumn insertions in (2) commute with the row insertions in (1), and a similar argument shows that the row insertions in (2) are well-defined. Hence $\mathcal{F}$ is well-defined as required in (i). Part (ii) is obvious from the definition of $\mathcal{F}$. □

9.6. **Tableau stability.** We now show that $\mathcal{F}$ is bijective in the case relevant to our stable partition system. We later quote the main part of this proof of the following lemma in the proof of the extension to the skew case in Proposition 11.13. Since this extension has other details that are somewhat fiddly, we do not attempt to continue the unified exposition. We remark that, by Lemma 9.8, the hypothesis that ω is $\big(\ell^- + 1, \ell^+\big)$-large implies the same condition on λ.

Proposition 9.19 (Tableau Stability). *Let κ^- and κ^+ be partitions. Fix $\ell^- = \ell(\kappa^-)$ and $\ell^+ = \ell(\kappa^+)$. Let λ be a partition and let ω be a $\big(\ell^- + 1, \ell^+)\big)$-large partition such that $\lambda \trianglelefteq \omega$. Let σ and π be partitions in the twisted* *interval*

$$\big[\lambda \oplus M(\kappa^-,\kappa^+), \omega \oplus M(\kappa^-,\kappa^+)\big]_{\unlhd}$$

for the ℓ^--twisted dominance order. Provided M is at least the maximum of

- $\mathrm{L}\big([\lambda^-,\omega^-]^{(\ell^-)}_{\blacktriangleleft}, \kappa^-\big)$
- $\mathrm{L}\big(\big[\lambda^+, \omega^+ + (|\lambda^+| - |\omega^+|)\big]_{\unlhd}, \kappa^+\big)$
- $\big(\max(\ell(\lambda^+), \ell^+) + |\omega^-| - |\overline{\lambda}^-| - \omega^-_{\ell^-}\big)/\kappa^-_{\ell^-}$

the map $\mathcal{F}$ is a well-defined bijection

$$\mathcal{F} : \mathrm{SSYT}\big(\sigma\big)_{(\pi^-,\pi^+)} \to \mathrm{SSYT}\big(\sigma \oplus (\kappa^-,\kappa^+)\big)_{(\pi^-+\kappa^-,\pi^++\kappa^+)}.$$

Proof. By Lemma 9.8 the partitions λ and σ are both (ℓ^-+1,ℓ^+)-large and so (ℓ^-,ℓ^+)-large. Hence, by Lemma 9.18(i), the map $\mathcal{F}$ is well-defined. Fix a semistandard signed tableau $t \in \mathrm{SSYT}\big(\sigma\big)_{(\pi^-,\pi^+)}$. By hypothesis the three bounds on M required to apply Lemma 9.16 all hold, and we have just seen the required largeness conditions hold. Therefore we have properties (i), (ii), (iii), (iv) and (v) in this lemma.

By (i) and (ii) for top positions, each of the $\kappa^-_r - \kappa^-_{r+1}$ new rows of length $r^- \le \ell^-$ with entries $-1,\ldots,-r^-$ are inserted below a row of t having $-r^-$ in column r^- and so have the same entries in their first r^- positions. These row insertions therefore preserve the semistandard condition for columns.

By (iv) and (v) for left positions, each new column of height $r^+ \le \ell^+$ with entries $1,\ldots,r^+$ is inserted to the right of a column having r^+ in row r^+ and so having the same entries as the inserted column in its first r^+ positions, or as a new column ℓ^-+1, immediately to the right of a column having only negative entries. These column insertions therefore preserve the semistandard condition for rows.

It is clear that the overall effect of these insertions is to change the signed weight of t from (π^-,π^+) to $(\pi^-+\kappa^-,\pi^++\kappa^+)$. Hence $\mathcal{F}$ has image in the set $\mathrm{SSYT}\big(\sigma \oplus (\kappa^-,\kappa^+)\big)_{(\pi^-+\kappa^-,\pi^++\kappa^+))}$ as claimed.

The map $\mathcal{F}$ is defined by inserting certain rows and columns into fixed positions in a tableau, so is clearly injective.

To see that $\mathcal{F}$ is surjective, let $u \in \mathrm{SSYT}\big(\sigma \oplus (\kappa^-,\kappa^+)\big)_{(\pi^-+\kappa^-,\pi^++\kappa^+)}$. Suppose that $\kappa^-_{r^-} > \kappa^-_{r^-+1}$. Then, by definition of the r^--top position, the row containing the r^--bottom position of u, and each of the $\kappa^-_{r^-} - \kappa^-_{r^-+1}$ rows weakly above it (including the row itself) has length r^-. By (i) and (ii) for bottom positions, the r^--bottom position in u contains $-r^-$; since boxes in column r^- contain entries at least $-r^-$ in the order in Definition 3.7, all entries in this column are $-r^-$. Therefore all $\kappa^-_{r^-} - \kappa^-_{r^-+1}$ rows have the form $-1,\ldots,-r^-$. Deleting these rows and shifting the remaining boxes in lower rows up gives a signed semistandard tableau because (as remarked at the start of the proof of Lemma 9.18), by Lemma 9.17, each r^--top position is immediately above a row of length at most r^- not meeting $[\sigma_\star]$. Therefore this deletion undoes the insertion map in (1). Our assumption that $\kappa^-_{r^-} > \kappa^-_{r^-+1}$ is now seen to be without loss of generality, since if equality holds then no rows were inserted. The argument for right positions and column deletion is very similar: if $\kappa^+_{r^+} > \kappa^+_{r^++1}$ then the column containing the r^+-right position of u and each of the $\kappa^+_{r^+} - \kappa^+_{r^++1}$ columns weakly left of it (including the column itself) has length r^+ and entries $1, \ldots, r^+$. Deleting these columns undoes the insertion map in (2). (Here it is obvious that deletion preserves the signed semistandard condition.) Hence $\mathcal{F}$ is surjective and so bijective. □

9.7. **Stable partition systems from twisted intervals.** We summarise this section in the following corollary. Recall that the first two bounds are defined in Definition 9.2. We remark that (as seen at the start of proof of Proposition 9.10), the hypotheses below imply, via Lemma 6.10, that λ is $(\ell^- + 1, \ell^+)$-large; thus adjoining to λ behaves as expected from Lemma 9.6, and we do no need an explicit hypothesis that λ is suitably large.

Corollary 9.20. *Let κ^-, κ^+ be partitions. Fix $\ell^- = \ell(\kappa^-)$ and $\ell^+ = \ell(\kappa^+)$. Let $g_\pi = e_{\pi^-} h_{\pi^+}$ for each $\pi \in \mathrm{Par}$. Let ω be a $\big(\ell^- + 1, \ell^+\big)$-large partition and let $\lambda \trianglelefteq \omega$ in the ℓ^--twisted dominance order. Let L be the maximum of the quantities*

- $\mathrm{L}\big([\lambda^-, \omega^-]^{(\ell^-)}_{\underline{\blacktriangleleft}}, \kappa^-\big)$,
- $\mathrm{L}\big([\lambda^+, \omega^+ + (|\lambda^+| - |\omega^+|)]_{\trianglelefteq}, \kappa^+\big)$,
- $\big(\omega^+_1 + \omega^+_2 - 2\lambda^+_1 + 2|\lambda^+| - 2|\omega^+|\big)/(\kappa^+_1 - \kappa^+_2)$,
- $\big(\max(\ell(\lambda^+), \ell^+) + |\omega^-| - |\lambda^-| - \omega^-_{\ell^-}\big)/\kappa^-_{\ell^-}$

omitting the third if $\kappa^+_1 = \kappa^+_2$ and the fourth if $\kappa^- = \varnothing$. Let

$$\mathcal{P}^{(M)} = \big[\lambda \oplus M(\kappa^-, \kappa^+), \omega \oplus M(\kappa^-, \kappa^+)\big]_{\trianglelefteq}$$

for each $M \in \mathbb{N}_0$. Then $(\mathcal{P}^{(M)})_{M \in \mathbb{N}_0}$ is a stable partition system with respect to the map $F : \mathrm{Par} \to \mathrm{Par}$ defined by $F(\sigma) = \sigma \oplus (\kappa^-, \kappa^+)$ and the twisted symmetric functions g_π. The system is stable for $M \geq L$.

Proof. We check the two conditions in the definition of a stable partition system in Definition 7.1. The four bounds above give the hypotheses required in Proposition 9.10 (Partition Stability). Therefore condition (a) holds for $M \geq K$. The hypotheses on M in Proposition 9.19 (Tableau Stability) are the first two bounds above and $M \geq \big(|\omega^-| - \omega^-_{\ell^-} + \ell^+ - |\lambda^-|\big)/\kappa^-_{\ell^-}$, which is implied by the fourth bound above. The condition that ω is (ℓ^-+1, ℓ^+)-large holds by assumption. Hence $|\mathrm{SSYT}(\sigma)_{(\pi^-, \pi^+)}| = |\mathrm{SSYT}(F(\sigma))_{(F(\pi)^-, F(\pi)^+)}|$ for all $\pi, \sigma \in \mathcal{P}(M)$ provided $M \geq K$. By Lemma 5.3 (Twisted Kostka Numbers) it follows that $\langle e_{\pi^-} h_{\pi^+}, s_\sigma \rangle = \langle e_{F(\pi)^-} h_{F(\pi)^+}, s_{F(\sigma)} \rangle$ for all $\pi, \sigma \in \mathcal{P}(M)$ provided $M \geq K$. Therefore condition (b) holds for $M \geq K$. □

10. Twisted weight bound for Theorem 1.1

To apply Corollary 9.20 in the proofs of our two main theorems we need an upper bound in the ℓ^--twisted dominance order on the constituents of an arbitrary plethysm. For instance, in the overview in §2, we implicitly used (see Example 6.15) the 1-twisted dominance order with the twisted intervals $\big[(6,2) \oplus M\big((1),(1)\big), (5,1,1,1) \oplus M\big((1),(1)\big)\big]_{\trianglerighteq}$, arguing that if s_λ is a constituent in $s_{(3,1,1^M)} \circ s_{(2)}$ then $\lambda \trianglelefteq (5,1,1,1) \oplus M\big((1),(1)\big)$. The aim of this section is to prove Corollary 10.10 which gives the upper bound we use for Theorem 1.1. En route we obtain Proposition 10.7 which is of independent interest. We show in Example 10.8 that, in the case of the plethysm $s_{(3,1,1^M)} \circ s_{(2)}$, Proposition 10.7 specializes to give the upper bound obtained earlier by ad-hoc arguments; Example 10.9 shows the connection between our upper bound and the extended example in §8.

10.1. **Weight large skew partitions.** There is an analogous technicality to that pointed out before Definition 3.1 about adjoining to partitions. Recall from Definition 4.3 and Lemma 4.4 that $t_{\ell^-}(\tau/\tau_\star)$ is the semistandard tableau of shape $\tau/\tau_\star$ of greatest signed weight in the ℓ^--signed dominance order. By Lemma 6.4 the signed weight $\big(\omega_{\ell^-}(\tau/\tau_\star)^-, \omega_{\ell^-}(\tau/\tau_\star)^+\big)$ of $t_{\ell^-}(\tau/\tau_\star)$ is the ℓ^--decomposition of a partition.

Definition 10.1. Fix $\ell^- \in \mathbb{N}_0$. Let $\tau/\tau_\star$ be a skew partition and let σ be the partition with ℓ^--decomposition $\big\langle \omega_{\ell^-}(\tau/\tau_\star)^-, \omega_{\ell^-}(\tau/\tau_\star)^+\big\rangle$. Let $a \in \mathbb{N}_0$. We say $\tau/\tau_\star$ is

(a) (a,ℓ^+)-*weight large for* ℓ^- if σ is (a,ℓ^+)-large,

(b) (ℓ^-,ℓ^+)-*weight large* if σ is (ℓ^-,ℓ^+)-large.

We state the definition in this form to emphasise the connection with Definition 3.1. When $\ell^- \geq 1$, the paragraph after this definition implies that the skew partition $\tau/\tau_\star$ is (ℓ^-,ℓ^+)-weight large if and only if part ℓ^- of $\omega_{\ell^-}(\tau/\tau_\star)^-$ is at least ℓ^+, or equivalently, if and only if $t_{\ell^-}(\tau/\tau_\star)$ has at least ℓ^+ entries of $-\ell^-$. This is the interpretation we need most often.

		1	**2**	1	1
	1	**2**	1		
1	**2**	1	2		
1					

Example 10.2. From Example 4.5, where $\ell^- = 2$, we see that $(6,4,4,1)$, $(6,4,4,1)/(1,1)$ and $(6,4,4,1)/(2,1)$ are $(2,\ell^+)$-weight large if and only if $\ell^+ \leq 3$ and $(6,4,4,1)/(3,3)$ is $(2,\ell^+)$-weight large if and only if $\ell^+ \leq 2$. This is most easily seen using the final characterisation just mentioned: for example the tableau $t_2\big((6,4,4,1)/(2,1)\big)$ shown in the margin evidently has three entries of -2. It is easily checked from the other tableaux in Example 4.5 that $(6,4,4,1)$ and $(6,4,4,1)/(1,1)$ are $(3,3)$-weight large for 2 because the relevant partitions σ in Definition 10.1 have $(3,3)$ as a box (or equivalently the greatest tableaux both have 3 as an entry) but $(6,4,4,1)/(2,1)$ is not, because, as the marginal tableau shows, 3 is not an entry. Working directly from Definition 10.1, we would instead compute

$$\big\langle \omega_{\ell^-}((6,4,4,1)/(2,1))^-, \omega_{\ell^-}((6,4,4,1)/(2,1))^+\big\rangle = \big\langle (4,3),(4,1)\big\rangle \leftrightarrow (6,3,2,1)$$

and note that $(6,3,2,1)$ does not have a box in position $(3,3)$. Thus $(6,4,4,1)/(2,1)$ is $(2,3)$-weight large and $(3,2)$-weight large for 2 but not $(3,3)$-weight large. Note also that while $(6,4,4,1)/(3,3)$ is not $(2,3)$-weight large, it is $(2,3)$-large, since $(6,4,4,1)_3 = 4 \geq 2$.

We have just seen that '(ℓ^-,ℓ^+)-large' does not imply '(ℓ^-,ℓ^+)-weight large'. The following lemma gives the complete picture.

Lemma 10.3. *Let $\tau/\tau_\star$ be a skew partition.*

(i) *If $\tau/\tau_\star$ is $(\ell^- + a(\tau_\star), \ell^+)$-large then $\tau/\tau_\star$ is (ℓ^-,ℓ^+)-weight large.*

(ii) *If* $\tau/\tau_\star$ *is* (ℓ^-, ℓ^+)*-weight large then* $\tau/\tau_\star$ *is* (ℓ^-, ℓ^+)*-large*
Moreover, if $\tau_\star = \varnothing$ *then the converses also hold.*

Proof. For (i), by hypothesis $[\tau]\backslash[\tau_\star]$ contains all the boxes $(i, a(\tau/\tau_\star)+j)$ for $1 \leq i \leq \ell^+$ and $1 \leq j \leq \ell^-$. As in the proof of Lemma 6.4, rows $1, 2, \ldots, \ell^+$ of $t_{\ell^-}(\tau/\tau_\star)$ begin $\boxed{\mathbf{1}}\boxed{\mathbf{2}}\boxed{\;\cdots\;}\boxed{\boldsymbol{\ell^-}}$. (These boxes form part of the heavy marked region $[\alpha]$ in Figure 11.1.) Hence $t_{\ell^-}(\tau/\tau_\star)$ has at least ℓ^+ entries of $-\ell^-$ and so $\tau/\tau_\star$ is (ℓ^-, ℓ^+)-weight large. For (ii), we have just seen that $t_{\ell^-}(\tau/\tau_\star)$ has at least ℓ^+ entries equal to $-\ell^-$. Hence there are at least ℓ^+ rows such that $(\tau - \tau_\star)_i \geq \ell^-$. It easily follows that $(\ell^+, \ell^-) \in [\tau]$ and so $\tau/\tau_\star$ is (ℓ^-, ℓ^+)-large. Finally, if $\tau_\star = \varnothing$ then $a(\tau_\star) = 0$ and so (i) and (ii) are opposite directions of the required implication. □

We also have the following remark, analogous to Remark 3.2.

Remark 10.4. Fix partitions κ^- and κ^+ and let $\ell^- = \ell(\kappa^-)$ and $\ell^+ = \ell(\kappa^+)$. When $\kappa^- \neq \varnothing$, each application of the map $\tau/\tau_\star \mapsto \tau/\tau_\star \oplus (\kappa^-, \kappa^+)$ inserts $\kappa^-_{\ell^-}$ new parts of length ℓ^-, and so increases the number of parts of length at least ℓ^- by at least $\kappa^-_{\ell^-}$. Therefore after $\lceil \ell(\tau_\star)/\kappa^-_{\ell^-} \rceil$ steps, the skew partition obtained has $(\ell(\tau_\star), \ell^-)$ as a box. On each subsequent step we insert $\kappa^-_{\ell^-}$ new boxes in column ℓ^- which in the greatest tableau all contain $-\ell^-$. Therefore the original skew partition $\tau/\tau_\star$ becomes (ℓ^-, ℓ^+)-weight large after at most $\lceil \ell(\tau_\star)/\kappa^-_{\ell^-} \rceil + \lceil \ell^+/\kappa^-_{\ell^-} \rceil$ steps. Each further step creates at least $\kappa^+_{\ell^+}$ new boxes containing ℓ^+. Therefore, when $\kappa^+ \neq \varnothing$, for any $a \in \mathbb{N}$, the original skew partition $\tau/\tau_\star$ becomes $(\ell^- + a, \ell^+)$-weight large, meaning that $(\ell^+, \ell^- + a)$ is a box of the partition corresponding to the signed weight of $t_{\ell^-}\big(\tau/\tau_\star \oplus M(\kappa^-, \kappa^+)\big)$, after at most $M = \lceil \ell(\tau_\star)/\kappa^-_{\ell^-} \rceil + \lceil \ell^+/\kappa^-_{\ell^-} \rceil + \lceil a/\kappa^+_{\ell^+} \rceil$ steps.

By this remark, there is no loss of generality in assuming in all the results below that the partitions involved are suitably weight large.

For example, take $\kappa^- = (2, 1, 1)$, $\kappa^+ = (1, 1)$ and $\tau/\tau_\star = (1, 1, 1)/(1, 1, 1)$, so that $\ell^- = 3$ and $\ell^+ = 2$. The tableaux in Figure 10.1 show that three applications of the adjoining map $\tau/\tau_\star \mapsto \tau/\tau_\star \oplus \big((2, 1, 1), (1, 1)\big)$ are necessary and sufficient to obtain a $(3, 2)$-weight large skew partition; this skew partition is also $(3, 3)$-weight large. One further application gives a $(3, 4)$-weight large skew partition. As is typically the case, this beats the bound in Remark 10.4, which specifies $\lceil 3/1 \rceil + \lceil 2/1 \rceil = 5$ adjoinings.

Finally we have the expected analogue of Lemma 9.6.

Lemma 10.5. *Let* κ^- *and* κ^+ *be partitions. Fix* $\ell^- = \ell(\kappa^-)$*. If* $\tau/\tau_\star$ *is a* $\big(\ell(\kappa^-), \ell(\kappa^+)\big)$*-weight large skew partition then, in the* ℓ^-*-decomposition of* $\omega_{\ell^-}\big(\tau/\tau_\star \oplus (\kappa^-, \kappa^+)\big)$ *we have*

$$\omega_{\ell^-}\big(\tau/\tau_\star \oplus (\kappa^-, \kappa^+)\big)^- = \omega_{\ell^-}(\tau/\tau_\star)^- + \kappa^-$$
$$\omega_{\ell^-}\big(\tau/\tau_\star \oplus (\kappa^-, \kappa^+)\big)^+ = \omega_{\ell^-}(\tau/\tau_\star)^+ + \kappa^+.$$

Proof. By Lemma 10.3(ii) and Lemma 9.6 we have $\big(\tau \oplus (\kappa^-, \kappa^+)\big)^- = \tau^- + \kappa^-$ and $\big(\tau \oplus (\kappa^-, \kappa^+)\big)^+ = \tau^+ + \kappa^+$. This implies the two equations. □

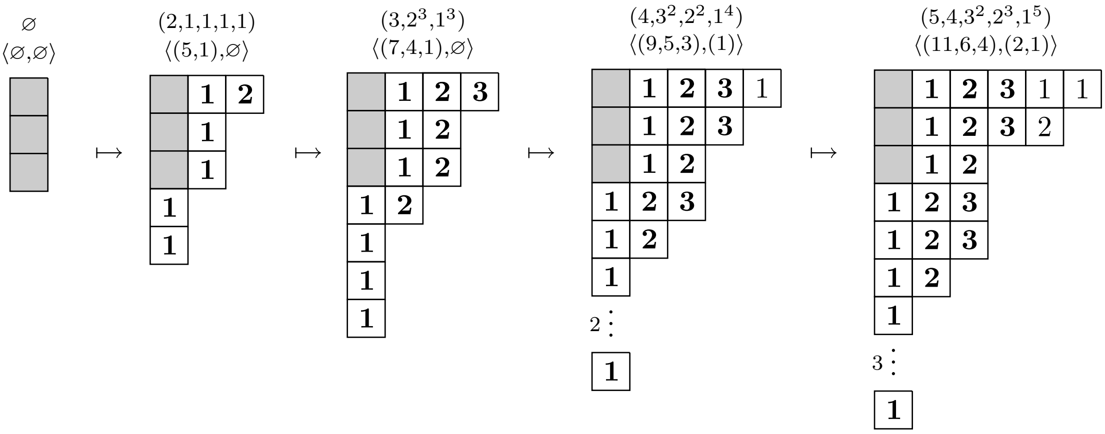


FIGURE 10.1. The map $\tau/\tau_\star \mapsto \tau/\tau_\star \oplus \big((2,1,1),(1,1)\big)$ applied repeatedly to the skew partition $(1,1,1)/(1,1,1)$, showing the tableaux $t_3\big((1,1,1) \oplus M\big((2,1,1),(1,1)\big)\big)$ for $M \in \{0,1,2,3,4\}$. The corresponding 'weight' partitions $\omega_3\big((1,1,1) \oplus M\big((2,1,1),(1,1)\big)\big)$ and their 3-decompositions are shown above the tableaux. In each subsequent step the weight partition grows by $\oplus\big((2,1,1),(1,1)\big)$; note that this is not the case until the weight partition becomes $(3,2)$-large, thus the technical nature of Remark 10.4.

10.2. **Bounding plethysms by greatest weights.** If $\langle \alpha^-, \alpha^+\rangle$ is an ℓ^--decomposition then so is $\langle n\alpha^-, n\alpha^+\rangle$ for any $n \in \mathbb{N}$. Therefore, by Lemma 6.4, the following definition is well posed.

Definition 10.6. Let $\ell^- \in \mathbb{N}_0$ and let $n \in \mathbb{N}_0$. Given a skew partition $\tau/\tau_\star$ we define $\omega^{(n)}_{\ell^-}(\tau/\tau_\star)$ to be the unique partition whose ℓ^--decomposition is $n\big\langle \omega_{\ell^-}(\tau/\tau_\star)^-, \omega_{\ell^-}(\tau/\tau_\star)^+\big\rangle$.

We give examples after the next proposition, which is the main result in this section, giving an upper bound in the ℓ^--twisted dominance order (see Definition 6.6) on the constituents of an arbitrary plethysm.

Proposition 10.7. *Let $\ell^- \in \mathbb{N}_0$. Let ρ be a partition of n and let $\tau/\tau_\star$ be a skew partition. If s_π is a constituent of $s_\rho \circ s_{\tau/\tau_\star}$ then $\pi \trianglelefteq \omega^{(n)}_{\ell^-}(\tau/\tau_\star)$ in the ℓ^--twisted dominance order.*

Proof. By Lemma 6.12, s_π is a summand of $e_{\pi^-}h_{\pi^+}$ with multiplicity 1. Hence, using Proposition 5.6 (Plethystic Signed Kostka Numbers) for the first equality below, we have

$$\big|\,\mathrm{PSSYT}(\rho,\tau/\tau_\star)_{(\pi^-,\pi^+)}\big| = \langle e_{\pi^-}h_{\pi^+}, s_\rho \circ s_{\tau/\tau_\star}\rangle \geq \langle s_\pi, s_\rho \circ s_{\tau/\tau_\star}\rangle \geq 1.$$

Let $T \in \mathrm{PSSYT}\big(\rho,\tau/\tau_\star\big)_{(\pi^-,\pi^+)}$ and let t be an inner $\tau/\tau_\star$-tableau in T. By Lemma 4.4 $\big(\omega_{\ell^-}(\tau/\tau_\star)^-, \omega_{\ell^-}(\tau/\tau_\star)^+\big)$ is the greatest weight (in the ℓ^--signed dominance order on $\mathcal{W}_{\ell^-} \times \mathcal{W}$ defined in Definition 4.1) of all signed weights of $\tau/\tau_\star$-tableaux. Thus, writing $\mathrm{swt}(t)$ for the signed weight of t, we have

$$\big(\mathrm{swt}(t)^-, \mathrm{swt}(t)^+\big) \trianglelefteq \big\langle \omega_{\ell^-}(\tau/\tau_\star)^-, \omega_{\ell^-}(\tau/\tau_\star)^+\big\rangle$$

where we regard either side as a composition, as in the definition of the ℓ^--twisted dominance order in Definition 6.6. Hence, summing over all inner

$\tau/\tau_\star$-tableaux in T, we have

$$\big\langle \pi^-, \pi^+ \big\rangle \trianglelefteq \big\langle n\omega_{\ell^-}(\tau/\tau_\star)^-, n\omega_{\ell^-}(\tau/\tau_\star)^+ \big\rangle.$$

(Note that $\langle \pi^-, \pi^+ \rangle$ is an ℓ^--decomposition simply because π is a partition.) By definition of the ℓ^--twisted dominance order this inequality holds if and only if $\pi \trianglelefteq \omega^{(n)}_{\ell^-}(\tau/\tau_\star)$, as required. □

By Remark 13.25, Proposition 10.7 also follows from a special case of Corollary 13.24; this alternative proof brings in many technicalities irrelevant to Theorem 1.1, and so we much prefer the proof above which is self-contained to this section. We pause to give two examples.

Example 10.8. Fix $\ell^- = 1$. Take $\rho = (3, 1, 1^M)$ and $\tau/\tau_\star = (2)$. Then $t_1\big((2)\big) = \boxed{\mathbf{1}}\boxed{1}$, and so $\omega_1\big((2)\big) = \big((1), (1)\big)$. Hence

$$\omega^{(4+M)}_1\big((2)\big) \leftrightarrow (4 + M)\big\langle 1, 1\big\rangle = \big\langle (4 + M), (4 + M)\big\rangle \leftrightarrow (5 + M, 1^{3+M}).$$

and $\omega^{(4+M)}_1\big((2)\big) = (5 + M, 1^{3+M})$. Note, as claimed at the start of this section, that the right-hand side is the partition used as the upper bound in §2.6 (see Example 6.15).

Example 10.9. Fix $\ell^- = 2$. Taking $\rho = (3 + M)$ and $\tau/\tau_\star = (4)$ in Proposition 10.7 we obtain

$$\operatorname{supp} s_{(3+M)} \circ s_{(4)} \subseteq \big\{\lambda \in \mathrm{Par}(12 + 4M) : \lambda \trianglelefteq \omega^{(3+M)}_2(4)\big\}$$

where $\trianglelefteq$ is the 2-twisted dominance order. Since $t_2\big((4)\big) = \boxed{\mathbf{1}}\boxed{\mathbf{2}}\boxed{1}\boxed{1}$ has signed weight $\big((1^2), (2)\big)$ we have

$$\begin{aligned}\omega^{(3+M)}_2\big((3 + M)\big) &\leftrightarrow (3 + M)\big\langle (1^2), (2)\big\rangle \\ &= \big\langle (3 + M, 3 + M), (6 + 2M)\big\rangle \leftrightarrow (8 + 2M, 2^{2+M})\end{aligned}$$

and so $\omega^{(3+M)}_2\big((3 + M)\big) = (8 + 2M, 2^{2+M})$ is an upper bound in the 2-twisted dominance order for the constituents of the plethysm $s_{(3+M)} \circ s_{(4)}$. In particular, if s_σ appears in $s_{(3+M)} \circ s_{(4)}$ then $\ell(\sigma) \leq 3 + M$, as used earlier in §8.2. Correspondingly, $(8 + 2M, 2^{2+M})$ is the upper bound in the twisted interval defining the stable partition system $\mathcal{P}^{(M)}$ used through §8: see §8.4 for the twisted interval interpretation.

The following corollary is used in Lemma 11.3 to verify condition (i) in the Signed Weight Lemma (Lemma 7.3).

Corollary 10.10 (Inner Twisted Weight Bound)**.** *Let κ^- and κ^+ be partitions. Fix $\ell^- = \ell(\kappa^-)$ and let $\ell^+ = \ell(\kappa^+)$. Let ρ be a partition of n and let $\mu/\mu_\star$ be an (ℓ^-, ℓ^+)-weight large skew partition. If s_σ is a constituent of the plethysm $s_\rho \circ s_{\mu/\mu_\star \oplus M(\kappa^-,\kappa^+)}$ then*

$$\sigma \trianglelefteq \omega^{(n)}_{\ell^-}(\mu/\mu_\star) \oplus nM(\kappa^-, \kappa^+).$$

Proof. By Proposition 10.7 taking $\tau/\tau_\star = \mu/\mu_\star \oplus M(\kappa^-, \kappa^+)$ we have

$$\sigma \trianglelefteq \omega^{(n)}_{\ell^-}\big(\mu/\mu_\star \oplus M(\kappa^-, \kappa^+)\big) \tag{10.1}$$

Since $\mu/\mu_\star$ is (ℓ^-,ℓ^+)-weight large, by Lemma 10.5 we have

$$\omega_{\ell^-}\big(\mu/\mu_\star \oplus M(\kappa^-,\kappa^+)\big) = \big(\omega_{\ell^-}(\mu/\mu_\star)^- + M\kappa^-, \omega_{\ell^+}(\mu/\mu_\star)^- + M\kappa^+\big)$$

and so

$$\omega^{(n)}_{\ell^-}\big(\mu/\mu_\star \oplus M(\kappa^-,\kappa^+)\big) = \omega^{(n)}_{\ell^-}\big(\mu/\mu_\star\big) \oplus nM(\kappa^-,\kappa^+).$$

Therefore (10.1) is equivalent to $\sigma \trianglelefteq \omega^{(n)}_{\ell^-}(\mu/\mu_\star) \oplus nM(\kappa^-,\kappa^+)$. □

This result should be compared to Corollary 13.24, which gives a more sophisticated bound used in the proof of Theorem 1.2.

11. Proof of Theorem 1.1

We begin in §11.1 by proving the second part of this theorem where the stable multiplicity is zero. We then use the Signed Weight Lemma (Lemma 7.3) to prove the remaining part of the theorem. In §11.2 we construct a suitable stable partition system. In §11.5 we construct a bijection satisfying (ii) in the Signed Weight Lemma. Finally in §11.6 we put together all the pieces proving Theorem 11.15 which restates Theorem 1.1 with an explicit bound.

11.1. **The vanishing case of Theorem 1.1.** We require the following statistic. Recall that $\langle\lambda^-,\lambda^+\rangle$ denotes the ℓ^--decomposition of a partition λ, as defined in Definition 6.1.

Definition 11.1. Let (κ^-,κ^+) and (η^-,η^+) be signed weights. Fix $\ell^- = \max(\ell(\eta^-),\ell(\kappa^-))$. Let λ and ω be partitions of the same size. We define $\mathrm{LZ}\big([\lambda,\omega]_\trianglelefteq, (\kappa^-,\kappa^+), (\eta^-,\eta^+)\big)$ to be the minimum of the quantities

- $\dfrac{\sum_{i=1}^k \omega_i^- - \sum_{i=1}^k \lambda_i^-}{\sum_{i=1}^k \eta_i^- - \sum_{i=1}^k \kappa_i^-}$.
- $\dfrac{|\omega^-| + \sum_{i=1}^k \omega_i^+ - |\lambda^-| - \sum_{i=1}^k \lambda_i^+}{|\eta^-| + \sum_{i=1}^k \eta_i^+ - |\kappa^-| - \sum_{i=1}^k \kappa_i^+}$

taken in each case over those k for which the denominator is strictly positive; if there are no such k, we leave $\mathrm{LZ}\big([\lambda,\omega]_\trianglelefteq, (\kappa^-,\kappa^+), (\eta^-,\eta^+)\big)$ undefined.

If, as in the first case of Theorem 1.1, we have $(\eta^-,\eta^+) \not\trianglelefteq (\kappa^-,\kappa^+)$ in the ℓ^--signed dominance order (see Definition 4.1), then it immediately follows from the definition of this order that $\mathrm{LZ}\big([\lambda,\omega]_\trianglelefteq, (\kappa^-,\kappa^+), (\eta^-,\eta^+)\big)$ is defined for any partitions λ and ω.

In the following proposition we prove a generalization of the final part of Theorem 1.1, with an explicit bound from Definition 11.1. Recall from Definition 10.6 that $\omega^{(n)}_{\ell^-}(\mu/\mu_\star)$ is the unique partition whose ℓ^--decomposition is $n\langle\omega_{\ell^-}(\tau/\tau_\star)^-, \omega_{\ell^-}(\tau/\tau_\star)^+\rangle$, where $\langle\omega_{\ell^-}(\tau/\tau_\star)^-, \omega_{\ell^-}(\tau/\tau_\star)^+\rangle$ is the signed weight of the ℓ^--greatest tableau: see Definition 4.3 and Lemma 4.4. See Definitions 3.1 and 10.1 for the definitions of 'large' and 'weight large'.

Proposition 11.2. *Let κ^- and κ^+ be partitions. Let $\ell^- = \ell(\kappa^-)$, Let η^- and η^+ be partitions with $\ell(\eta^-) \le \ell^-$. Let $\ell^+ = \max(\ell(\kappa^+),\ell(\eta^+))$ and let $\mu/\mu_\star$*

be an (ℓ^-, ℓ^+)-weight large skew partition. Let λ be a (ℓ^-, ℓ^+)-large partition. For each $M \in \mathbb{N}_0$, let $\nu^{(M)}$ be a partition of n. If $(\eta^-, \eta^+) \not\trianglelefteq (\kappa^-, \kappa^+)$ then

$$\big\langle s_{\nu^{(M)}} \circ s_{\mu/\mu_\star \oplus M(\kappa^-,\kappa^+)}, s_{\lambda \oplus nM(\eta^-,\eta^+)} \big\rangle = 0$$

for all

$$M > \mathrm{LZ}\big([\lambda, \omega^{(n)}_{\ell^-}(\mu/\mu_\star)]_\trianglelefteq, (\kappa^-, \kappa^+), (\eta^-, \eta^+)\big)/n.$$

Proof. By Corollary 10.10, applied with $\rho = \nu^{(M)}$, if s_σ is a constituent of the plethysm $s_{\nu^{(M)}} \circ s_{\mu/\mu_\star \oplus M(\kappa^-,\kappa^+)}$ then $\sigma \trianglelefteq \omega^{(n)}_{\ell^-}(\mu/\mu_\star) \oplus nM(\kappa^-, \kappa^+)$. Therefore, by Definition 10.6 and the definition of the ℓ^--twisted dominance order (see Definition 6.6) we have

$$(\sigma^-, \sigma^+) \trianglelefteq n\big(\omega_{\ell^-}(\mu/\mu_\star)^-, \omega_{\ell^-}(\mu/\mu_\star)^+\big) + nM(\kappa^-, \kappa^+) \tag{11.1}$$

in the ℓ^--signed dominance order (see Definition 4.1). Since λ is (ℓ^-, ℓ^+)-large and $\ell(\eta^-) \leq \ell^-$ and $\ell(\eta^+) \leq \ell^+$ by hypothesis, we may apply Lemma 9.6 to get that the ℓ^--decomposition of $\lambda \oplus nM(\eta^-, \eta^+)$ is $\big\langle \lambda^- + nM\eta^-, \lambda^+ + nM\eta^+ \big\rangle$. We now substitute $(\lambda^-, \lambda^+) + nM(\eta^-, \eta^+)$ for (σ^-, σ^+) in (11.1) to obtain the inequality

$$(\lambda^-, \lambda^+) + nM(\eta^-, \eta^+) \trianglelefteq n\big(\omega_{\ell^-}(\mu/\mu_\star)^-, \omega_{\ell^-}(\mu/\mu_\star)^+\big) + nM(\kappa^-, \kappa^+) \tag{11.2}$$

Thus, by Lemma 6.7(a), we have, for each $k \leq \ell^-$,

$$\sum_{i=1}^k \lambda_i^- + nM \sum_{i=1}^k \eta_i^- \leq \sum_{i=1}^k n\omega_{\ell^-}(\mu/\mu_\star)_i^- + nM \sum_{i=1}^k \kappa_i^-$$

and so

$$nM\Big(\sum_{i=1}^k \eta_i^- - \sum_{i=1}^k \kappa_i^-\Big) \leq \sum_{i=1}^k n\omega_{\ell^-}(\mu/\mu_\star)_i^- - \sum_{i=1}^k \lambda_i^-.$$

Hence, if $\sum_{i=1}^k \eta_i^- > \sum_{i=1}^k \kappa_i^-$ then nM is at most the relevant bound from the first case of Definition 11.1. The proof for the second family of bounds is very closely analogous: by (11.2) and Lemma 6.7(b) we obtain

$$\begin{aligned}|\lambda^-| + \sum_{i=1}^k \lambda_i^+ + nM \sum_{i=1}^k \eta_i^+ \\ \leq n|\omega_{\ell^-}(\mu/\mu_\star)^-| + \sum_{i=1}^k n\omega_{\ell^-}(\mu/\mu_\star)_i^+ + nM \sum_{i=1}^k \kappa_i^+ + nM\big(|\eta^+| - |\kappa^+|\big)\end{aligned}$$

and so using $|\eta^+| - |\kappa^+| = -|\eta^-| + |\kappa^-|$ we have

$$\begin{aligned}nM\Big(\sum_{i=1}^k \eta_i^+ - \sum_{i=1}^k \kappa_i^+ + |\eta^-| - |\kappa^-|\Big) \\ \leq n|\omega_{\ell^-}(\mu/\mu_\star)^-| + \sum_{i=1}^k n\omega_{\ell^-}(\mu/\mu_\star)_i^+ - |\lambda^-| - \sum_{i=1}^k \lambda_i^+\end{aligned}$$

showing that nM is at most the relevant bound from the second case of Definition 11.1. □

11.2. **Stable partition system for Theorem 1.1.** The stable partition system we require to prove the main part of Theorem 1.1 again comes from Corollary 10.10.

Lemma 11.3. *Let κ^- and κ^+ be partitions. Fix $\ell^- = \ell(\kappa^-)$ and $\ell^+ = \ell(\kappa^+)$. Let ρ be a partition of n and let $\mu/\mu_\star$ be an $(\ell^- + 1, \ell^+)$-weight large for ℓ^- skew partition. Let λ be a partition of $n|\mu/\mu_\star|$ such that $\lambda \trianglelefteq \omega^{(n)}_{\ell^-}(\mu/\mu_\star)$, where $\trianglelefteq$ is the ℓ^--twisted dominance order. Let*

$$\mathcal{P}^{(M)} = [\lambda \oplus nM(\kappa^-, \kappa^+), \omega^{(n)}_{\ell^-}(\mu/\mu_\star) \oplus nM(\kappa^-, \kappa^+)]_{\trianglelefteq}.$$

Then $(\mathcal{P}^{(M)})_{M\in\mathbb{N}_0}$ is a stable partition system for the symmetric functions $g_\pi = e_{\pi^-} h_{\pi^+}$. Moreover, if $\pi \in \mathcal{P}^{(M)}$ and s_σ is a summand of $e_{\pi^-} h_{\pi^+}$ appearing in the plethysm $s_\rho \circ s_{\mu/\mu_\star \oplus M(\kappa^-,\kappa^+)}$ then $\sigma \in \mathcal{P}^{(M)}$.

Proof. By the hypothesis that $\mu/\mu_\star$ is (ℓ^-+1, ℓ^+)-weight large for ℓ^-, the partition $\omega_{\ell^-}(\mu/\mu_\star)$ is $(\ell^- + 1, \ell^+)$-large (see Definition 10.1) and so $\omega^{(n)}_{\ell^-}(\mu/\mu_\star)$ is also $(\ell^- + 1, \ell^+)$-large. We therefore may apply Corollary 9.20 to deduce that the partition system is stable. For the final claim, by Lemma 6.12, we have $\sigma \trianglerighteq \pi$. By Corollary 10.10, we have $\sigma \trianglelefteq \omega^{(n)}_{\ell^-}(\mu/\mu_\star) \oplus nM(\kappa^-, \kappa^+)$. Hence $\sigma \in \mathcal{P}^{(M)}$. □

11.3. **Positions for plethystic tableaux: motivating example.** The aim in the next three subsections is to prove Proposition 11.13, the plethystic analogue of Proposition 9.19 (Tableau Stability), using the map $\mathcal{G}$ on plethystic semistandard signed tableaux defined in Definition 11.9. We begin with a motivating example that gives a good idea how to define $\mathcal{G}$ using the earlier map $\mathcal{F}$ on semistandard tableaux from Definition 9.14. We use this example throughout this section.

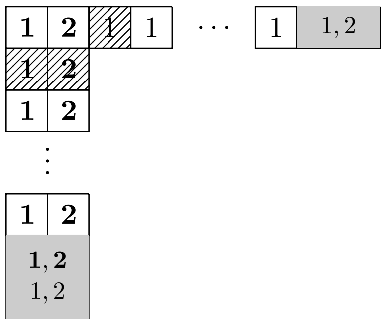
Example 11.4. The special case of Theorem 1.1 taking $\nu = (2)$, $\mu = (2, 2)$, $\mu_\star = \varnothing$, $(\kappa^-, \kappa^+) = \big((1, 1), (1)\big)$ and $\lambda = (8 - b, b)$ with $b \in \{2, 3, 4\}$ is that

$$\langle s_{(2)} \circ s_{(2+M,2,2^M)}, s_{(8-b+2M,b,2^{2M})}\rangle \tag{11.3}$$

is ultimately constant. In our proof using the Signed Weight Lemma (Lemma 7.3) we must verify condition (ii), that

$$\big|\mathrm{PSSYT}\big((2), (2 + M, 2, 2^M\big)_{((2+2M,2+2M),(6-b+2M,b-2))}\big| \tag{11.4}$$

is constant for all M sufficiently large. (Here $\big((2 + 2M, 2 + 2M), (6 - b + 2M, b - 2)\big)$ is the 2-decomposition of $(8 - b, b) \oplus 2M\big((1, 1), (1)\big)$: see Definition 6.1.) In any plethystic semistandard tableau $\boxed{s \mid t}$ of signed weight $\big((2 + 2M, 2 + 2M), (6 - b + 2M, b - 2)\big)$, the inner tableaux s and t have $2 + 2M$ entries of $\mathbf{1}$ and $\mathbf{2}$ and $6 - b + 2M$ entries of 1 between them. By the signed semistandard condition in Definition 3.10, the entries of $\mathbf{1}$ and $\mathbf{2}$ lie in the first two columns. Thus when M is large, both s and t have the form



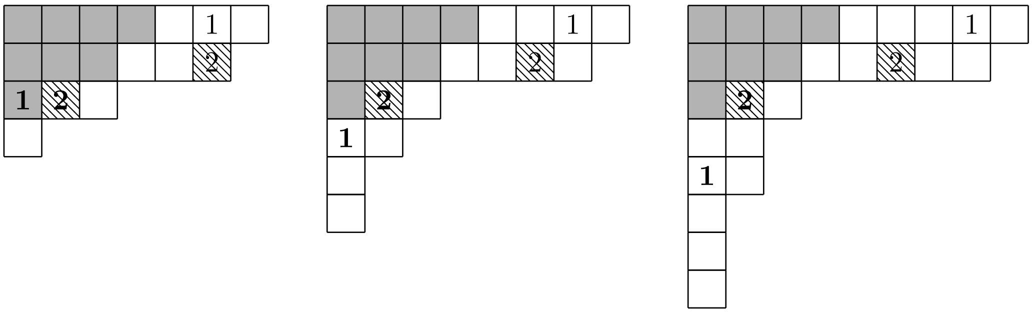
where the two shaded regions in the bottom 2 rows and rightmost 2 columns are marked with the possible entries. (As ever negative entries may be repeated in a column, and positive entries repeated in row.) Clearly, almost all the entries of t and u are determined by their weight. In particular both s and t are obtained from a tableau for the case $M-1$ by inserting $\boxed{\mathbf{1}\,|\,\mathbf{2}}$ as a new complete second row and $\boxed{1}$ as a new complete third column. By Definition 9.11, the 2-top and 1-left positions of $(2+M, 2^M)$ are both $(1,2)$. Thus these insertions are exactly as specified by the map $\mathcal{F}$ from Definition 9.14. We have shown that provided M is sufficiently large, the map

$$\boxed{s \,|\, t} \longmapsto \boxed{\mathcal{F}(s) \,|\, \mathcal{F}(t)}$$

is surjective (with inverse defined by deleting the hatched boxes and performing suitable shifts) and so the cardinality in (11.4) is ultimately constant. In fact $\mathcal{F}(s)$ and $\mathcal{F}(t)$ are semistandard provided that the 2-top positions of both s and t both contain $-\mathbf{2}$. As we show in Example 11.8, using Lemma 11.7 in the following subsection, this holds provided $M \geq \max(3, b+2)$,

11.4. **Lemma on positions for plethystic tableaux.** The critical positions are defined in Definition 9.11 and were seen in the non-skew case in the previous subsection. To remind the reader of the general definition we begin with an example in the skew case.

Example 11.5. We take $\kappa^- = (2,1)$ and $\kappa^+ = (1,1)$ so $\ell^- = \ell^+ = 2$. Let $\mu/\mu_\star = (7,6,3,1)/(4,3,1)$. The diagrams below show the 1-left, 2-left, 1-top and 2-top positions in the partitions $\mu/\mu_\star \oplus M(\kappa^-,\kappa^+)$ for $M \in \{0,1,2\}$. Following our usual convention, top positions, relevant to the insertion of negative entries, are marked by bold numbers. The skew partition $\mu/\mu_\star$ is $(a(\mu_\star)+\ell^-, \ell^+)$-large as it contains $(2,6)$, and $(\ell^-, \ell(\mu_\star))$-large as it contains $(3,2)$; all these positions are contained within $[\mu]$. Moreover, also as promised by Lemma 9.17, the top positions are no higher than row $\max(\ell(\mu_\star), \ell(\mu^+), \ell^+)$ and left positions are no further left than column $\ell^- + a(\mu_\star)$. The relevant boxes $(2,6)$ and $(3,2)$ are hatched, as in Figure 11.1.



We remark that if we changed $\mu_\star = (4,3,1)$ to (a) with $a \in \{0,1,2,3,4\}$ then the 1-top and 2-top positions remain unchanged, because they always lie in rows weakly below $\max(\ell(\mu_\star), \ell^+, \ell(\mu^+))$, and this statistic remains 3 since $\mu^+ = (5,4,1)$. The 1-left position is also unchanged, but the 2-left position is now $\bigl(2, \ell^- + \max(a, \mu^+_3)\bigr) = \bigl(2, 2 + \max(a,1)\bigr)$ which is $(2,3)$ if and only if $a \le 1$. Note that if $a \le 1$ then it is possible to insert a column of height 2 immediately right of column 3, and the entries of this column *must* be positive.

Definition 11.6. Let κ, μ and λ be partitions. Let $A \in \mathbb{N}_0$. We define $\mathrm{LP}\bigl(n, \mu : \lambda, \kappa : A\bigr)$ to be 0 if $\kappa = \varnothing$ and otherwise to be the maximum of

$$\frac{n\sum_{i=1}^k \mu_i - \sum_{i=1}^k \lambda_i - \mu_k + \max(A, \mu_{k+1})}{\kappa_k - \kappa_{k+1}}.$$

for $1 \le k \le \ell(\kappa)$, omitting any expressions with zero denominator.

The following lemma is the analogue of Lemma 9.16. The statement, apart from the change of bounds, is very similar, but the proof is somewhat easier, apart from some technicalities arising from the skew case, because the shape is known to be $\mu/\mu_\star \oplus M(\kappa^-, \kappa^+)$, rather than an arbitrary partition in an interval for the ℓ^--twisted dominance order. The relevant positions are defined in Definition 9.11.

Lemma 11.7. *Let κ^-, κ^+ be partitions. Fix $\ell^- = \ell(\kappa^-)$ and $\ell^+ = \ell(\kappa^+)$. Let $\mu/\mu_\star$ be an $\bigl(\ell^- + a(\mu_\star), \ell^+\bigr)$-large and $\bigl(\ell^-, \ell(\mu_\star)\bigr)$-large skew partition. Let λ and ω be (ℓ^-, ℓ^+)-large partitions of $n|\mu/\mu_\star|$. Let*

$$\pi \in \bigl[\lambda \oplus nM(\kappa^-, \kappa^+), \omega \oplus nM(\kappa^-, \kappa^+)\bigr]_{\unlhd}.$$

Let $T \in \mathrm{PSSYT}\bigl(\rho, \mu/\mu_\star \oplus M(\kappa^-, \kappa^+)\bigr)_{(\pi^-, \pi^+)}$ and let t be a $\mu/\mu_\star \oplus M(\kappa^-, \kappa^+)$-inner tableau of T. Let L be the maximum of

- $\mathrm{LP}\bigl(n, \mu' : \lambda^-, \kappa^- : \max(\ell(\mu_\star), \ell(\mu^+), \ell^+)\bigr)$
- $\mathrm{LP}\bigl(n, \mu : \lambda^+, \kappa^+ : a(\mu_\star) + \ell^-\bigr) + |\lambda^+| - |\omega^+|$.

If $M - 1 \ge L$ then

(i) *the r^--bottom position of t contains $-r^-$ if $r^- < \ell^-$ and $\kappa^-_{r^-} > \kappa^-_{r^-+1}$;*
(ii) *the ℓ^--bottom position of t contains $-\ell^-$;*
(iii) *if $\kappa^- \neq \varnothing$ and* either *$\kappa^+ \neq \varnothing$* or *$\mu_\star \neq \varnothing$* or *$\mu^+ \neq \varnothing$ then the box $\bigl(\max(\ell(\mu_\star), \ell(\mu^+), \ell^+), \ell^-\bigr)$ of t contains a negative entry;*
(iv) *the r^+-right position of t contains r^+ if $r^+ < \ell^+$ and $\kappa^+_{r^+} > \kappa^+_{r^++1}$;*
(v) *the ℓ^+-right position of t contains ℓ^+.*

Moreover if $M \geq L$ then the same results hold replacing 'bottom' with 'top' and 'right' with 'left', except that

(ii) *if $\mu_\star = \varnothing$ and $\kappa^+ = \varnothing$ and $\mu^+ = \varnothing$ then the ℓ^--top position is $(0, \ell^-)$;*

(iv) *and* (v) *if $\mu_{r^++1} \leq \ell^- + a(\mu_\star)$ and so the r^+-left-position is $\big(r^+, \ell^- + a(\mu_\star)\big)$, then it may contain $-\ell^-$.*

Proof. Since μ is $\big(\ell^- + a(\mu_\star), \ell^+\big)$-large, it is (ℓ^-, ℓ^+)-large. Therefore, by Lemma 9.6, we have $\big(\mu \oplus M(\kappa^-, \kappa^+)\big)^- = \mu^- + M\kappa^-$ and $\big(\mu \oplus M(\kappa^-, \kappa^+)\big)^+ = \mu^+ + M\kappa^+$. For use throughout the proof we define a subpartition α of $\mu \oplus M(\kappa^-, \kappa^+)$ containing $\mu_\star$ by

$$\alpha_i = \min(\mu_{\star i} + \ell^-, \mu_i) \text{ for } 1 \leq i \leq \ell(\mu) + Ma(\kappa^-). \tag{11.5}$$

Thus $[\alpha/\mu_\star]$ consists of the first ℓ^- boxes in each row of $\mu/\mu_\star \oplus M(\kappa^-, \kappa^+)$, or the whole row if it has fewer than ℓ^- boxes. We show $[\alpha/\mu_\star]$ in Figure 11.1. As a final preliminary, for ease of reference, we record the following immediate corollary of Lemma 9.17: the r^--top position of $\mu/\mu_\star \oplus M(\kappa^-, \kappa^+)$ is

$$\begin{cases} \big(\mu'_{r^-+1} + M\kappa^-_{r^-+1}, r^-\big) & \text{if } r^- < \ell^- \\ \big(\max(\ell(\mu_\star), \ell(\mu^+), \ell^+), \ell^-\big) & \text{if } r^- = \ell^-. \end{cases} \tag{11.6}$$

and the r^+-left position of $\mu/\mu_\star \oplus M(\kappa^-, \kappa^+)$ is

$$\begin{cases} \big(r^+, \mu_{r^++1} + M\kappa^+_{r^++1}\big) & \text{if } r^+ < \ell^+ \\ \big(\ell^+, \max(\ell^- + a(\mu_\star), \mu_{\ell^++1})\big) & \text{if } r^+ = \ell^+. \end{cases} \tag{11.7}$$

For (i), we have $r^- < \ell^-$ so may suppose $\kappa^- \neq \varnothing$. If s is an arbitrary $\mu/\mu_\star \oplus M(\kappa^-, \kappa^+)$-inner tableau in T then the total number of entries of s in the set $\{-1, \ldots, -r^-\}$ is at most $\sum_{j=1}^{r^-} \mu'_j + M \sum_{j=1}^{r^-} \kappa^-_j$. To simplify some arithmetic steps later, we write this quantity as $D^- + \mu'_{r^-} + M\kappa^-_{r^-}$ where

$$D^- = \sum_{j=1}^{r^--1} \mu'_j + M \sum_{j=1}^{r^--1} \kappa^-_j \tag{11.8}$$

is (as we have just seen) an upper bound on the number of entries of s in $\{-1, \ldots, -(r^- - 1)\}$. By (11.6) the r^--top position of s is $\big(\mu'_{r^-+1} + M\kappa^-_{r^-+1}, r^-\big)$. We assume, for a contradiction, that, in *some* tableau t in T, this position has either a positive entry, or some $-q$ with $-q > -r^-$ in the order in Definition 3.7, meaning that $q > r^-$. Define a subpartition τ of α by

$$\tau_i = \begin{cases} \min(\mu_{\star i} + r^-, \mu_i) & \text{if } 1 \leq i < \mu'_{r^-+1} + M\kappa^-_{r^-+1} \\ \min(\mu_{\star i} + r^- - 1, \mu_i) & \text{if } i \geq \mu'_{r^-+1} + M\kappa^-_{r^-+1}. \end{cases} \tag{11.9}$$

Thus $[\tau/\mu_\star]$ consists of the first r^- boxes in each row of $\mu/\mu_\star \oplus M(\kappa^-, \kappa^+)$, or the whole row if it has fewer than r^- boxes, *except* that the boxes of $\mu/\mu_\star$ at or below the r^--top position known, by our assumption, not to contain $-r^-$, are excluded. By construction τ contains $\mu_\star$ and, by our assumption, all the entries of t in $\{-1, \ldots, -r^-\}$ are contained in $[\tau]$. (This is the blue shaded region in Figure 11.1.) Hence using (11.8), the total number of entries of t in

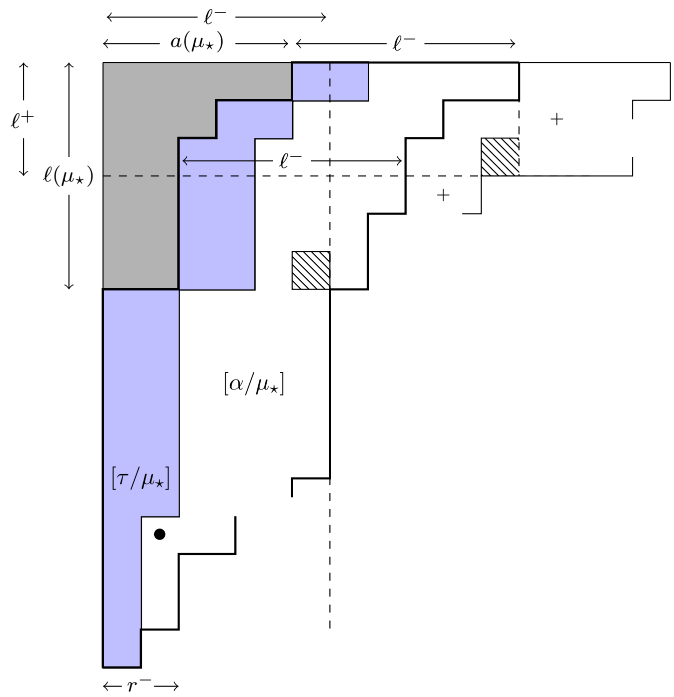


FIGURE 11.1. Entries in a tableau $t \in \mathrm{SSYT}\big(\mu/\mu_\star \oplus M(\kappa,\kappa^+)\big)_{(\pi^-,\pi^+)}$ when $\mu/\mu_\star$ is $(\ell^- + a(\mu/\mu_\star), \ell^+)$-large and $(\ell^-, \ell(\mu_\star))$-large, and so contains the hatched boxes. The case $\ell(\mu/\mu_\star) > \ell(\mu^+) > \ell^+$, in which the skew part of the partition is most important in determining the quantity $\max(\ell(\mu_\star), \ell(\mu^+), \ell^+)$ defining the row of the ℓ^--top position is shown. The heavy lines show the region $[\alpha/\mu_\star]$ defined in (11.5) that contains all negative entries of t in the proof of Lemma 11.7(i). It contains the region $[\tau/\mu_\star]$ shaded in blue which contains all entries of $\{-1,\ldots,-r^-\}$, under the assumption in this part of the proof that the r^--top position marked $\bullet$ does not contain $-r^-$. In the figure we have taken $r^- = 2$.

the set $\{-1,\ldots,-r^-\}$ is strictly less than $D^- + \mu'_{r^-+1} + M\kappa^-_{r^-+1}$. The $n-1$ tableaux other than t forming T have at most $(n-1)D^- + (n-1)\big(\mu'_{r^-} + M\kappa^-_{r^-}\big)$ entries in $\{-1,\ldots,-r^-\}$. Therefore T has strictly less than

$$\begin{aligned}(n-1)D^- + (n-1)\big(\mu'_{r^-} + M\kappa^-_{r^-}\big) + D^- + \mu'_{r^-+1} + M\kappa^-_{r^-+1} \\ = n\sum_{j=1}^{r^-}\mu'_j - \mu'_{r^-} + nM\sum_{j=1}^{r^-}\kappa^-_j - M\kappa^-_{r^-} + \mu'_{r^-+1} + M\kappa^-_{r^-+1} \qquad (11.10)\end{aligned}$$

entries in the set $\{-1,\ldots,-r^-\}$. On the other hand, as $\pi \trianglerighteq \lambda \oplus nM(\kappa^-,\kappa^+)$, and λ is (ℓ^-,ℓ^+) large, it follows from Lemma 6.7(a) and Lemma 9.6 that there are at least

$$\sum_{j=1}^{r^-}\lambda^-_j + nM\sum_{j=1}^{r^-}\kappa^-_j \qquad (11.11)$$

entries of T in $\{-1,\dots,-r^-\}$. From (11.10) and (11.11) we obtain

$$n\sum_{j=1}^{r^-}\mu'_j-\mu'_{r^-}+\mu'_{r^-+1}-\sum_{j=1}^{r^-}\lambda^-_j>M(\kappa^-_{r^-}-\kappa^-_{r^-+1}). \tag{11.12}$$

This contradicts the first bound. Therefore (i) holds for top positions. For bottom positions we mimic the proof of Lemma 9.16, and run the same argument, replacing each μ'_{r^-+1} with $\mu'_{r^-+1}+\kappa^-_{r^-}-\kappa^-_{r^-+1}$, and obtain (11.12) with $\kappa^-_{r^-}-\kappa^-_{r^-+1}$ subtracted from the right-hand side, which therefore becomes $(M-1)(\kappa^-_{r^-}-\kappa^-_{r^-+1})$. We then get a contradiction as before from the first bound.

For (ii), we may again suppose $\kappa^-\neq\varnothing$; then by (11.6) the ℓ^--top position of t is $\big(\max(\ell(\mu_\star),\ell(\mu^+),\ell^+),\ell^-\big)$. We may suppose that either $\mu_\star\neq\varnothing$ or $\kappa^-\neq\varnothing$ or $\mu^+\neq\varnothing$, so that this is a box of $\mu/\mu_\star$. The proof is then almost identical to (i) using (11.6) to replace every appearance of μ'_{r^-+1} in the argument for (i) with $\max(\ell(\mu_\star),\ell(\mu^+),\ell^+)$. This also proves (iii), since the hypotheses for (iii) imply that $\big(\max(\ell(\mu_\star),\ell(\mu^+),\ell^+),\ell^-\big)$ is a box of $\mu/\mu^\star$.

Comparing (11.6) and (11.7) for $r^-<\ell^-$ and $r^+<\ell^+$, we see that they are symmetric with respect to conjugation. While a further non-symmetric change is necessary later on, this indicates the most conceptual way to prove (iv). Read the proof of (i), replacing μ' with μ and κ^- with κ^+. Thus the subpartition τ is now defined by

$$\tau'_j=\begin{cases}\min(\mu_\star{}'_j+r^+,\mu_j) & \text{if } 1\le j<\mu_{r^++1}+M\kappa^+_{r^++1}\\ \min(\mu_\star{}'_i+r^+-1,\mu'_i) & \text{if } j\ge\mu'_{r^++1}+M\kappa^+_{r^++1}\end{cases}$$

and the analogue of (11.10) is that there are strictly less than

$$n\sum_{i=1}^{r^+}\mu_j-\mu_{r^+}+nM\sum_{i=1}^{r^+}\kappa^+_j-M\kappa^+_{r^+}+\mu_{r^++1}+M\kappa^+_{r^++1} \tag{11.13}$$

entries of T in $\{1,\dots,r^+\}$. The only non-symmetric change is that from $\pi\trianglerighteq\lambda\oplus nM(\kappa^-,\kappa^+)$ and Lemma 6.7(b) we now get

$$\pi^++\big(|\lambda^+|+nM|\kappa^+|-|\pi^+|\big)\trianglerighteq\lambda^++nM\kappa^+$$

where $|\pi^+|\le|\lambda^+|+nM|\kappa^+|$. We must therefore bring in the upper bound $\pi\trianglelefteq\omega\oplus nM(\kappa^-,\kappa^+)$ to get, again by Lemma 6.7(b), $|\pi^+|\ge|\omega^+|+nM|\kappa^+|$. Hence

$$|\lambda^+|+nM|\kappa^+|-|\pi^+|\le\big(|\lambda^+|+nM|\kappa^+|\big)-\big(|\omega^+|+nM|\kappa^+|\big)=|\lambda^+|-|\omega^+|.$$

The replacement for (11.11) is therefore that there are at least

$$\sum_{i=1}^{r^+}\lambda^+_i+nM\sum_{i=1}^{r^+}\kappa^+_i-|\lambda^+|+|\omega^+| \tag{11.14}$$

entries of T in $\{1,\dots,r^+\}$. From (11.13) and (11.14) we get

$$n\sum_{i=1}^{r^+}\mu_i-\mu_{r^+}+\mu_{r^++1}-\sum_{i=1}^{r^+}\lambda^+_j+|\lambda^+|-|\omega^+|>M(\kappa^+_{r^+}-\kappa^+_{r^++1}). \tag{11.15}$$

This contradicts the second bound. The modifications for right positions are precisely analogous to the negative case; the relevant quantity to subtract from the right-hand side of (11.15) is $\kappa^+_{r^+} - \kappa^+_{r^++1}$, and as before we get a contradiction from the second bound.

For (v) we first note that, by (11.7), the ℓ^+-left position of t is $\bigl(\ell^+, \max(\ell^- + a(\mu_\star), \mu_{\ell^++1})\bigr)$. Since μ is $\bigl(\ell^- + a(\mu_\star), \ell^+\bigr)$-large, we have $(\mu - \mu_\star)_{\ell^+} \geq \ell^- + a(\mu_\star) - \mu_{\star\ell^+} \geq \ell^-$. Therefore if this position contains a negative entry, equality holds and the entry is $-\ell^-$. In the remaining case, and for the ℓ^+-right position, the proof of (iv) adapts routinely. This completes the proof. □

1	**2**
1	**2**

Example 11.8. Take $\mu = (2,2)$, $\mu_\star = \varnothing$, $(\kappa^-, \kappa^+) = \bigl((1,1),(1)\bigr)$ and $\lambda = (8-b, b) \leftrightarrow \bigl\langle (2,2), (6-b, b-2)\bigr\rangle$ with $b \in \{2,3,4\}$ and $n = 2$ as in Example 11.4. Since $\omega_2\bigl((2,2)\bigr) = \bigl((2,2), \varnothing\bigr)$ is the signed weight of the 2-greatest semistandard signed tableau $t_2\bigl((2,2)\bigr)$ shown in the margin, we have

$$\omega_2^{(2)}\bigl((2,2)\bigr) \leftrightarrow 2\bigl\langle (2,2), \varnothing\bigr\rangle = \bigl\langle (4,4), \varnothing\bigr\rangle$$

and so we take $\omega = (2,2,2,2)$. Since $\kappa^-_1 = \kappa^-_2$, the important cases of Lemma 11.7 are (ii) and (v). We saw in Example 11.4 that the 2-top position and 1-left position are both $(1,2)$, and so the 2-bottom position is $(2,2)$ and the 1-right position is $(1,3)$. From (ii) we get that the 2-bottom position $(1,2)$ contains **2** in every $(2+M, 2, 2^M)$-tableau entry in a plethystic semistandard signed tableau of outer shape (2) provided

$$M - 1 \geq \mathrm{LP}(2, (2,2) : (2,2), (1,1) : 1) = \frac{2(2{+}2) - (2{+}2) - 2 + \max(1,0)}{1-0} = 3.$$

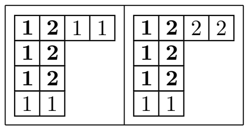
(Note that (ii) was proved using only the first bound in the statement of Lemma 11.7.) From (v), using that $|\lambda^+| = 4$ and $|\omega^+| = 0$, we get that the 1-right position $(1,3)$ contains 1 in every $(2+2M, 2, 2^M)$-tableau entry in a plethystic semistandard signed tableau of outer shape (2) provided

$$\begin{aligned} M - 1 &\geq \mathrm{LP}(2, (2,2) : (6-b, b-2), (1) : 2) + 4 - 0 \\ &= \frac{2.2 - (6-b) - 2 + \max(2,2)}{1} + 4 = \frac{b+2}{1} = b+2. \end{aligned}$$

(Again note that (v) was proved using only the second bound in the statement of Lemma 11.7.) Therefore provided $M - 1 \geq \max(3, b{+}2)$ the hatched boxes in the diagram in Example 11.4 contain **1** **2** and 1 and the insertion map from plethystic semistandard signed tableaux defined for M to plethystic semistandard signed tableaux defined for $M+1$ is surjective for $M \geq \max(3, b+2)$, as stated in Example 11.4. The table below shows the number of plethystic semistandard signed tableaux of outer shape (2), inner shape $(2{+}M, 2, 2^M)$ and signed weight $\bigl((2{+}2M, 2{+}2M), (6{-}b{+}2M, b{-}2)\bigr)$ as in (11.4) and the stable values of $\langle s_{(2)} \circ s_{(2+M,2,2^M)}, s_{(8-b+2M,b,2^{2M})}\rangle$ as in (11.3), for each $0 \leq M \leq 3$ and $b \in \{2,3,4\}$.

b	$M=0$	$M=1$	$M=2$	$M=3$	(11.4)	(11.3)	$b+2$	$\max(3,b-2)$
2	1	1	1	1	1	0	4	3
3	4	5	5	5	5	0	5	3
4	10	19	20	20	20	1	6	3

As expected the values are constant for $M \geq \max(3, b+2)$. For example, taking $M = 2$, the plethystic semistandard signed tableau in the margin is the unique element of $\mathrm{PSSYT}\big((2),(4,2,2,2)\big)_{(6,6),(6,2)}$ having 2 in the 1-right box $(1,3)$ of an inner $(4,2,2,2)$-tableau entry, and so not in the image of the insertion map from the 19 plethystic tableaux for $M=1$ to the 20 plethystic tableaux for $M = 2$. The insertion map is then surjective at each subsequent step. The final column in the table above is relevant to Example 11.14 below.



11.5. **The $\mathcal{G}$ insertion map on plethystic tableaux.** We now define the plethystic extension of the insertion map $\mathcal{F}$ in Definition 9.14. Recall from after Definition 3.9 that $\mathrm{PYT}(\rho, \mu/\mu_\star)$ denotes the set of plethystic signed tableaux of shape ρ having entries from the set $\mathrm{YT}(\mu/\mu_\star)$ of all signed tableaux of shape $\mu/\mu_\star$. Given a plethystic signed tableau $T \in \mathrm{PYT}(\rho, \mu/\mu_\star)$, we define its *conjugate* $T' \in \mathrm{PYT}(\rho', \mu/\mu_\star)$ by $T'(i,j) = T(j,i)$ for $(i,j) \in [\rho]$. Note that the conjugation is defined with respect to the outer Young diagram $[\rho]$; it does not change the shape of the inner tableaux of T.

Definition 11.9. Let κ^- and κ^+ be partitions. Let $\mu/\mu_\star$ be a $\big(\ell(\kappa^-) + a(\mu_\star), \ell(\kappa^+)\big)$-large and $\big(\ell(\kappa^-), \ell(\mu_\star)\big)$-large skew partition. Let ρ be a partition. Let $\rho^\dagger = \rho$ if $|\kappa^-|$ is even and let $\rho^\dagger = \rho'$ if $|\kappa^-|$ is odd. We define

$$\mathcal{G} : \mathrm{PSSYT}(\rho, \mu/\mu_\star) \to \mathrm{PYT}\big(\rho^\dagger, \mu/\mu_\star \oplus (\kappa^-, \kappa^+)\big)$$

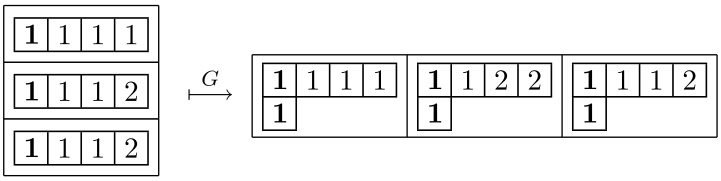
by applying $\mathcal{F}$ to each inner $\mu/\mu_\star$-tableau entry of $T \in \mathrm{PSSYT}(\rho, \mu/\mu_\star)$ to obtain $U \in \mathrm{PYT}\big(\rho, \mu/\mu_\star \oplus (\kappa^-, \kappa^+)\big)$. If $|\kappa^-|$ is even then we define $\mathcal{G}(T) = U$; if $|\kappa^-|$ is odd then we define $\mathcal{G}(T) = U'$.

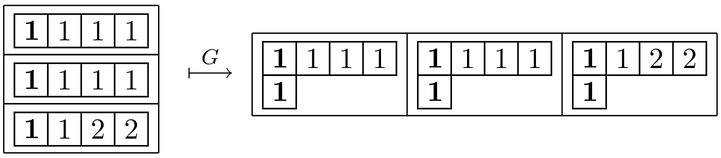
By Lemma 9.18(i), using the largeness hypotheses on $\mu/\mu_\star$, the map $\mathcal{G}$ is well-defined.

The following example is relevant to the special case of Theorem 1.1, taking $\nu = (n)$, $\mu = (m)$, $(\kappa^-, \kappa^+) = \big((1), \varnothing\big)$, that $\langle s_{(n)^{(M)}} \circ s_{(m,1^M)}, s_{\lambda \sqcup (1^{nM})}\rangle$ is ultimately constant for any partition λ of mn, where $(n)^{(M)} = (n)$ if M is even and $(n)^{(M)} = (1^n)$ if M is odd. In fact, as we mentioned in the discussion of Theorem 1.1 in [8] in §1.7, provided $\ell(\lambda) \geq n$, the plethysm coefficient is constant for all $M \in \mathbb{N}_0$. Correspondingly, one can check that $\mathcal{G} : \mathrm{PSSYT}\big((n),(m,1^M)\big)_{(\lambda^-,\lambda^+)} \to \mathrm{PYT}\big((1^n),(m,1^{M+1})\big)$ is a bijection onto $\mathrm{PSSYT}\big((1^n),(m,1^{M+1})\big)_{(\lambda^- + (n), \lambda^+)}$.

Example 11.10. We take $\rho = (1^3)$, $\mu = (4)$ and $\lambda = (8,3,1)$. Take $\ell^- = 1$ and note that λ has 1-decomposition $\big\langle (3),(7,2)\big\rangle$. The map $\mathcal{G} : \mathrm{PSSYT}\big((1^3),(4)\big) \to \mathrm{PSSYT}\big((3),(4,1)\big)$ in Definition 11.9 for $(\kappa^-, \kappa^+) =$

$\big((1), \varnothing\big)$ is shown below on both elements of $\mathrm{PSSYT}\big((1^3),(4)\big)_{((3),(7,2))}$:





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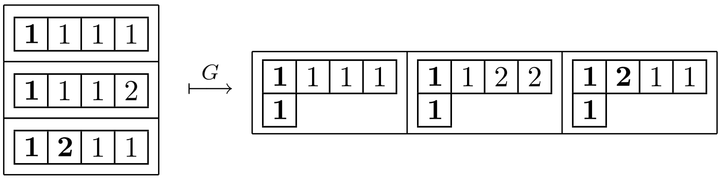
As expected from the fact that $\mathcal{G}$ is a bijection onto $\mathrm{PSSYT}\big((3),(4,1)\big)$, the image is $\mathrm{PSSYT}\big((3),(4,1)\big)_{((6),(7,2))}$. Note that the conjugation in the outer shape is essential: in each plethystic tableau there is a repeated inner tableau, and this is permitted because each has sign -1 in the plethystic tableau of outer shape (1^3) and each has sign $+1$ in the plethystic tableaux of outer shape (3). Moreover, the example



shows that the convention that, in the right-hand plethystic semistandard signed tableau, negative inner tableaux are greater than positive inner tableaux, is essential to make the plethystic tableau semistandard.

The following two lemmas and proposition generalize Example 11.10 and show that $\mathcal{G}$ respects the semistandard condition on inner tableaux in different positions in a plethystic semistandard signed tableau, up to the technical order reversal required in Lemma 11.12(iii). Recall that the signed colexicographic order $<$ is defined in Definition 3.8.

Lemma 11.11. *Let $\tau/\tau_\star$ be a skew partition and let s and t be semistandard signed tableaux of shape $\tau/\tau_\star$ and the same sign such that $s < t$ in the signed colexicographic order.*

(i) *Let $\tilde{s}$ and $\tilde{t}$ be the signed tableaux of shape $\tau \sqcup (r^-)/\tau_\star$ obtained from s and t by inserting a single new row with entries $-1, \ldots, -r^-$ into each. If $\tilde{s}$ and $\tilde{t}$ are semistandard then $\tilde{s} < \tilde{t}$.*

(ii) *Let $\tilde{s}$ and $\tilde{t}$ be the signed tableaux of shape $\tau + (1^{r^+})/\tau_\star$ obtained from s and t by inserting a single new column with entries $1, 2, \ldots, r^+$. If $\tilde{s}$ and $\tilde{t}$ are semistandard then $\tilde{s} < \tilde{t}$.*

Proof. Let m be the rightmost column in which the multisets of entries in s and t differ. For (i), since a single new entry of $-k$ is added to column k for each $k \le r^-$ in both s and t, it is clear that m is again the rightmost column in which the multisets of entries in $\tilde{s}$ and $\tilde{t}$ differ. Since $\tilde{s}$ and $\tilde{t}$ have the same sign, the relative order of $\tilde{s}$ and $\tilde{t}$ is determined by the multisets $C(s)$

and $C(t)$ of entries of s and t in column m. If $m > r^-$ then $C(\tilde{s}) = C(s)$ and $C(\tilde{t}) = C(t)$ and hence the greatest entry (taken with multiplicity) still lies in $\tilde{t}$, and hence $\tilde{s} < \tilde{t}$. If $m \le r^-$ then $C(\tilde{s}) = C(s) \cup \{-m\}$ and $C(\tilde{t}) = C(t) \cup \{-m\}$, and again $\tilde{s} < \tilde{t}$. For (ii), suppose that the new column with entries $-1, \ldots, -r^-$ was inserted as column c, moving the existing columns $c, c+1, \ldots$ one box to the right. Thus if $m \ge c$ then we compare $\tilde{s}$ and $\tilde{t}$ on column $m+1$ and get $\tilde{s} < \tilde{t}$. Otherwise $m < c$ and since the inserted column c is equal in $\tilde{s}$ and $\tilde{t}$, we still compare on column m and again get $\tilde{s} < \tilde{t}$. □

As mentioned earlier, the following lemma re-uses a large part of the proof of Proposition 9.19 (Tableau Stability). Again we use the signed colexicographic order $<$ from Definition 3.8.

Lemma 11.12. *Let κ^- and κ^+ be partitions. Let $\mu/\mu_\star$ be a $\big(\ell(\kappa^-) + a(\mu_\star),\ \ell(\kappa^+)\big)$-large and $\big(\ell(\kappa^-), \ell(\mu_\star)\big)$-large skew partition. Let λ and ω be $\big(\ell(\kappa^-), \ell(\kappa^+)\big)$-large partitions of $n|\mu/\mu_\star|$. Let*

$$\pi \in \big[\lambda \oplus nM(\kappa^-, \kappa^+), \omega \oplus nM(\kappa^-, \kappa^+)\big]_{\unlhd}.$$

Let $T \in \mathrm{PSSYT}\big(\rho, \mu/\mu_\star \oplus M(\kappa^-, \kappa^+)\big)_{(\pi^-, \pi^+)}$ and let s and t be $\mu/\mu_\star \oplus M(\kappa^-, \kappa^+)$-inner tableaux of T such that $s < t$. Suppose that M is at least the maximum of

- $\mathrm{LP}\big(n, \mu' : \lambda^-, \kappa^- : \max(\ell(\mu_\star), \ell(\mu^+), \ell(\kappa^+))\big)$
- $\mathrm{LP}\big(n, \mu : \lambda^+, \kappa^+ : a(\mu_\star) + \ell(\kappa^-)\big) + |\lambda^+| - |\omega^+|$.

Then

(i) *We have $\mathcal{F}(s), \mathcal{F}(t) \in \mathrm{SSYT}\big(\mu/\mu_\star \oplus (M+1)(\kappa^-, \kappa^+)\big)$.*

(ii) *If $|\kappa^-|$ is even then $\mathcal{F}(s)$ and $\mathcal{F}(t)$ have the same sign and $\mathcal{F}(s) < \mathcal{F}(t)$.*

(iii) *If $|\kappa^-|$ is odd and s and t have the same sign then $\mathcal{F}(s)$ and $\mathcal{F}(t)$ have the same sign and $\mathcal{F}(s) < \mathcal{F}(t)$. Otherwise s is negative, t is positive, $\mathcal{F}(s)$ is positive, $\mathcal{F}(t)$ is negative, and $\mathcal{F}(s) > \mathcal{F}(t)$.*

Proof. We have all the hypotheses for Lemma 11.7. Using the results on left and top positions from this lemma, the proof of Proposition 9.19 generalizes to show that $\mathcal{F}(s)$ and $\mathcal{F}(t)$ are semistandard $\mu/\mu_\star \oplus (M+1)(\kappa^-, \kappa^+)$-tableaux. The only extra points to note are that, by Lemma 9.17, each top position is in row $\ell(\mu_\star)$ or lower, and each left position is in column $\ell(\kappa^-) + a(\mu_\star)$ or further right, and hence the inserted columns do not meet $\mu_\star$, *and*, by (iii) in the lemma, setting $\ell^- = \ell(\kappa^-)$, inserting or deleting a new row with entries $-1, \ldots, -\ell^-$ immediately below the ℓ^--top position gives a well-defined semistandard signed tableau. (This is where we need that this position is in row $\ell(\mu^+)$ or lower, as remarked on in the caption to Figure 9.2 and seen in the proof of the earlier lemma.) This proves (i).

Suppose that s and t have the same sign. Then, by Lemma 11.11, applied to each row and column insertion in turn, we have $\mathcal{F}(s) < \mathcal{F}(t)$. To prove (ii) and (iii) in the remaining case where s and t have opposite sign, observe that since $s < t$, it follows immediately from Definition 3.8 that s is negative and t is positive. If $|\kappa^-|$ is even then, by Lemma 9.18(ii), $\mathcal{F}$ is sign preserving, and so $\mathcal{F}(s)$ is negative and $\mathcal{F}(t)$ is positive, and $\mathcal{F}(s) < \mathcal{F}(t)$, proving (ii). Finally for (iii), when $|\kappa^-|$ is odd then, again by Lemma 9.18(ii), $\mathcal{F}$ is sign reversing, and so $\mathcal{F}(s)$ is positive and $\mathcal{F}(t)$ is negative, and $\mathcal{F}(s) > \mathcal{F}(t)$. □

We are now ready to prove the analogue of Proposition 9.19 (Tableau Stability). Again we re-use part of its proof. This is the point where we need the sign-reversed colexicographic order on semistandard signed tableaux, defined in Definition 3.8 and the corresponding set $\mathrm{PSSYT}^{\mp}$ of sign-reversed plethystic semistandard signed tableaux defined in Definition 3.10 and motivated in Example 11.10. Recall, as seen in this example, in $\mathrm{PSSYT}^{\mp}$, negative inner tableaux are *greater* than positive inner tableaux. The map $\mathcal{G}$ is defined in Definition 11.9.

Proposition 11.13 (Inner stability for plethystic tableaux)**.** *Let κ^- and κ^+ be partitions. Let ρ be a partition of n. Let $\mu/\mu_\star$ be a $\big(\ell(\kappa^-)+a(\mu_\star), \ell(\kappa^+)\big)$-large and $\big(\ell(\kappa^-), \ell(\mu_\star)\big)$-large skew partition. Let $\rho^\dagger = \rho$ if $|\kappa^-|$ is even and let $\rho^\dagger = \rho'$ if $|\kappa^-|$ is odd. Let ω be a $(\ell(\kappa^-), \ell(\kappa^+))$-large partition and let $\lambda \trianglelefteq \omega$ in the $\ell(\kappa^-)$-twisted dominance order. Let π be a partition in the interval*

$$\big[\lambda \oplus nM(\kappa^-,\kappa^+), \omega \oplus nM(\kappa^-,\kappa^+)\big]_{\trianglelefteq}$$

for the $\ell(\kappa^-)$-twisted dominance order. If M is at least

- $\mathrm{LP}\big(n, \mu' : \lambda^-, \kappa^- : \max(\ell(\mu_\star), \ell(\mu^+), \ell(\kappa^+))\big)$
- $\mathrm{LP}\big(n, \mu : \lambda^+, \kappa^+ : a(\mu_\star) + \ell(\kappa^-)\big) + |\lambda^+| - |\omega^+|$.

then the map $\mathcal{G}$ is a well-defined bijection

$$\begin{aligned}\mathcal{G} :\ & \mathrm{PSSYT}\big(\rho, \mu/\mu_\star \oplus M(\kappa^-,\kappa^+)\big)_{(\pi^-,\pi^+)} \\ \longrightarrow & \begin{cases} \mathrm{PSSYT}\big(\rho^\dagger, \mu/\mu_\star \oplus (M+1)(\kappa^-,\kappa^+)\big)_{(\pi^-+n\kappa^-,\pi^++n\kappa^+)} & \text{if } |\kappa^-| \text{ is even} \\ \mathrm{PSSYT}^{\mp}\big(\rho^\dagger, \mu/\mu_\star \oplus (M+1)(\kappa^-,\kappa^+)\big)_{(\pi^-+n\kappa^-,\pi^++n\kappa^+)} & \text{if } |\kappa^-| \text{ is odd.}\end{cases}\end{aligned}$$

Proof. Note that we have all the hypotheses for Lemma 11.12. Let $T \in \mathrm{PSSYT}\big(\rho, \mu/\mu_\star \oplus M(\kappa^-,\kappa^+)\big)_{(\pi^-,\pi^+)}$. By this lemma, after applying $\mathcal{F}$ to each inner $\mu/\mu_\star \oplus M(\kappa^-,\kappa^+)$-tableau in T we have a plethystic signed tableau U of shape ρ having well-defined entries from $\mathrm{SSYT}^{\pm}\big(\mu/\mu_\star \oplus (M+1)(\kappa^-,\kappa^+)\big)$. By Lemma 11.12(i), U has signed weight $(\pi^- + n\kappa^-, \pi^+ + n\kappa^+)$, as required.

Suppose that $|\kappa^-|$ is even. Then $\mathcal{G}(T) = U$. By Lemma 11.12(ii), $\mathcal{F}$ preserves strict equality in the signed colexicographic order, it follows that $U \in \mathrm{PSSYT}\big(\rho, \mu/\mu_\star \oplus M(\kappa^-,\kappa^+)\big)_{(\pi^-+n\kappa^-,\pi^++n\kappa^+)}$, as required.

Now suppose that $|\kappa^-|$ is odd. Then $\mathcal{G}(T) = U'$. By Lemma 11.12(ii), $\mathcal{F}$ preserves strict inequality in the signed colexicographic order on inner tableaux of the same sign, it follows that the conjugate plethystic signed tableau U' is semistandard with respect to inner $\mu/\mu_\star \oplus (M+1)(\kappa^-,\kappa^+)$-tableaux *of the same sign.* Moreover, since in T, equal positive inner tableaux are repeated only in the same row, and equal negative inner tableaux are repeated only in the same column, the same holds in U, swapping 'row' and 'column'.

Let β be the subpartition of ρ such that the negative $\mu/\mu_\star \oplus M(\kappa^-,\kappa^+)$-tableau entries in T lie in $[\beta]$. In U, since $\mathcal{F}$ is sign reversing, the positive inner $\mu/\mu_\star \oplus (M+1)(\kappa^-,\kappa^+)$-tableau lie in $[\beta]$ and the negative inner $\mu/\mu_\star \oplus (M+1)(\kappa^-,\kappa^+)$-tableau entries lie in $[\rho/\beta]$. Therefore the conjugate plethystic signed tableau U' is semistandard with respect to the sign-reversed colexicographic order; that is $U' \in \mathrm{PSSYT}^{\mp}\big(\rho^\dagger, \mu/\mu_\star \oplus (M+1)(\kappa^-,\kappa^+)\big)_{(\pi^- + n\kappa^-, \pi^+ + n\kappa^+)}$ as required.

We have now shown that the image $\mathcal{G}(T)$ is in the set specified in the proposition. Let t be an inner $\mu/\mu_\star \oplus (M+1)(\kappa^-,\kappa^+)$-tableau entry of $\mathcal{G}(T)$. Using the results on right and bottom positions from Lemma 11.7, and noting that the removed rows are strictly below row $\ell(\mu_\star)$ and the removed columns are strictly to the right of column $\ell(\kappa^-) + a(\mu_\star)$, it follows as in the proof of Proposition 9.19 that t is in the image of $\mathcal{F}$. Hence $\mathcal{G}$ is surjective. □

We remark that the bounds in Proposition 11.13 are typically not optimal.

Example 11.14. We continue Example 11.8 to show how the general bounds coming ultimately from Lemma 11.7 can be sharpened by considering negative and positive entries together. From the diagram in Example 11.4 we see that the entries of 1 in the two semistandard tableaux s and t forming

$$T \in \mathrm{PSSYT}\big((2), (2+N, 2, 2^N)\big)_{((2+2N, 2+2N), (6-b+2N, b-2))}$$

lie either in the bottom two rows of their first two columns, or in the N boxes ending their first rows. At most two 1s can be in the first two columns of each. Therefore each of s and t has at most $2+N$ entries of 1, and if the 1-right position $(1,3)$ in either s or t does not contain 1 then the total number of entries of 1 in T is at most $(2+N)+2 = 4+N$. Therefore $4+N \geq 6-b+2N$ and we deduce that $N \leq b-2$. Hence if $N \geq b-1$ then the 1-right position $(1,3)$ in both tableaux s and t contains 1. Therefore, provided $M \geq b-2$, and $M \geq 3$ (as we needed from Lemma 11.7), insertion of $\boxed{1}$ and $\boxed{\mathbf{1}}\boxed{\mathbf{2}}$ defines a surjective map

$$\begin{aligned}&\mathrm{PSSYT}\big((2), (2+M, 2, 2^M)\big)_{((2+2M, 2+2M), (6-b+2M, b-2))}\\ &\quad\longrightarrow \mathrm{PSSYT}\big((2), (2+(M+1), 2, 2^{M+1})\big)_{((2+2(M+1), 2+2(M+1)), (6-b+2(M+1), b-2))}.\end{aligned}$$

(Note that in this context N becomes $M+1$, hence $M \geq b-2$ and $N \geq b-1$.) This gives the improved bound $\max(3, b-2)$ shown in the table in Example 11.8. Note that the bound from Proposition 9.19 is $M \geq 2$, so this bound still holds.

11.6. **Proof of Theorem 1.1.** We prove Theorem 1.1 with an explicit stability bound. By Remarks 3.2 and 10.4 there is no loss of generality in the 'largeness' hypotheses in the theorem. The L and LP bounds are defined in Definitions 9.2 and 11.6, respectively. (Remark 9.1 explains the small difference in notation for the intervals in the first two bounds.) As long promised, we use the Signed Weight Lemma (Lemma 7.3) for the main part of the proof. An example of the six bounds, proving (1.2) in §1.7, is given after the proof.

Theorem 11.15 (Signed inner stability with bound)**.** *Let κ^- and κ^+ be partitions. Let ν be a partition of n. Fix $\ell^- = \ell(\kappa^-)$ and $\ell^+ = \ell(\kappa^+)$. Let $\mu/\mu_\star$ be a $\bigl(\ell^- + a(\mu_\star), \ell^+\bigr)$-large and $\bigl(\ell^-, \ell(\mu_\star)\bigr)$-large and $\bigl(\ell^- + 1, \ell^+\bigr)$-weight large for $\ell(\kappa^-)$ skew partition. Let ω be the partition $\omega^{(n)}_{\ell^-}(\mu/\mu_\star)$ of $n|\mu/\mu_\star|$ defined in Definition 10.6. Let λ be an (ℓ^-, ℓ^+)-large partition. Let L be the maximum of*

- $\mathrm{L}\bigl([\lambda^-, \omega^-]^{(\ell^-)}_{\underline{\blacktriangleleft}}, \kappa^-\bigr)/n$,
- $\mathrm{L}\bigl([\lambda^+, \omega^+ + (|\lambda^+| - |\omega^+|)]_{\unlhd}, \kappa^+\bigr)/n$
- $\bigl(\omega^+_1 + \omega^+_2 - 2\lambda^+_1 + 2|\lambda^+| - 2|\omega^+|\bigr)/n(\kappa^+_1 - \kappa^+_2)$,
- $\bigl(\max(\ell(\lambda^+), \ell^+) + |\omega^-| - |\lambda^-| - \omega^-_{\ell^-}\bigr)/n\kappa^-_{\ell(\kappa^-)}$.
- $\mathrm{LP}\bigl(n, \mu' : \lambda^-, \kappa^- : \max(\ell(\mu_\star), \ell(\mu^+), \ell(\kappa^+)\bigr)$
- $\mathrm{LP}\bigl(n, \mu : \lambda^+, \kappa^+ : a(\mu_\star) + \ell(\kappa^-)\bigr) + |\lambda^+| - |\omega^+|$

omitting the third if $\kappa^+_1 = \kappa^+_2$ and the fourth if $\kappa^- = \varnothing$. Then

$$\bigl\langle s_{\nu^{(M)}} \circ s_{\mu/\mu_\star \oplus M(\kappa^-, \kappa^+)}, s_{\lambda \oplus nM(\kappa^-, \kappa^+)} \bigr\rangle$$

is constant for $M \geq L$, where if $|\kappa^-|$ is even then $\nu^{(M)} = \nu$ for all M and if $|\kappa^-|$ is odd then $\nu^{(M)} = \nu$ if M is even and $\nu^{(M)} = \nu'$ if M is odd. Moreover if $\lambda \not\trianglelefteq \omega$ in the ℓ^--twisted dominance order then the plethysm coefficient is 0 *for all $M \in \mathbb{N}_0$.*

Proof. We apply the Signed Weight Lemma (Lemma 7.3). For $M \in \mathbb{N}_0$ set

$$\mathcal{P}^{(M)} = \bigl[\lambda \oplus M(\kappa^-, \kappa^+), \omega^{(n)}_{\ell^-}(\mu/\mu_\star) \oplus M(\kappa^-, \kappa^+)\bigr]_{\unlhd}.$$

If $\lambda \not\trianglelefteq \omega^{(n)}_{\ell^-}(\mu/\mu_\star)$ then Lemma 9.6 implies that

$$\lambda \oplus M(\kappa^-, \kappa^+) \not\trianglelefteq \omega^{(n)}_{\ell^-}(\mu/\mu_\star) \oplus M(\kappa^-, \kappa^+)$$

for all $M \in \mathbb{N}_0$ and hence, by Proposition 10.7, $\langle s_{\nu^{(M)}} \circ s_{\mu/\mu_\star \oplus M(\kappa^-, \kappa^+)}, s_{\lambda \oplus nM(\kappa^-, \kappa^+)}\rangle = 0$ for all $M \in \mathbb{N}_0$. Thus all the plethysm coefficients are zero, as claimed in the final part of the statement.

We may therefore assume that $\lambda \unlhd \omega^{(n)}_{\ell^-}(\mu/\mu_\star)$. By the hypothesis that $\mu/\mu_\star$ is $(\ell^- + 1, \ell^+)$-weight large, the partition is $\omega^{(n)}_{\ell^-}(\mu/\mu_\star)$ is $(\ell^- + 1, \ell^+)$-large, which is the other hypothesis required in Corollary 9.20. Hence, by this corollary, $(\mathcal{P}^{(M)})_{M \in \mathbb{N}_0}$ is a stable partition system with respect to the map $\pi \mapsto \pi \oplus (\kappa^-, \kappa^+)$ and the twisted symmetric functions $g_\pi = e_{\pi^-} h_{\pi^+}$. We take the subsystem $(\mathcal{Q}^{(M)})_{M \in \mathbb{N}_0}$ where $\mathcal{Q}^{(M)} = \mathcal{P}^{(nM)}$. Up to the factor $1/n$, the first four bounds in our hypotheses are those in Corollary 9.20. Therefore $\mathcal{Q}^{(M)}$ is stable for $M \geq L$.

We are now ready to verify the conditions in the Signed Weight Lemma (Lemma 7.3) taking $\nu^{(M)}$ as already defined, $\mu/\mu_\star{}^{(M)} = \mu/\mu_\star \oplus M(\kappa^-, \kappa^+)$, and $\mathcal{Q}^{(M)}$ as our stable partition system. Since $\lambda \oplus nM(\kappa^-, \kappa^+) \in \mathcal{Q}^{(M)}$, this implies the theorem.

Condition (i) *in the Signed Weight Lemma.* By Lemma 11.3 the stable partition system $\mathcal{Q}^{(M)}$ satisfies condition (i) of the Signed Weight Lemma (Lemma 7.3) for the plethysms $s_{\nu^{(M)}} \circ s_{\mu/\mu_\star \oplus M(\kappa^-, \kappa^+)}$.

Condition (ii) *in the Signed Weight Lemma.* We must verify that

$$\begin{aligned}\bigl|\mathrm{PSSYT}\bigl(\nu^{(M)}, \mu/\mu_\star{}^{(M)}\bigr)_{(\pi^-,\pi^+)}\bigr| \\ = \bigl|\mathrm{PSSYT}\bigl(\nu^{(M+1)}, \mu/\mu_\star{}^{(M+1)}\bigr)_{(\pi^-+n\kappa^-,\pi^++n\kappa^+)}\bigr|\end{aligned} \tag{11.16}$$

for all $M \in \mathbb{N}_0$. By hypothesis $\mu/\mu_\star$ is $\bigl(\ell(\kappa^-) + a(\mu_\star), \ell(\kappa^+)\bigr)$-large and $\bigl(\ell(\kappa^-), \ell(\mu_\star)\bigr)$-large as required in Proposition 11.13. The final two bounds on M in the statement are those required by Proposition 11.13. Fix $M \in \mathbb{N}$ at least these bounds and let $\rho = \nu^{(M)}$. By this proposition we have

$$\bigl|\mathrm{PSSYT}\bigl(\rho, \mu/\mu_\star \oplus M(\kappa^-,\kappa^+)_{(\pi^-,\pi^+)}\bigr|$$
$$= \begin{cases} \bigl|\mathrm{PSSYT}\bigl(\rho^\dagger, \mu/\mu_\star \oplus (M{+}1)(\kappa^-,\kappa^+)\bigr)_{(\pi^-+n\kappa^-,\pi^++n\kappa^+)}\bigr| & \text{if } |\kappa^-| \text{ is even} \\ \bigl|\mathrm{PSSYT}^{\mp}\bigl(\rho^\dagger, \mu/\mu_\star \oplus (M{+}1)(\kappa^-,\kappa^+)\bigr)_{(\pi^-+n\kappa^-,\pi^++n\kappa^+)}\bigr| & \text{if } |\kappa^-| \text{ is odd}\end{cases}$$

for all $\pi \in \mathcal{Q}^{(M)}$. If $|\kappa^-|$ is even then $\nu^{(M+1)} = \rho^\dagger = \rho = \nu^{(M)}$ and we have (11.16). Otherwise we use the final part of Lemma 5.5 to obtain

$$\begin{aligned}\bigl|\mathrm{PSSYT}^{\mp}\bigl(\rho', \mu/\mu_\star \oplus (M+1)(\kappa^-,\kappa^+)_{(\pi^-+n\kappa^-,\pi^++n\kappa^+)}\bigr| \\ = \bigl|\mathrm{PSSYT}\bigl(\rho', \mu/\mu_\star \oplus (M+1)(\kappa^-,\kappa^+)_{(\pi^-+n\kappa^-,\pi^++n\kappa^+)}\bigr|,\end{aligned}$$

and since $\rho' = \nu^{(M+1)}$ and $\rho = \nu^{(M)}$, we again get (11.16). Therefore the stable partition system $\mathcal{Q}^{(M)}$ satisfies condition (ii) of the Signed Weight Lemma. □

Example 11.16. We use Theorem 11.15 to show that the plethysm coefficients $\langle s_{\nu^\dagger} \circ s_{\mu'+(1^{\ell(\mu')})\sqcup(1^M)}, s_{\lambda'+(1^{n\ell(\mu')})\sqcup(1^{nM})}\rangle$ in (1.2) relevant to [8, Theorem 1.1] are constant for all $M \geq 0$. (All we need about $\nu^\dagger$ is that it is a partition of n.) Take $\kappa^- = (1)$ and $\kappa^+ = \varnothing$ in Theorem 11.15 and replace ν in the theorem with $\nu^\dagger$ and μ in the theorem with $\mu' + (1^{\ell(\mu')})$. Since $\ell^- = 1$, the negative part of a partition α is simply $(\ell(\alpha))$ and the 1-decomposition is

$$\bigl\langle(\ell(\alpha)), (\alpha_1 - 1, \ldots, \alpha_k - 1)\bigr\rangle \tag{11.17}$$

where k is greatest such that $\alpha_k \geq 2$. Therefore the negative part of $\lambda' + (1^{n\ell(\mu')})$ is $\max(\ell(\lambda'), n\ell(\mu'))$. Denote this quantity P. Since the greatest signed tableau $t_1(\mu' + (1^{\ell(\mu')}))$ of shape $\mu' + (1^{\ell(\mu')})$ defined in Definition 4.2 has $\ell(\mu')$ entries of -1, we have $\omega^- = (n\ell(\mu'))$. Therefore the first bound in Theorem 11.15 is

$$\text{Ł}\bigl([(P), n\ell(\mu')]^{(1)}_{\underline{\blacktriangleleft}}, (1)\bigr) = \frac{n\ell(\mu') - P - n\ell(\mu')}{n} = -\frac{P}{n} < 0.$$

(Note that the exceptional case in Definition 9.2 applies, giving a smaller bound than that obtained by using L_1.) Since $\kappa^+ = \varnothing$, the second and third bounds vanish. Now observe that, by (11.17), if $\ell(\lambda') < n\ell(\mu')$ then the positive part of $\lambda' + (1^{n\ell(\mu')})$ is λ', while if $\ell(\lambda') \geq n\ell(\mu')$ then the positive part has length at most $\ell(\lambda')$. Therefore $\ell(\lambda^+) \leq \ell(\lambda')$ and the fourth bound is at most

$$\frac{\max(\ell(\lambda'), 0) + n\ell(\mu') - P - n\ell(\mu')}{n} \leq \frac{\ell(\lambda') - P}{n} \leq 0.$$

Since the positive part of $\mu' + (1^{\ell(\mu')})$ is μ', the fifth bound is

$$\begin{aligned}\mathrm{LP}\big(n, (\mu' + (1^{\ell(\mu')}))' : (P), (1) : \max(0, \ell(\mu'), 0)\big) \\ = \frac{n\ell(\mu') - P - \ell(\mu') + \max(\ell(\mu'), \ell(\mu'), 0)}{1-0} \leq 0.\end{aligned}$$

Since $\kappa^+ = \varnothing$, the sixth bound vanishes. Hence, as claimed earlier in §1.7, the plethysm coefficient is constant for all $M \geq 0$. Finally, note that a semistandard signed tableau of shape $\mu' + (1^{\ell(\mu')})$ can have at most $\ell(\mu')$ entries of -1. Therefore if $\ell(\lambda') > n\ell(\mu')$, and so $P > n\ell(\mu')$, the set $\mathrm{PSSYT}\big(\nu^\dagger, \mu + (1^{\ell(\mu')})\big)_{((\ell(\lambda')),\pi^+)}$ is empty for any partition π^+. Correspondingly, since $\ell(\lambda') > n\ell(\mu')$ implies that then $\lambda \not\trianglelefteq \omega$ in the 1-twisted dominance order, it follows from the final part of Theorem 11.15 that the plethysm coefficient vanishes when $M = 0$. Since this is its constant value, it vanishes for all $M \in \mathbb{N}_0$.

We end this section with a generalization of the final part of the example above. Observe that when $\mu_\star = \varnothing$, the greatest signed weight $\omega_{\ell^-}(\mu)$ is simply the ℓ^--decomposition of μ and so it is immediate from Definition 10.6 that the partition $\omega^{(n)}_{\ell^-}(\mu)$ has ℓ^--decomposition $n\langle \mu^-, \mu^+\rangle$. Hence, by the definition of the ℓ-twisted dominance order in Definition 6.6 we have $\lambda \trianglelefteq \omega^{(n)}_{\ell^-}(\mu)$ if and only if $(\lambda^-, \lambda^+) \trianglelefteq n(\mu^-, \mu^+)$, where $\trianglelefteq$ is the ℓ-signed dominance order of Definition 4.1. Therefore, the final part of Theorem 11.15 implies that, unless $(\lambda^-, \lambda^+) \trianglelefteq n(\mu^-, \mu^+)$,

$$\big\langle s_\nu \circ s_{\mu \oplus M(\kappa^-,\kappa^+)}, s_{\lambda \oplus nM(\kappa^-,\kappa^+)}\big\rangle = 0$$

for all $M \in \mathbb{N}_0$. This justifies the remark after Theorem 1.1 in the introduction.

12. The positive case of Theorem 1.1

In this section we state the case of Theorem 11.15 when $\kappa^- = \varnothing$, and then the still more special case where $\mu_\star = \varnothing$; as we saw in the survey in §1.7, this special case implies many of the stability results on plethysm coefficients in the literature. Moreover, as we mentioned in Remark 4.16, by applying the ω involution, these special cases easily imply the analogous special cases where $\kappa^+ = \varnothing$. The L and LP bounds are defined in Definitions 9.2 and 11.6, respectively. By Definition 3.5, $\mathrm{wt}(t)$ is the positive part of the signed weight of a tableau having only positive integer entries.

Corollary 12.1. *Let κ be a partition. Let ν be a partition of n, and let $\mu/\mu_\star$ be a $\big(a(\mu_\star), \ell(\kappa)\big)$-large skew partition. Let λ be a partition of $n|\mu/\mu_\star|$ with $\ell(\lambda) \geq \ell(\kappa)$. Let t be the semistandard tableau of shape $\mu/\mu_\star$ having 1, 2, ..., $\mu'_j - \mu'_{\star j}$ as its entries in column j, for each $j \leq a(\mu)$. Suppose that $\mathrm{wt}(t)$ is $\big(a(\mu_\star)+1, \ell(\kappa)\big)$-large. Set $\omega = n\,\mathrm{wt}(t)$. Then $\langle s_\nu \circ s_{\mu_\star + M\kappa}, s_{\lambda + nM\kappa}\rangle$ is constant for*

$$M \geq \max\Big(\frac{\mathrm{L}\big([\lambda, \omega]_\trianglelefteq, \kappa\big)}{n}, \mathrm{LP}\big(n, \mu : \lambda, \kappa : a(\mu_\star)\big)\Big)$$

If $\lambda \not\trianglelefteq \omega$ *then the plethysm coefficient is* 0 *for all* $M \in \mathbb{N}_0$*. Moreover if* $\eta \not\trianglelefteq \kappa$ *then* $\langle s_\nu \circ s_{\mu/\mu_\star + M\kappa}, s_{\lambda+nM\eta}\rangle = 0$ *for all* $M > L$ *where* L *is the minimum of*

$$\frac{\sum_{i=1}^k \omega_i - \sum_{i=1}^k \lambda_i}{n\bigl(\sum_{i=1}^k \eta_i - \sum_{i=1}^k \kappa_i\bigr)}$$

taken over those k *such that the denominator is strictly positive.*

Proof. The final part is immediate from Proposition 11.2 applied with $\kappa^- = \eta^- = \varnothing$ and $\kappa^+ = \kappa$, $\eta^+ = \eta$. For the main part we apply Theorem 11.15 with $\kappa^- = \varnothing$ and $\kappa^+ = \kappa$. By Definition 4.2, ω is $\omega_0^{(n)}(\mu/\mu_\star)$ and so by Lemma 4.4 we have $\mathrm{wt}(t) = \omega_0(\mu/\mu_\star)^+$. Therefore, by Definition 10.1, the condition that $\mu/\mu_\star$ is $(a(\mu_\star)+1, \ell(\kappa))$-weight large for 0 in Theorem 11.15 is equivalent to $\mathrm{wt}(t)$ being $(a(\mu_\star)+1, \ell(\kappa))$-large. The condition that λ is $\bigl(1, \ell(\kappa)\bigr)$-large simplifies to $\ell(\lambda) \geq \ell(\kappa)$ Of the six bounds, the first is 0, the second simplifies to $\mathrm{L}([\lambda,\omega]_\trianglelefteq, \kappa)$, the third to $(\omega_1 + \omega_2 - 2\lambda_1)/n(\kappa_1 - \kappa_2)$ which is either one of the bounds contributing to $\mathrm{L}([\lambda,\omega]_\trianglelefteq, \kappa)$, or ignored because $\kappa_1 = \kappa_2$, the fourth and fifth bounds are 0 and the sixth simplifies to $\mathrm{LP}\bigl(n, \mu : \lambda, \kappa : a(\mu_\star)\bigr)$. □

Corollary 12.2. *Let* κ *be a partition. Let* ν *be a partition of* n*, let* μ *be a partition of* m *and let* λ *be a partition of* mn *such that* $\ell(\mu) \geq \ell(\kappa)$ *and* $\ell(\lambda) \geq \ell(\kappa)$*. Then* $\langle s_\nu \circ s_{\mu+M\kappa}, s_{\lambda+nM\kappa}\rangle$ *is constant for* M *at least the maximum of*

$$\frac{n\sum_{i=1}^r \mu_i - \sum_{i=1}^r \lambda_i - \mu_r + \mu_{r+1}}{\kappa_r - \kappa_{r+1}}$$

for $1 \leq r \leq \ell(\kappa)$*, where any terms with zero denominator are ignored. If* $\lambda \not\trianglelefteq n\mu$ *then the plethysm coefficient is* 0 *for all* $M \in \mathbb{N}_0$*. Moreover if* $\eta \not\trianglelefteq \kappa$ *then* $\langle s_\nu \circ s_{\mu+M\kappa}, s_{\lambda+nM\eta}\rangle = 0$ *for all* $M > L$ *where* L *is the minimum of*

$$\frac{n\sum_{i=1}^k \mu_i - \sum_{i=1}^k \lambda_i}{n\bigl(\sum_{i=1}^k \eta_i - \sum_{i=1}^k \kappa_i\bigr)}$$

taken over those k *such that the denominator is strictly positive.*

Proof. We apply Corollary 12.1. By Lemma 10.3, since $\mu_\star = \varnothing$ then the largeness conditions in this corollary become that μ and λ are $\bigl(1, \ell(\kappa)\bigr)$-large, or equivalently, $\ell(\mu) \geq \ell(\kappa)$ and $\ell(\lambda) \geq \ell(\kappa)$. Note that, again since $\mu_\star = \varnothing$, we have $\mathrm{wt}(t) = \mu$ and so the partition ω in the corollary is simply $n\mu$, and so by part of the corollary, if $\lambda \not\trianglelefteq n\mu$ then the plethysm coefficients are zero. If $n = 1$ then the plethysm coefficient is $\langle s_{\mu+M\kappa}, s_{\lambda+M\kappa}\rangle$, which is obviously constant. When $n \geq 2$, the bound $M \geq \mathrm{LP}\bigl(n, \mu : \lambda, \kappa : 0\bigr)$ (see Definition 11.6) is equivalent to

$$M(\kappa_k - \kappa_{k+1}) \geq n\sum_{i=1}^k \mu_i - \sum_{i=1}^k \lambda_i - \mu_k + \mu_{k+1}$$

for each $1 \le k \le \ell(\kappa)$. Using that $\omega = n\mu$, the bound we require, namely that $M \ge \mathrm{L}([\lambda,\omega]_{\unlhd},\kappa)/n$ is equivalent to

$$M(\kappa_k - \kappa_{k+1}) \ge \frac{1}{n}\Big(2\sum_{i=1}^{k-1} n\mu_i + n\mu_k + n\mu_{k+1} - 2\sum_{i=1}^{k}\lambda_i\Big)$$

again for each $1 \le k \le \ell(\kappa)$. Fixing k, the difference of the two right-hand sides is

$$(n-2)\sum_{i=1}^{k-1}\mu_i + (n-2)\mu_k - \frac{n-2}{n}\sum_{i=1}^{k}\lambda_i = \frac{n-2}{n}\sum_{i=1}^{k}(n\mu_i - \lambda_i)$$

which is non-negative because $\lambda \unrhd n\mu$. Therefore the hypotheses of this corollary imply that $M \ge \mathrm{L}([\lambda,\omega]_{\unlhd},\kappa)/n$, as required to apply Corollary 12.1. The final claim of this corollary is immediate from Corollary 12.1. □

We remark that if $\kappa = (1^R)$ then by one further specialization we obtain that $\langle s_\nu \circ s_{\mu+(M^R)}, s_{\lambda+n(M^R)}\rangle$ is constant for $M \ge n\sum_{i=1}^R \mu_i - \sum_{i=1}^R \lambda_i - \mu_R + \mu_{R+1}$. Except for the assumptions that $\ell(\lambda) \ge R$ and $\ell(\mu) \ge R$, which as we mentioned following Remark 3.2 can be dropped, this recovers Theorem 1.2 in [8].

13. Twisted weight bound for Theorem 1.2

This section is the analogue of §10, culminating in Corollary 13.24, the analogue of Corollary 10.10, giving an upper bound (in a sense made precise in the corollary) on the constituents s_σ of the plethysm $s_{\nu^{(M)}} \circ s_{\mu/\mu_\star}$ such that $\sigma \unrhd \lambda \oplus M(\kappa^-,\kappa^+)$ in the $\ell(\kappa^-)$-twisted dominance order. Here, as in Theorem 1.2, $\nu^{(M)} = \nu + (M^R)$ if the strongly maximal signed weight (κ^-,κ^+) has sign $+1$ and $\nu^{(M)} = \nu \sqcup (R^M)$ if the strongly maximal signed weight (κ^-,κ^+) has sign -1. We outline the strategy of the proof in §13.2 after the essential preliminaries in the following subsection.

13.1. **Adapted signed colexicographic order for a strongly maximal signed weight.** By Lemma 4.11, if (κ^-,κ^+) is a strongly maximal signed weight then there is a unique semistandard signed tableau family of signed weight (κ^-,κ^+) of the same shape, size and type as (κ^-,κ^+). By Definition 4.10 the sign of (κ^-,κ^+) is the common sign of the tableaux in this family; in the sense of Definition 4.7, the family has row-type if the common sign is -1 and column-type if the common sign is $+1$. Recall that plethystic semistandard signed tableaux were defined in Definition 3.10; in this definition the inner tableaux are ordered by the signed colexicographic order (see Definition 3.8).

Definition 13.1. Let (κ^-,κ^+) be a strongly maximal signed weight of sign ε. Let $\mathcal{M}_{(\kappa^-,\kappa^+)}$ be the unique semistandard signed tableau family of signed weight (κ^-,κ^+). Let $T_{(\kappa^-,\kappa^+)}$ be the unique plethystic semistandard tableau of outer shape ρ and inner shape $\mu/\mu_\star$ having as its entries the elements of $\mathcal{M}_{(\kappa^-,\kappa^+)}$, where $\rho = (1^R)$ if $\varepsilon = +1$ and $\rho = (R)$ if $\varepsilon = -1$.

As we mentioned in Remark 5.7, in this section we use the freedom in Lemma 5.5 to define plethystic semistandard signed tableaux using an order on their inner tableaux adapted to the relevant strongly maximal semistandard signed tableau family.

Definition 13.2 (Adapted colexicographic order)**.** Let (κ^-, κ^+) be a strongly maximal signed weight of sign ε. Let $\leq$ be the signed colexicographic order if $\varepsilon = -1$ and the sign-reversed colexicographic order if $\varepsilon = +1$. The (κ^-, κ^+)-*adapted colexicographic order*, denoted $\leq_\kappa$, is the total order on semistandard signed tableaux of shape $\mu/\mu_\star$ defined by $s \leq_\kappa t$ if $s \in \mathcal{M}_{(\kappa^-,\kappa^+)}$ and $t \notin \mathcal{M}_{(\kappa^-,\kappa^+)}$; in the remaining cases where either both or neither of s and t are in $\mathcal{M}_{(\kappa^-,\kappa^+)}$, we set $s \leq_\kappa t$ if and only if $s \leq t$.

Thus the elements in $\mathcal{M}_{(\kappa^-,\kappa^+)}$ always come first in the adapted colexicographic order for (κ^-, κ^+), and since Moreover, thanks to the choice of the signed colexicographic order when (κ^-, κ^+) has sign -1 (and so the elements of $\mathcal{M}_{(\kappa^-,\kappa^+)}$ are all negative) and the sign-reversed colexicographic order when the (κ^-, κ^+) has sign $+1$ (and so the elements of $\mathcal{M}_{(\kappa^-,\kappa^+)}$ are all positive), the next greatest elements are the remaining semistandard signed $\mu/\mu_\star$-tableaux of the same sign as those in $\mathcal{M}_{(\kappa^-,\kappa^+)}$.

Remark 13.3. The plethystic semistandard signed tableau $T_{(\kappa^-,\kappa^+)}$ defined in Definition 13.1 (in which the signed colexicographic order is used to order inner tableau entries) is (κ^-, κ^+)-adapted; in fact, since all its inner tableau entries have the same sign, it is semistandard for any of our orders on semistandard signed tableaux.

These observations can easily be verified in following two examples.

Example 13.4. Let $(\kappa^-, \kappa^+) = \big(\varnothing, (4,1,1)\big)$ of shape (2), size 3 and sign $+1$. This is the strongly maximal signed weight seen in Example 4.12, for which the unique semistandard signed tableau family is $\mathcal{M}_{(\kappa^-,\kappa^+)} = \big\{\boxed{1|1}, \boxed{1|2}, \boxed{1|3}\big\}$ of column-type. In the adapted colexicographic order for κ we have

$$\boxed{1|1} <_\kappa \boxed{1|2} <_\kappa \boxed{1|3} <_\kappa \boxed{2|2} <_\kappa \boxed{2|3} <_\kappa \ldots <_\kappa \boxed{\mathbf{1}|1} <_\kappa \boxed{\mathbf{1}|2} <_\kappa \ldots$$

whereas in the sign-reversed colexicographic order, we have

$$\boxed{1|1} < \boxed{1|2} < \boxed{2|2} < \boxed{1|3} < \boxed{2|3} < \ldots < \boxed{\mathbf{1}|1} < \boxed{\mathbf{1}|2} < \ldots.$$

Example 13.5. By Lemma 4.17, if (κ^-, κ^+) is the signed weight of a strongly c^+-maximal singleton semistandard signed tableau family of shape $\mu/\mu_\star$ then the family is $\{t_{\ell(\kappa^-)}(\mu/\mu_\star)\}$; since by Remark 4.6, $t_{\ell(\kappa^-)}(\mu/\mu_\star)$ is the unique least semistandard signed tableau in the signed colexicographic order if $\varepsilon = +1$ and in the sign-reversed colexicographic order if $\varepsilon = -1$, in this special case the adapted order agrees with the usual order.

Definition 13.6. Given a strongly maximal signed weight (κ^-, κ^+), let $\mathrm{PSSYT}_\kappa(\rho, \mu/\mu_\star)$ be the set of all plethystic semistandard of outer shape ρ and inner shape $\mu/\mu_\star$ with negative entries from $\big\{-1, \ldots, -\ell(\kappa^-)\big\}$, defined as in Definition 3.10, but using the (κ^-, κ^+)-adapted colexicographic order to order the inner $\mu/\mu_\star$-tableaux. We say that such plethystic semistandard signed tableaux are (κ^-, κ^+)-*adapted*. We write $\mathrm{PSSYT}_\kappa(\rho, \mu/\mu_\star)_{(\pi^-,\pi^+)}$ for the adapted plethystic semistandard signed tableaux in $\mathrm{PSSYT}_\kappa(\rho, \mu/\mu_\star)$ whose signed weight is (π^-, π^+).

This is the obvious extension of the notation in Definitions 3.9 and 3.11.

13.2. **Overview and running example.** Fix a strongly maximal signed weight (κ^-, κ^+). We shall substantially simplify the exposition in this section and from §13.4 onwards by stating and proving all results only in the case when (κ^-, κ^+) has sign $+1$. The modifications for sign -1 are routine and are given briefly in §13.7 at the end of this section.

Remark 13.7. The underlying principle in this section and the next is *provided M is sufficiently large, every (κ^-, κ^+)-adapted plethystic semistandard signed tableau in* $\mathrm{PSSYT}_\kappa\bigl(\nu + M(1^R), \mu/\mu_\star\bigr)_{(\lambda^-+M\kappa^-,\lambda^++M\kappa^+)}$ *has the elements of $\mathcal{M}_{(\kappa^-,\kappa^+)}$, which form the inner tableaux of the plethystic semistandard signed tableau $T_{(\kappa^-,\kappa^+)}$, in the top R positions of almost all its columns.* In Definition 13.10 we say that such columns are 'typical'. In particular, as we make precise in Corollary 13.19, the number of columns whose top R positions contain semistandard signed tableaux whose total signed weight is *not* dominated in the ℓ^--signed dominance order by (κ^-, κ^+) is bounded *independently* of M.

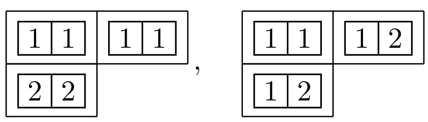
We see this principle in the first running example begun below, proving the special case of Theorem 1.2 that $\langle s_{(2,1)+M(1,1,1)} \circ s_{(2)}, s_{(4,2)+M(4,1,1)}\rangle$ is ultimately constant.

Example 13.8. Let $(\kappa^-, \kappa^+) = \bigl(\varnothing, (4,1,1)\bigr)$ be the strongly maximal signed weight of the column-type tableau family $\bigl\{\,\boxed{1\,|\,1}\,,\ \boxed{1\,|\,2}\,,\ \boxed{1\,|\,3}\,\bigr\}$ in Example 13.4 of shape (2), size 3 and sign $+1$. To apply the Signed Weight Lemma (Lemma 7.3) to prove that $\langle s_{(2,1)+M(1,1,1)} \circ s_{(2)}, s_{(4,2)+M(4,1,1)}\rangle$ is ultimately constant, it is natural to look for a stable partition system $\bigl(\mathcal{P}^{(M)}\bigr)_{M\in\mathbb{N}_0}$ such that $(4,2) + M(4,1,1) \in \mathcal{P}^{(M)}$ for each $M \in \mathbb{N}_0$ and, for condition (ii), such that

$$|\,\mathrm{PSSYT}_{(\varnothing,(4,1,1))}\bigl((2,1) + M(1,1,1), (2)\bigr)_{(\varnothing,\pi)}|$$

is ultimately constant for all $\pi \in \mathcal{P}^{(M)}$. Note that here the first subscript refers to adapted plethystic semistandard signed tableaux (in the sense of Definition 13.6), whose inner (2)-tableaux are ordered according to the $\bigl(\varnothing, (4,1,1)\bigr)$-sign-reversed colexicographic order. In this case however, since there are no negative entries, the distinction between the sign-reversed colexicographic order and the usual colexicographic order is irrelevant.

In the special case where $\pi = (4,2) + M(4,1,1)$ the $\bigl(\varnothing, (4,1,1)\bigr)$-adapted plethystic semistandard signed tableaux in this set are, when $M = 0$,



and, when $M = 1$,

$$\begin{array}{|c|c|c|}\hline \boxed{1\,1} & \boxed{1\,1} & \boxed{1\,1} \\ \hline \boxed{1\,2} & \boxed{1\,2} \\ \cline{1-2} \boxed{2\,3} \\ \cline{1-1}\end{array}\ ,\quad \begin{array}{|c|c|c|}\hline \boxed{1\,1} & \boxed{1\,1} & \boxed{1\,1} \\ \hline \boxed{1\,2} & \boxed{1\,3} \\ \cline{1-2} \boxed{2\,2} \\ \cline{1-1}\end{array}\ ,\quad \begin{array}{|c|c|c|}\hline \boxed{1\,1} & \boxed{1\,1} & \boxed{1\,1} \\ \hline \boxed{1\,2} & \boxed{2\,2} \\ \cline{1-2} \boxed{1\,3} \\ \cline{1-1}\end{array}\ ,\quad \begin{array}{|c|c|c|}\hline \boxed{1\,1} & \boxed{1\,1} & \boxed{1\,2} \\ \hline \boxed{1\,2} & \boxed{1\,2} \\ \cline{1-2} \boxed{1\,3} \\ \cline{1-1}\end{array}\ .$$

There are four plethystic semistandard signed tableaux when $M = 2$. They are obtained by inserting the tableau $T_{(\kappa^-,\kappa^+)}$ shown in the margin as a new first column into each of the four plethystic semistandard signed tableaux for $M = 1$. Note that since $\boxed{1\,3} <_\kappa \boxed{2\,2}$ in the $\big(\varnothing, (4,1,1)\big)$-adapted colexicographic order, this preserves the semistandard condition even when we insert into the second plethystic semistandard tableau for $M = 1$.

$$\begin{array}{|c|}\hline \boxed{1\,1} \\ \hline \boxed{1\,2} \\ \hline \boxed{1\,3} \\ \hline\end{array}$$

In the previous example we saw that inserting $T_{(\kappa^-,\kappa^+)}$ as a new column of height R in a (κ^-, κ^+)-adapted plethystic semistandard signed tableau of weight $\lambda \oplus M(\kappa^-, \kappa^+)$ give a bijection establishing hypothesis (ii) in the Signed Weight Lemma (Lemma 7.3). But still it is not obvious how to choose $\mathcal{P}^{(M)}$, or that the same bijection will work when $\lambda \oplus M(\kappa^-, \kappa^+)$ is replaced with an arbitrary π in the relevant $\mathcal{P}^{(M)}$. We continue the example to show one difficulty, circumvented using the final result of this section (see Corollary 13.24).

Example 13.9. Since the greatest partition (6) in the dominance order is obviously an upper bound for the constituents of $s_{(2,1)} \circ s_{(2)}$, Example 13.8 suggests we might apply the Signed Weight Lemma (Lemma 7.3) with the stable partition system

$$\begin{aligned}&\big[(4,2) + M(4,1,1), (6) + M(4,1,1)\big]_{\trianglelefteq} \\ &\qquad = \big\{(4,2) + M(4,1,1), (5,1) + M(4,1,1), (6) + M(4,1,1)\big\}.\end{aligned}$$

for $M \in \mathbb{N}_0$. (Here $\trianglelefteq$ is the usual dominance order: by Remark 6.8, this is the 0-twisted dominance order, so we have $\ell^- = 0$ and the symmetric functions in the lemma are h_π for $\pi \in \mathrm{Par}$.) However condition (i) in the Signed Weight Lemma fails when $M = 1$: we have $(8,3,1) \in [(4,2) + (4,1,1), (6) + (4,1,1)]_{\trianglelefteq}$ and since $(8,4) \trianglerighteq (8,3,1)$ and $s_{(8,4)}$ is a constituent of $s_{(3,2,1)} \circ s_{(2)}$ — for instance, this follows from the generalized Cayley–Sylvester formula (5.3) in §5.4 — we have

$$s_{(8,4)} \in \mathrm{supp}(h_{(8,3,1)}) \cap \mathrm{supp}(s_{(3,2,1)} \circ s_{(2)}).$$

But $(8,4) \notin [(4,2) + (4,1,1), (6) + (4,1,1)]_{\trianglelefteq}$ since $(8,4)$ and $(10,1,1)$ are incomparable. It might seem that the problem is that our chosen upper bounds $(6) + M(4,1,1)$ are too small to contain all partitions in the support of $s_{(2,1)+M(1,1,1)} \circ s_{(2)}$. However, one can show using Theorem 1.5 of [8] that the maximal constituents of $s_{(2,1)+M(1,1,1)} \circ s_{(2)}$ are precisely the partitions

$$\big\{(5,1) + (M-a)(4,1,1) + a(3,3) : 0 \le a \le M\big\} \tag{13.1}$$

and since $(4,1,1)$ and $(3,3)$ are incomparable in the dominance order, there is *no* stable partition system of intervals

$$[\lambda + (M-S)(4,1,1), \omega + (M-S)(4,1,1)]_{\trianglelefteq} \tag{13.2}$$

with λ and ω partitions of $6S$ that contains all the maximal constituents of $s_{(2,1)+M(1,1,1)} \circ s_{(2)}$ for all $M \geq S$, or even for all M sufficiently large. (Beginning with partition of $6S$ gives ample freedom to avoid technical issues to do with 'largeness', in the sense of Definition 3.1 and Definition 10.1, so this is not the problem.) Perhaps surprisingly, *we conclude that it is essential to use the lower bound as well.* And because a plethystic semistandard signed tableau of shape $(2,1) + M(1,1,1)$ and signed weight $\bigl(\varnothing, (4,2) + M(4,1,1)\bigr)$ may have an exceptional column in the sense of Definition 13.10 (see Example 13.20), the stable partition system we define has to start with partitions of 12. (This corresponds to taking $S = 1$ in (13.2).) Since $(4,2) + (4,1,1) = (8,3,1)$ we therefore consider the intervals

$$\bigl[(8,3,1) + N(4,1,1), (12) + N(4,1,1)\bigr]_{\unlhd} \tag{13.3}$$

for $N \in \mathbb{N}_0$. If $\pi \in \bigl[(8,3,1) + N(4,1,1), (12) + N(4,1,1)\bigr]_{\unlhd}$ and

$$\sigma \in \operatorname{supp}(h_\pi) \cap \operatorname{supp}(s_{(2,1)+(N+1)(1,1,1)} \circ s_{(2)}$$

then, by Lemma 6.12 applied to $\operatorname{supp}(h_\pi)$ we have

$$\sigma \unrhd \pi \unrhd (8,3,1) + N(4,1,1) \tag{13.4}$$

and, by (13.1), we have

$$\begin{aligned}\sigma &\unlhd (9,2,1) + (N-a)(4,1,1) + a(3,3) \\ &= (9-a, 2+2a, 1-a) + N(4,1,1)\end{aligned} \tag{13.5}$$

for some a with $0 \leq a \leq N$. If $a = 0$ then, by (13.5), $\sigma \unlhd (9,2,1) + N(4,1,1) \unlhd (12) + N(4,1,1)$. Similarly if $a = 1$ then $\sigma \unlhd (8,4) + N(4,1,1) \unlhd (12) + N(4,1,1)$, and, despite involving the problematic partition $(8,4)$, thanks to our choice in (13.3), σ is in the interval for all $N \in \mathbb{N}_0$. Finally if $a \geq 2$ then we must have $N \geq 1$ and we get $\sigma_1 \leq 9 - a + 4N$ which, using the *lower bound* on the intervals in (13.3) — justified by (13.4) obtained using Lemma 6.12 — that $\sigma \unrhd (8,3,1) + N(4,1,1)$, contradicts that $\sigma_1 \geq 8 + 4N$. Therefore condition (i) in the Signed Weight Lemma (Lemma 7.3) holds for all $N \in \mathbb{N}_0$. We note that this contradiction was obtained by comparing in the dominance order just on the first part, and correspondingly, $(4,1,1)$ is a strongly 1-maximal signed weight.

We continue this example in Example 13.12.

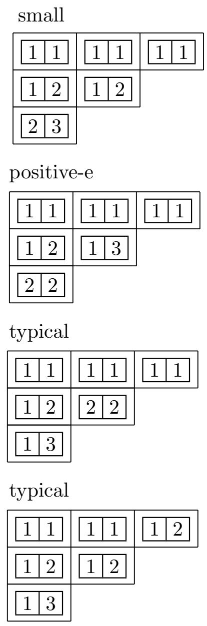
13.3. **Exceptional columns and rows.** We define the *signed weight* of a subset $\mathcal{B}$ of the boxes of a plethystic semistandard tableau to be the sum of the weights of the inner tableaux in $\mathcal{B}$. In the following definition we use the ℓ^--signed dominance order on the set $\mathcal{W}_{\ell^-} \times \mathcal{W}$ defined in Definition 4.1 to compare (ϕ^-, ϕ^+) and (κ^-, κ^+). Adapted plethystic semistandard signed tableaux are defined in Definition 13.6.

Definition 13.10. Let (κ^-, κ^+) be a strongly c^+-maximal signed weight of shape $\mu/\mu_\star$ and size R. Let T be a (κ^-, κ^+)-adapted plethystic semistandard signed tableau of inner shape $\mu/\mu_\star$. When (κ^-, κ^+) has sign $+1$, we say that a column of T of height at least R whose top R boxes have signed

weight (ϕ^-, ϕ^+) is *small* if $(\phi^-, \phi^+) \lhd (\kappa^-, \kappa^+)$, *typical* if the top R boxes in the column form the plethystic semistandard signed tableau $T_{(\kappa^-, \kappa^+)}$ and *exceptional* if $(\phi^-, \phi^+) \not\trianglelefteq (\kappa^-, \kappa^+)$. In the latter case we say the column is
(a) *large-exceptional* if $\ell(\phi^+) > \ell(\kappa^+)$;
(b) *negative-exceptional* if $|\phi^-| < |\kappa^-|$;
(c) *positive-exceptional* if $|\phi^-| + \sum_{i=1}^{c^+} \phi_i^+ < |\kappa^-| + \sum_{i=1}^{c^+} \kappa_i^+$.
When (κ^-, κ^+) has sign -1 we make the analogous definitions replacing 'column' with 'row', now considering the leftmost R boxes in the row.

The relevant strongly maximal signed weight (κ^-, κ^+) in this definition will always be clear from the context. We shall prove in Lemma 13.14(i) that a column is either small, typical or exceptional, and in Lemma 13.14(ii) that an exceptional column is either large-exceptional, negative-exceptional or positive-exceptional. Note the latter three cases are not mutually exclusive: in fact any combination of them may hold.

Remark 13.11. If $R = 1$ then, by Lemma 4.17, the unique strongly maximal semistandard signed tableau family of shape $\mu/\mu_\star$ is $\{t_{\ell^-}(\mu/\mu_\star)\}$. By Lemma 4.4, $\{t_{\ell^-}(\mu/\mu_\star)\}$ has the greatest signed weight, in the ℓ^--signed dominance order on all $\mu/\mu_\star$-tableaux with entries from $\{-1, \ldots, -\ell^-\} \cup \mathbb{N}$. Therefore in the notation of Definition 13.10, we always have $(\phi^-, \phi^+) \trianglelefteq (\kappa^-, \kappa^+)$ where $(\kappa^-, \kappa^+) = \big(\omega_{\ell^-}(\mu/\mu_\star)^-, \omega_{\ell^-}(\mu/\mu_\star)^+\big)$ is the signed weight of $t_{\ell^-}(\mu/\mu_\star)$, and so there are no exceptional columns or rows. It is instructive to see how the remaining results in this section specialize to easy corollaries of Lemma 4.4 in this case: we summarise the situation in Remark 13.25 at the end of this section.



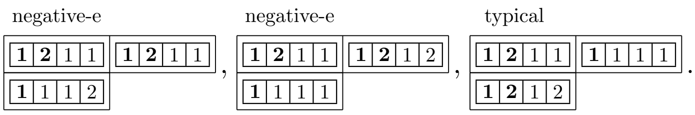
Example 13.12. Of the four $(\varnothing, (4,1,1))$-adapted plethystic semistandard signed tableaux in the set $\mathrm{PSSYT}_{(\varnothing,(4,1,1))}\big((3,2,1),(2)\big)_{(\varnothing,(8,3,1)}$ shown at the end of Example 13.8, and repeated in the margin for ease of reference, the first column of the first tableau has signed weight $\big(\varnothing,(3,2,1)\big) \trianglelefteq \big(\varnothing,(4,1,1)\big)$ so is small. The first column of the second has signed weight $\big(\varnothing,(3,3)\big)$, which is incomparable with the strongly 1-maximal signed weight $\big(\varnothing,(4,1,1)\big)$; this column is therefore exceptional and since the sums on the left- and right-sides of (c) are 3 and 4 respectively it is positive-exceptional. The final two tableaux each have first column $T_{(\varnothing,(4,1,1))}$ of signed weight $\big(\varnothing,(4,1,1)\big)$; these two columns are typical. The boxes not in the column of height 3 are not classified by Definition 13.10.

This example is continued in Example 13.20. We now give a further example to show the full generality of Definition 13.10.

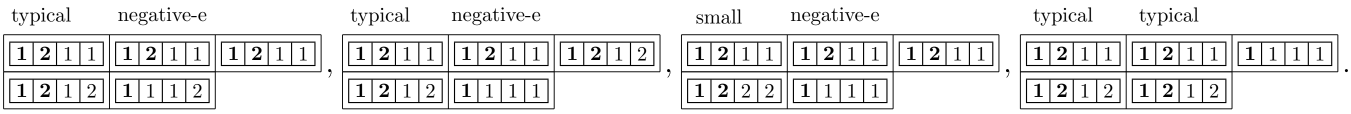
Example 13.13. Let $(\kappa^-, \kappa^+) = \big((2,2),(3,1)\big)$. By Example 4.18(ii), this is the strongly 1-maximal weight of the column-type tableau family $\mathcal{M}_{((2,2),(3,1))}$ of shape (4), size 2 and sign $+1$ shown below

$$\Big\{ \boxed{\mathbf{1}\,|\,\mathbf{2}\,|\,1\,|\,1}\,,\ \boxed{\mathbf{1}\,|\,\mathbf{2}\,|\,1\,|\,2} \Big\}\ .$$

The special case $\nu = (2,1)$ and $\mu/\mu_\star = (4)/\varnothing$ of Theorem 1.2 is that the plethysm coefficients $\langle s_{(2,1)+(M,M)} \circ s_{(4)}, s_{\lambda + M(3,1)\sqcup(2^M)} \rangle$ are ultimately constant for all partitions λ of 12. First we take $\lambda = (8,3,1)$ with 2-decomposition $\big\langle (3,2),(6,1)\big\rangle$. There are three $((2,2),(3,1))$-adapted plethystic semistandard tableaux in $\mathrm{PSSYT}_{((2,2),(3,1))}\big((2,1),(4)\big)_{((3,2),(6,1))}$, namely

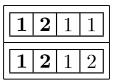


In each of the first two, the first column is negative-exceptional, being deficient in negative entries. In the third the first column is typical. Since the second column is singleton, it is not classified by Definition 13.10. Growing by $\nu \mapsto \nu + (1,1)$ and $\lambda \mapsto \lambda \oplus \big((2,2),(3,1)\big)$ as in Theorem 1.2, we find that the four plethystic semistandard signed tableaux in the set $\mathrm{PSSYT}_{((2,2),(3,1)}\big((3,2),(4)\big)_{((5,4),(9,2))}$ are



For instance, in the third tableau the signed weight of the first column is $\big((2,2),(2,2)\big) \lhd \big((2,2),(3,1)\big)$ and so this column is small. We remark that in fact

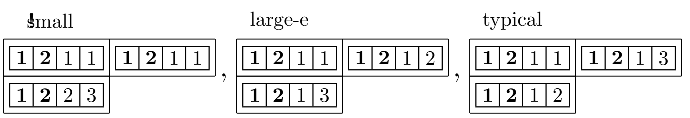
$$|\,\mathrm{PSSYT}_{((2,2),(3,1)}\big((2,1)+M(1,1),(4)\big)_{((3+2M,2+2M),(6+3M,1+M))}| = 4$$



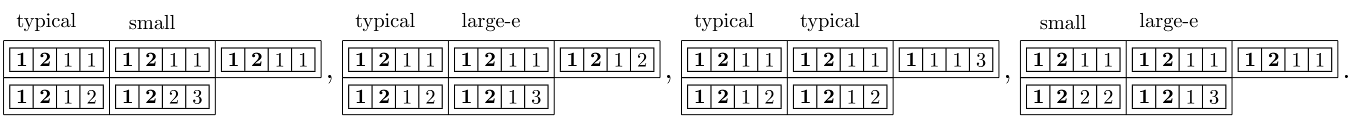
for all $M \geq 1$; a bijective proof is given by insertion of the plethystic semistandard signed tableau shown in the margin corresponding to $\mathcal{M}_{((2,2),(3,1))}$ as a new first column; this is the $\mathcal{H}$ map in the proof of Theorem 14.7. (In this case, unlike Example 13.8, the $\big((2,2),(3,1)\big)$-adapted colexicographic order defined in Definition 13.2 coincides with the usual sign-reversed colexicographic order of Definition 3.8, and so using *either* order, the insertion map preserves semistandardness.) Computation by computer algebra shows that the constant value of the plethysm coefficient is 2.

In this case there are no large-exceptional columns because the maximum positive entry permitted by the signed weight $\big((3+2M,2+2M),(6+3M,1+M)\big)$, namely 2, is also the length of the positive part of the strongly maximal weight, namely $\ell\big((3,1)\big) = 2$.

Taking instead $\lambda = (6,3,3)$ with 2-decomposition $\big\langle (3,3),(4,1,1)\big\rangle$, the three elements of $\mathrm{PSSYT}_{((2,2),(3,1))}\big((2,1),(4)\big)_{((3,3),(4,1,1))}$ are



and the four elements $\mathrm{PSSYT}_{((2,2),(3,1)}\big((3,2),(4)\big)_{((5,5),(7,2,1))}$ are



We leave it to the reader to verify the annotations above and to show that the same insertion map gives a bijective proof that

$$|\,\mathrm{PSSYT}_{((2,2),(3,1)}\big((2,1)+M(1,1),(4)\big)_{((3+2M,3+2M),(4+3M,1+M,1))}| = 4$$

for all $M \geq 1$. Here it is helpful to note that since the negative part of the signed weight is $(3+2M, 3+2M)$, every inner tableau has the form $\boxed{\mathbf{1}}\boxed{\mathbf{2}}\boxed{\cdot}\boxed{\cdot}$; this explains the absence of negative-exceptional columns in this case. Computation by computer algebra shows that the constant value of the plethysm coefficient is 1.

13.4. **Exceptional column bound.** The aim of this subsection is to prove Lemma 13.17, making the underlying principle in Remark 13.7 precise. As promised earlier, to simplify the exposition, from now on we assume that the strongly maximal signed weight has sign $+1$. See §13.7 for the modifications for sign -1.

In the next lemma it is important to note that while the strongly maximal signed weights defined in Definition 4.10 are of tableau families having all elements of the same sign (by (b) the sign is $+1$ for column-type and -1 for row-type), the comparison in Definition 4.8 is with all families of the relevant shape and size, with negative entries from the prescribed set $\big\{-1,\ldots,-\ell(\kappa^-)\big\}$ — see the italicised end to the paragraph after Definition 4.8. We remind the reader that, by Remark 13.3, $T_{(\kappa^-,\kappa^+)}$ is semistandard in the (κ^-,κ^+)-adapted colexicographic order.

Lemma 13.14. *Let (κ^-,κ^+) be a strongly maximal signed weight of shape $\mu/\mu_\star$, size R and sign $+1$. Let T be a (κ^-,κ^+)-adapted plethystic semistandard signed tableau of inner shape $\mu/\mu_\star$. Let (ϕ^-,ϕ^+) be the signed weight of the top R boxes in a column of T.*

(i) *If $(\phi^-,\phi^+) = (\kappa^-,\kappa^+)$ then the top R boxes in the column form the plethystic semistandard signed tableau $T_{(\kappa^-,\kappa^+)}$.*

(ii) *If $(\phi^-,\phi^+) \not\trianglelefteq (\kappa^-,\kappa^+)$, then the column is large-exceptional, negative-exceptional or positive-exceptional.*

Proof. Part (i) follows from the uniqueness of the plethystic semistandard signed tableau family corresponding to a strongly maximal signed weight, proved in Lemma 4.11. For (ii) let $(\phi^-,\phi^+) \not\trianglelefteq (\kappa^-,\kappa^+)$ and suppose that the column is not large-exceptional. Let (ψ^-,ψ^+) be a maximal signed weight in the dominance order on $\mathcal{W}_{\ell(\kappa^-)} \times \mathcal{W}$ of a column-type semistandard signed tableau family of shape $\mu/\mu_\star$ and size R such that $(\phi^-,\phi^+) \trianglelefteq (\psi^-,\psi^+)$, as in Definition 4.8. By hypothesis, $(\psi^-,\psi^+) \neq (\kappa^-,\kappa^+)$. Since the column is not large-exceptional, by Lemma 4.13, either (4.2) holds and we have

$$|\phi^-| \leq |\psi^-| < |\kappa^-|$$

and the column is negative-exceptional or (4.3) holds and

$$|\phi^-| + \textstyle\sum_{i=1}^{c^+} \phi_i^+ \leq |\psi^-| + \sum_{i=1}^{c^+} \psi_i^+ < |\kappa^-| + \sum_{i=1}^{c^+} \kappa_i^+$$

and the column is positive-exceptional. □

To state our bound on the number of exceptional columns we need the statistics on partitions in the following two definitions.

Definition 13.15. Given a partition ν and $R \in \mathbb{N}$, we define $B_R(\nu) = |\nu| - R\nu_R$.

In the general setting of §13.7 this statistic is $B^+_R(\nu)$, and $B^-_R(\nu)$ is the row version of Definition 13.15 needed when (κ^-, κ^+) has sign -1.

Equivalently $B_R(\nu)$ is the number of boxes (i, j) of $[\nu]$ such that either $i > R$ or $\nu'_j < R$; these are precisely the boxes not in the top R positions of a column of height at least R. We shall use many times below that

$$B_R\big(\nu + M(1^R)\big) = B_R(\nu) \tag{13.6}$$

for all $M \in \mathbb{N}_0$, as can be seen from Figure 13.1.

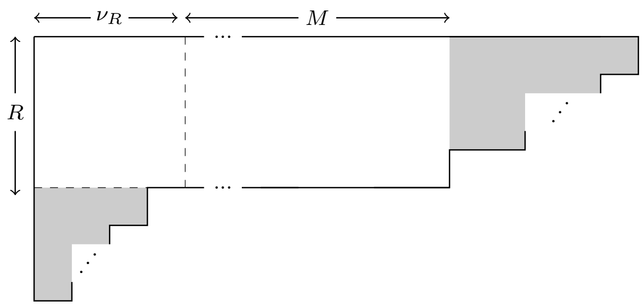


FIGURE 13.1. The partition $\nu + M(1^R)$ with the boxes not in the top R positions of a column of height at least R shaded. These are the boxes counted by $B_R\big(\nu+M(1^R)\big)$. The diagram also shows that $B_R\big(\nu+M(1^R)\big) = B_R(\nu)$ and that we can visualize the boxes added by the summand $M(1^R)$ as lying in columns $\nu_R + 1, \ldots, \nu_R + M$.

Note also that if T is a plethystic semistandard signed tableau of outer shape ν and inner shape $\mu/\mu_\star$ then, by Lemma 4.4, the contribution from the boxes counted by $B_R(\nu)$ in this tableau to the signed weight of T is at most

$$B_R(\nu)\big(\omega_{\ell^-}(\tau/\tau_\star)^-, \omega_{\ell^-}(\tau/\tau_\star)^+\big) \tag{13.7}$$

where $\big(\omega^-_\ell(\mu/\mu_\star), \omega^+_\ell(\mu/\mu_\star)\big)$ is the greatest signed weight of Definition 4.3.

Definition 13.16. Given a skew partition $\mu/\mu_\star$ and $\ell^- \in \mathbb{N}_0$ and $c^+ \in \mathbb{N}_0$, we define $A^-(\mu/\mu_\star) = |\omega_{\ell^-}(\mu/\mu_\star)^-|$ and $A^+(\mu/\mu_\star) = |\omega_{\ell^-}(\mu/\mu_\star)^-| + \sum_{i=1}^{c^+} \omega_{\ell^-}(\mu/\mu_\star)^+_i$.

Note that $A^-(\mu/\mu_\star)$ is the number of negative entries in the ℓ^--negative greatest tableau $t_{\ell^-}(\mu/\mu_\star)$ of signed weight $\big(\omega_{\ell^-}(\mu/\mu_\star)^-, \omega_{\ell^-}(\mu/\mu_\star)^+\big)$; as we saw in Lemma 4.4 this is the greatest signed weight in the ℓ^--signed dominance order (see Definition 4.1) of a semistandard signed tableau of shape $\mu/\mu_\star$. In particular, no semistandard signed tableau of shape $\mu/\mu_\star$ with entries from $\big\{-1, \ldots, -\ell^-\big\}$ can have more than $A^-(\mu/\mu_\star)$ negative entries. The set $\mathrm{PSSYT}_\kappa\big(\nu + M(1^R), \mu/\mu_\star\big)_{(\pi^-,\pi^+)}$ in the following lemma is defined in Definition 13.6.

Lemma 13.17 (Bound on exceptional columns)**.** *Let (κ^-,κ^+) be a strongly c^+-maximal signed weight of shape $\mu/\mu_\star$, size R and sign $+1$. Fix $\ell^- = \ell(\kappa^-)$. Let ν be a partition. Let (λ^-,λ^+) and $(\pi^-,\pi^+) \in \mathcal{W}_{\ell^-} \times \mathcal{W}$ be signed weights. Let*

$$T \in \mathrm{PSSYT}_\kappa\big(\nu + M(1^R), \mu/\mu_\star\big)_{(\pi^-,\pi^+)}.$$

If $(\pi^-,\pi^+) \trianglerighteq (\lambda^-,\lambda^+) + M(\kappa^-,\kappa^+)$ then T has at most

(i) $\sum_{i=\ell(\kappa^+)+1}^{\ell(\lambda^+)} \lambda^+_i$ *large-exceptional columns;*

(ii) $B_R(\nu)A^-(\mu/\mu_\star) + \nu_R|\kappa^-| - |\lambda^-|$ *negative-exceptional columns;*

(iii) $B_R(\nu)A^+(\mu/\mu_\star) + \nu_R\big(|\kappa^-| + \sum_{i=1}^{c^+}\kappa^+_i\big) - \big(|\lambda^-| + \sum_{i=1}^{c^+}\lambda^+_i\big)$ *positive-exceptional columns that each are neither large-exceptional nor negative-exceptional.*

Proof. Consider the integer entries of the inner $\mu/\mu_\star$-tableaux in T. Exactly $\sum_{i=\ell(\kappa^+)+1}^{\ell(\lambda^+)} \pi^+_i$ of these entries are strictly greater than $\ell(\kappa^+)$. Since $(\pi^-,\pi^+) \trianglerighteq (\lambda^-,\lambda^+) + M(\kappa^-,\kappa^+)$, we have

$$\sum_{i=\ell(\kappa^+)+1}^{\ell(\lambda^+)} \pi^+_i \le \sum_{i=\ell(\kappa^+)+1}^{\ell(\lambda^+)} \lambda^+_i.$$

Therefore there at most $\sum_{i=\ell(\kappa^+)+1}^{\ell(\lambda^+)} \lambda^+_i$ such entries. Now (i) follows since, by Definition 13.10(a), each large-exceptional column has at least one of them. By Lemma 13.14, each remaining exceptional column is either positive-exceptional or negative-exceptional.

To prove (ii) and (iii), we consider the integer entries of T in the sets $\{-1,\ldots,-\ell^-\}$ and $\{-1,\ldots,-\ell^-,1,\ldots,c^+\}$, respectively, and the inner $\mu/\mu_\star$-tableaux in which they lie. There are $B_R(\nu)$ such entries not lying in the top R boxes of a column having at least R boxes. By the remark immediately before this lemma, the $\mu/\mu_\star$-tableaux in these boxes have between them, at most $B_R(\nu)A^-(\mu/\mu_\star)$ entries in $\{-1,\ldots,-\ell^-\}$. Moreover, by Lemma 4.4, each such $\mu/\mu_\star$-tableau has signed weight bounded above (in the ℓ^--signed dominance order in Definition 4.1) by $\big(\omega_{\ell(\kappa^-)}(\mu/\mu_\star)^-, \omega_{\ell(\kappa^-)}(\mu/\mu_\star)^+\big)$, and so there are at most $B_R(\nu)A^+(\mu/\mu_\star)$ entries in $\{-1,\ldots,-\ell^-,\ 1,\ldots,c^+\}$ in these $\mu/\mu_\star$-tableaux. Each remaining $\mu/\mu_\star$-tableau entry lies in the top R boxes of a column of T having at least R boxes. As can be seen from Figure 13.1, there are $\nu_R + M$ such columns.

For (ii), suppose that $E^-(T)$ of these columns are negative-exceptional. In a non-negative-exceptional column whose top R entries have signed weight (ϕ^-,ϕ^+), we have, by Definition 13.10(b), $|\phi^-| = |\kappa^-|$. Hence the $\mu/\mu_\star$-tableaux in the top R rows of the non-negative-exceptional columns have, between them, exactly $\big(\nu_R + M - E^-(T)\big)|\kappa^-|$ entries in $\{-1,\ldots,-\ell^-\}$. By (4.2) in Lemma 4.13, in each negative-exceptional column there are at most $|\kappa^-| - 1$ entries in $\{-1,\ldots,-\ell^-\}$. Summing these bounds gives

$$\begin{aligned}\sum_{i=1}^{\ell^-} \pi^-_i &\le B_R(\nu)A^-(\mu/\mu_\star) + \big(\nu_R + M - E^-(T)\big)|\kappa^-| + E^-(T)\big(|\kappa^-| - 1\big)\\ &= B_R(\nu)A^-(\mu/\mu_\star) + (\nu_R + M)|\kappa^-| - E^-(T).\end{aligned}$$

On the other hand, since $(\pi^-,\pi^+) \trianglerighteq (\lambda^-,\lambda^+) + M(\kappa^-,\kappa^+)$ we have

$$|\pi^-| \geq |\lambda^-| + M|\kappa^-|.$$

Combining the two displayed inequalities and cancelling $M|\kappa^-|$ we get

$$E^-(T) \leq B_R(\nu)A^-(\mu/\mu_\star) + \nu_R|\kappa^-| - |\lambda^-|$$

as required.

For (iii), suppose there are $E^+(T)$ exceptional columns of T of height at least R that are not large-exceptional and not negative-exceptional. By Lemma 13.14(ii) these columns are positive-exceptional. Let (ϕ^-,ϕ^+) be the signed weight of such a column. Using (4.3) in Lemma 4.13 and Definition 13.10(c), we have $|\phi^-| + \sum_{i=1}^{c^+}\phi_i^+ < |\kappa^-| + \sum_{i=1}^{c^+}\kappa_i^+$. The analogous inequalities are therefore

$$|\pi^-| + \sum_{i=1}^{c^+}\pi_i^+ \leq B_R(\nu)A^+(\mu/\mu_\star) + (\nu_R + M)\bigl(|\kappa^-| + \sum_{i=1}^{c^+}\kappa_i^+\bigr) - E^+(T)$$

and

$$|\pi^-| + \sum_{i=1}^{c^+}\pi_i^+ \geq |\lambda^-| + \sum_{i=1}^{c^+}\lambda_i^+ + M\bigl(|\kappa^-| + \sum_{i=1}^{c^+}\kappa_i^+\bigr).$$

Combining these two inequalities and cancelling $M\bigl(|\kappa^-| + \sum_{i=1}^{c^+}\kappa_i^+\bigr)$ we get

$$E^+(T) \leq B_R(\nu)A^+(\mu/\mu_\star) + \nu_R\bigl(|\kappa^-| + \sum_{i=1}^{c^+}\kappa_i^+\bigr) - |\lambda^-| - \sum_{i=1}^{c^+}\lambda_i^+$$

again as required. □

Motivated by this result we make the following definition. The statistics $B_R(\nu)$, $A^-(\mu/\mu_\star)$ and $A^+(\mu/\mu_\star)$ are defined in Definitions 13.15 and 13.16.

Definition 13.18. Let (κ^-,κ^+) be a strongly c^+-maximal signed weight of shape $\mu/\mu_\star$, size R and sign $+1$. Let ν be a non-empty partition and let λ be a partition of $|\nu||\mu/\mu_\star|$. Fix $\ell^- = \ell(\kappa^-)$. Define

$$E^- = B_R(\nu)A^-(\mu/\mu_\star) + \nu_R|\kappa^-| - |\lambda^-|,$$
$$E^+ = B_R(\nu)A^+(\mu/\mu_\star) + \nu_R\bigl(|\kappa^-| + \textstyle\sum_{i=1}^{c^+}\kappa_i^+\bigr) - \bigl(|\lambda^-| + \sum_{i=1}^{c^+}\lambda_i^+\bigr)$$

and if $R \geq 2$,

$$E_{c^+,(\kappa^-,\kappa^+)}(\nu,\mu/\mu_\star : \lambda) = \max\bigl(E^- + E^+ + \textstyle\sum_{i=\ell(\kappa^+)+1}^{\ell(\lambda^+)}\lambda_i^+, 0\bigr).$$

If $R = 1$ we instead define $E_{c^+,(\kappa^-,\kappa^+)}(\nu,\mu/\mu_\star : \lambda) = 0$. Finally we set $E_{c^+,(\kappa^-,\kappa^+)}(\varnothing,\mu/\mu_\star : \varnothing) = -1$.

Note that since $R = (|\kappa^-| + |\kappa^+|)/|\mu/\mu_\star|$, there is no need to state R explicitly in the definition of $E_{c^+,(\kappa^-,\kappa^+)}(\nu,\mu/\mu_\star : \lambda)$. The reason for the technical final case when $\nu = \varnothing$ will be seen in the proof of Corollary 13.24.

Corollary 13.19. *Let (κ^-, κ^+) be a strongly c^+-maximal signed weight of shape $\mu/\mu_\star$, size R and sign $+1$. Let ν be a partition. Let λ be an $\big(\ell(\kappa^-), \ell(\kappa^+)\big)$-large partition. Suppose that $\pi \trianglerighteq \lambda \oplus M(\kappa^-, \kappa^+)$ in the $\ell(\kappa^-)$-twisted dominance order. If $T \in \mathrm{PSSYT}_\kappa(\nu + M(1^R), \mu/\mu_\star)_{(\pi^-,\pi^+)}$ then T has at most $E_{c^+,(\kappa^-,\kappa^+)}(\nu, \mu/\mu_\star : \lambda)$ exceptional columns.*

Proof. By Lemma 9.6 the $\ell(\kappa^-)$-decomposition of λ is $\langle \lambda^- + M\kappa^-, \lambda^+ + M\kappa^+\rangle$. Therefore $\pi \trianglerighteq \lambda \oplus M(\kappa^-, \kappa^+)$ is equivalent, by the definition of the $\ell(\kappa^-)$-twisted dominance order in Definition 6.6, to $(\pi^-, \pi^+) \trianglerighteq (\lambda^- + M\kappa^-, \lambda^+ + M\kappa^+)$. If $R \geq 2$ the corollary is now immediate from Lemma 13.17, given the definition of $E_{c^+,(\kappa^-,\kappa^+)}(\nu, \mu/\mu_\star : \lambda)$ in Definition 13.18. When $R = 1$ it follows from Remark 13.11. □

See Example 13.22 for the bound in this corollary in the running example of the 'signed' case begun in Example 13.13.

Example 13.20. Continuing our running 'unsigned' example (see Examples 13.8, 13.9 and 13.12), let (κ^-, κ^+) be the strongly 1-maximal signed weight $\big(\varnothing, (4,1,1)\big)$ of shape (2), size 3 and sign $+1$. (Note that this size can be computed directly from $\big(\varnothing, (4,1,1)\big)$ knowing the shape using the remark immediately after Definition 13.18.) Thus $\mu/\mu_\star = (2)/\varnothing$ and $R = 3$. We have $\ell^- = 0$ and $c^+ = 1$. Generalizing slightly, to show more clearly the effect of columns of height at least R in ν, let $\nu = (2,1) + C(1,1,1)$. The relevant statistics are $B_3\big((2,1) + C(1,1,1)\big) = 3$, and, since $\omega_0\big((2)\big) = (2)$ corresponding to the greatest tableau $\boxed{1}\boxed{1}$, we have $A^-(\mu/\mu_\star) = 0$ and $A^+(\mu/\mu_\star) = 2$. The quantities E^- and E^+ in Definition 13.18 are

$$
\begin{aligned}
E^- &= 0, \\
E^+ &= 3 \times 2 + 4C - 0 - \lambda_1 = 6 + 4C - \lambda_1
\end{aligned}
$$

and so $E_{1,(\varnothing,(4,1,1))}\big((2,1), (2) : \lambda\big) = 0 + 6 + 4C - \lambda_1 + \lambda_4 + \cdots$ for any partition λ of $6 + 4C$. Taking $C = 0$ and $\lambda = (4,2)$ as earlier we have $E_{1,(\varnothing,(4,1,1))}\big((2,1), (2) : (4,2)\big) = 2$ and so, by Lemma 13.17, a $\big(\varnothing, (4,1,1)\big)$-adapted plethystic semistandard signed tableaux lying in the set

$$
\mathrm{PSSYT}_{(\varnothing,(4,1,1))}\big((2,1) + M(1,1,1), (2)\big)_{(\varnothing,\pi^+)}
$$

where $\pi^+ \trianglerighteq (4,2) + M(4,1,1)$ may have at most two exceptional columns. For a general C, we note that if $\lambda = (4,2) + C(4,1,1)$ then $\lambda_1 = 4C + 4$ and so

$$
E_{1,(\varnothing,(4,1,1))}\big((2,1) + C(1,1,1), (2) : (4,2) + C(4,1,1)\big) = 2
$$

giving the same bound; this is expected from the proof of Lemma 13.17, because the contribution $(4,1,1)$ to λ *can only* come from a typical column, equal to the plethystic semistandard signed tableau shown in the margin. Of course if C is large and λ is not of this special form then there may be many more exceptional columns.

1 1
1 2
1 3

We now show by a direct argument that if $\pi^+ \trianglerighteq (4,2) + M(4,1,1)$ and $T \in \mathrm{PSSYT}_{(\varnothing,(4,1,1))}\big((2,1) + M(1,1,1), (2)\big)_{(\varnothing,\pi^+)}$ then T has at most one exceptional column. Thus as is usually the case, the bound from Corollary 13.19 is not optimal. The key observation is that each typical column contain four 1s as entries of its inner (2)-tableaux, and since the maximum positive entry is 3 and there are no negative entries, a column that is not typical, i.e. not equal to the plethystic semistandard signed tableau $T_{(\varnothing,(4,1,1))}$ shown earlier in the margin, has at most three 1s. The three boxes in T not in columns of length 3 contribute at most five 1s. (This can easily be seen from the tableaux in Example 13.8.) Therefore if there are N non-typical columns, T has at most $4M - N + 5$ entries of 1. On the other hand, since $\pi^+ \trianglerighteq (4,2) + M(4,1,1)$, we have $\pi^+_1 \geq 4 + 4M$. Therefore $4M - N + 5 \geq 4 + 4M$ and so $N \leq 1$. Since exceptional columns are non-typical, this implies there is at most one exceptional column, as claimed, and moreover, this exceptional column is positive-exceptional. Since the signed weight of the column is not dominated by $\big(\varnothing, (4,1,1)\big)$, it is necessarily equal to the plethystic semistandard signed tableau $T_{(\varnothing,(3,3))}$ shown in the margin, defined by the strongly 2-maximal semistandard signed tableau family of signed weight $\big(\varnothing,(3,3)\big)$. We continue this line of argument in Example 13.26.

1	1
1	2
2	2

We finish this example in Example 13.26 below.

13.5. **Signed weight bound in the ℓ^--signed dominance order.** We now turn the bound on the number of exceptional columns in Lemma 13.17 into an upper bound on signed weights in the ℓ^--signed dominance order in Definition 4.1. We continue to simplify the exposition by assuming the strongly maximal signed weight has sign $+1$. See §13.7 for the modifications for sign -1. Recall from Definition 4.3 and Lemma 4.4 that $\big(\omega_{\ell^-}(\mu/\mu_\star)^-, \omega_{\ell^-}(\mu/\mu_\star)^+\big)$ is the greatest signed weight in the ℓ^--signed dominance order of a semistandard signed tableau of shape $\mu/\mu_\star$. The set $\mathrm{PSSYT}_\kappa\big(\nu + M(1^R), \mu/\mu_\star\big)_{(\pi^-,\pi^+)}$ is defined in Definition 13.6. Small columns were defined in Definition 13.10 and the statistic $B_R(\nu)$ in Definition 13.15. An example is given after the proposition.

Proposition 13.21. *Let (κ^-, κ^+) be a strongly c^+-maximal signed weight of shape $\mu/\mu_\star$, size R and sign $+1$. Fix $\ell^- = \ell(\kappa^-)$. Let ν be a partition and let $\mu/\mu_\star$ be a skew partition. Set $E = E_{c^+,(\kappa^-,\kappa^+)}(\nu, \mu/\mu_\star : \lambda)$. Let $M \in \mathbb{N}_0$. Suppose that $T \in \mathrm{PSSYT}_\kappa\big(\nu + M(1^R), \mu/\mu_\star\big)_{(\pi^-,\pi^+)}$ where*

$$(\pi^-, \pi^+) \trianglerighteq (\lambda^-, \lambda^+) + M(\kappa^-, \kappa^+)$$

in the ℓ^--signed dominance order. Suppose that T has d small columns and that their top R boxes have signed weights $(\phi^-_1, \phi^+_1), \ldots, (\phi^-_d, \phi^+_d)$. If $M \geq -\nu_R + d + E$,

$$\begin{aligned}(\pi^-, \pi^+) \trianglelefteq \big(B_R(\nu) + ER\big)&\big(\omega_{\ell^-}(\mu/\mu_\star)^-, \omega_{\ell^-}(\mu/\mu_\star)^+\big) \\ &+ (\phi^-_1, \phi^+_1) + \cdots + (\phi^-_d, \phi^+_d) + \big(\nu_R - d - E + M\big)(\kappa^-, \kappa^+).\end{aligned}$$

and if $M \geq E - \nu_R$, the weaker bound

$$(\pi^-, \pi^+) \trianglelefteq \big(B_R(\nu) + ER\big)\big(\omega_{\ell^-}(\mu/\mu_\star)^-, \omega_{\ell^-}(\mu/\mu_\star)^+\big) + \big(\nu_R - E + M\big)(\kappa^-, \kappa^+)$$

also holds.

Proof. The second bound on (π^-, π^+) follows easily from the first since, by the definition of small in Definition 13.18, we have $(\phi_i^-, \phi_i^+) \trianglelefteq (\kappa^-, \kappa^+)$ for all i, and so the first bound is maximized when $d = 0$. It remains to prove the first bound.

If $R = 1$ then by Remark 13.11, $(\kappa^-, \kappa^+) = \bigl(\omega_{\ell^-}(\mu/\mu_\star)^-, \omega_{\ell^-}(\mu/\mu_\star)^+\bigr)$ and there are no exceptional columns. The claim is therefore that

$$(\pi^-, \pi^+) \trianglelefteq \bigl(B_1(\nu) + \nu_1 - d + M\bigr)\bigl(\omega_{\ell^-}(\mu/\mu_\star)^-, \omega_{\ell^-}(\mu/\mu_\star)^+\bigr) + (\phi_1^-, \phi_1^+) + \cdots + (\phi_d^-, \phi_d^+).$$

By Lemma 4.4, each box that is not the topmost box in one of the d non-small columns of T contributes at most $\bigl(\omega_{\ell^-}(\mu/\mu_\star)^-, \omega_{\ell^-}(\mu/\mu_\star)^+\bigr)$ to the signed weight of T. There are $B_1\bigl(\nu + (M)\bigr) + \nu_1 - d + M$ such boxes. Since by (13.6), we have $B_1\bigl(\nu + (M)\bigr) = B_1(\nu)$, the first bound now follows.

Now suppose that $R \geq 2$. Suppose that T has e exceptional columns, so T has eR boxes in the top R rows of exceptional columns. By (13.6) and (13.7) and Lemma 4.4, these boxes, together with the $B_R\bigl(\nu + M(1^R)\bigr) = B_R(\nu)$ boxes not in the top R positions of a column of length R, contribute at most

$$\bigl(B_R(\nu) + eR\bigr)\bigl(\omega_{\ell^-}(\mu/\mu_\star)^-, \omega_{\ell^-}(\mu/\mu_\star)^+\bigr) \tag{13.8}$$

to the signed weight of T. Each of the $\nu_R - d - e + M$ columns of height at least R that is both non-exceptional and non-small is typical (see Lemma 13.14) and so contributes (κ^-, κ^+) to the signed weight of T. There are also d small columns which contribute $(\phi_1^-, \phi_1^+) + \cdots + (\phi_d^-, \phi_d^+)$. Taken together these

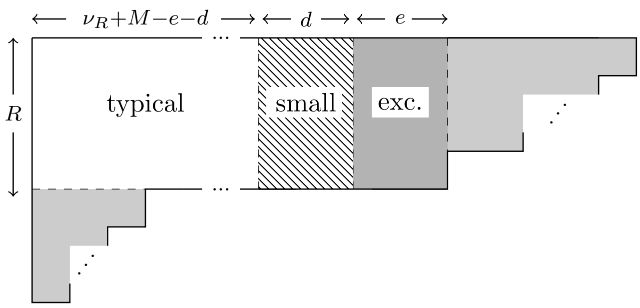


FIGURE 13.2. A plethystic semistandard signed tableau of outer shape $\nu + M(1^R)$ showing the contributions to the signed weight identified in the proof of Proposition 13.21. Each of the $B_R(\nu) + eR$ shaded boxes has an inner $\mu/\mu_\star$-tableau whose contribution is bounded by $\bigl(\omega_{\ell^-}(\mu/\mu_\star)^-, \omega_{\ell^-}(\mu/\mu_\star)^+\bigr)$. The hatched small columns contribute (ϕ_j^-, ϕ_j^+) for $1 \leq j \leq d$. The white boxes are in typical columns, the top R boxes in each contributing (κ^-, κ^+). (It is possible that some of the e exceptional columns appear to the left of the d small columns.) Note that as M varies, all but a constant number of boxes are in typical columns and so their signed weight (per column) is bounded by the stronger bound (κ^-, κ^+) rather than the weaker bound (per box) from $\bigl(\omega_{\ell^-}(\mu/\mu_\star)^-, \omega_{\ell^-}(\mu/\mu_\star)^+\bigr)$.

columns therefore contribute

$$\big(\nu_R + M - e - d\big)(\kappa^-,\kappa^+) + (\phi_1^-,\phi_1^+) + \cdots + (\phi_d^-,\phi_d^+) \tag{13.9}$$

to the signed weight of T. This is shown diagrammatically in Figure 13.2.

The sum of (13.8) and (13.9) is an upper bound on the signed weight of T. Since (κ^-,κ^+) is the signed weight of a tableau of outer shape (1^R) and inner shape $\mu/\mu_\star$, it follows, again by Lemma 4.4, that

$$(\kappa^-,\kappa^+) \trianglelefteq R\big(\omega_{\ell^-}(\mu/\mu_\star)^-, \omega_{\ell^-}(\mu/\mu_\star)^+\big).$$

We conclude that (for fixed d), this upper bound is maximized in the $\ell(\kappa^-)$-signed dominance order when e is as large as possible. By Lemma 13.17 we have $e \le E_{c^+,(\kappa^-,\kappa^+)}(\nu,\mu/\mu_\star : \lambda)$. Therefore we obtain an upper bound on (π^-,π^+) by substituting E for e in the sum of (13.8) and (13.9). (Note that by hypothesis $\nu_R - d - E + M \ge 0$ so the right-hand side is a valid signed weight, having non-negative entries.) This proves the first bound. □

Example 13.22. As in Example 13.13 we take the strongly 1-maximal signed weight $\big((2,2),(3,1)\big)$ of shape (2), size 2 and sign $+1$, and take $\nu = (2,1)$. We have $B_R(\nu) = B_2\big((2,1)\big) = 1$. Since $\big(\omega_{\ell^-}\big((4)\big)^-, \omega_{\ell^-}\big((4)\big)^+\big) = \big((1,1),(2)\big)$, and from Definition 13.16 we have $A^-\big((4)\big) = 2$, $A^+\big((4)\big) = 4$ and from Definition 13.18 we have

$$\begin{aligned} E^- &= 1.2 + 1.4 - |\lambda^-| = 6 - |\lambda^-|, \\ E^+ &= 1.4 + 1.(4+3) - \big(|\lambda^-| + \lambda_1^+\big) = 11 - |\lambda^-| - \lambda_1^+ \end{aligned}$$

and so, if λ is a partition of 12 having exactly k parts of size at least 2, we have

$$\begin{aligned} &E_{1,((2,2),(3,1))}\big((2,1),(4) : \lambda\big) \\ &\qquad = 17 - 2|\lambda^-| - (\lambda_1 - 2) + \lambda_3^+ + \cdots \\ &\qquad = 17 - 2|\lambda^-| - (\lambda_1 - 2) + (\lambda_3 - 2) + \cdots + (\lambda_k - 2). \end{aligned}$$

We denote this quantity by E as usual.

The plethystic semistandard signed tableaux relevant to the plethysm coefficients $\langle s_{(2,1)+M(1,1)} \circ s_{(4)}, s_{\lambda \oplus M((2,2),(3,1))}\rangle$ for $M \in \mathbb{N}_0$ lie in the set $\mathrm{PSSYT}_{((2,2),(3,1))}\big((2,1)+M(1^3),(4)\big)_{(\pi^-,\pi^+)}$. Since $\nu_R = \nu_2 = 1$, the weaker bound from Proposition 13.21 is that if $M \ge E-1$ and $(\pi^-,\pi^+) \trianglerighteq (\lambda^-,\lambda^+) + M\big((2,2),(3,1)\big)$ then

$$(\pi^-,\pi^+) \trianglelefteq (1+2E)\big((1,1),(2)\big) + (1 - E + M)\big((2,2),(3,1)\big)$$

in the 2-signed dominance order. We consider the two cases in the earlier Example 13.13. If $\lambda = (8,3,1)$ with 2^--decomposition $\big\langle(3,2),(6,1)\big\rangle$ then $E^- = 1$, $E^+ = 0$ and $E = 1$ and so the upper bound in the 2-signed dominance order is that if $(\pi^-,\pi^+) \trianglerighteq \big((3,2),(6,1)\big) + M\big((2,2),(3,1)\big)$ then

$$\begin{aligned} (\pi^-,\pi^+) &\trianglelefteq 3\big((1,1),(2)\big) + M\big((2,2),(3,1)\big) \\ &= \big((3,3),(6)\big) + M\big((2,2),(3,1)\big) \end{aligned}$$

for $M \geq 0$. If instead $\lambda = (6,3,3)$ with 2^--decomposition $\bigl\langle (3,3),(4,1,1)\bigr\rangle$ then $E^- = 0$, $E^+ = 1$ but because the proof of Lemma 13.17 allows, as it must in general, one exceptional column for each integer entry in $\{3,4,\ldots\}$, of which there are $\lambda^+_3 = 1$, we have $E = 0+1+1 = 2$ and the upper bound in the 2-signed dominance order is that if $(\pi^-,\pi^+) \trianglerighteq \bigl((3,3),(4,1,1)\bigr) + M\bigl((2,2),(3,1)\bigr)$ then

$$\begin{aligned}(\pi^-,\pi^+) &\trianglelefteq (1+2.2)\bigl((1,1),(2)\bigr) + (1-2+M)\bigl((2,2),(3,1)\bigr)\\ &= \bigl((3,3),(7,-1)\bigr) + M\bigl((2,2),(3,1)\bigr)\end{aligned}$$

for $M \geq 1$, where just for this inequality, to facilitate comparison, we allow a negative entry in what would normally be a signed weight. Note in each case that the upper bound is conditional on the lower bound, as we saw is necessary in Example 13.9.

We conclude this example in Example 14.6 in which small columns also must be considered.

13.6. **Signed weight bound in the ℓ-twisted dominance order.** We are now almost ready to prove Corollary 13.24; it is the analogue of Proposition 10.7 and Corollary 10.10. First though we must address the technical point that to apply Corollary 9.20 we require an $\bigl(\ell(\kappa^-)+1,\ell(\kappa^+)\bigr)$-large partition as the upper bound.

Lemma 13.23. *Let (κ^-,κ^+) be a strongly maximal signed weight. Fix $\ell^- = \ell(\kappa^-)$. Let $\mu/\mu_\star$ be a skew partition. Given any $K \in \mathbb{N}$ and $W \in \mathbb{N}_0$, the pair*

$$K\langle \kappa^-,\kappa^+\rangle + W\bigl\langle \omega_{\ell^-}(\mu/\mu_\star)^-, \omega_{\ell^-}(\mu/\mu_\star)^-\bigr\rangle$$

is the ℓ^--decomposition of the $\bigl(\ell^-+1,\ell(\kappa^+)\bigr)$-large partition

$$\kappa \oplus (K-1)(\kappa^-,\kappa^+) \oplus W\bigl(\omega_{\ell^-}(\mu/\mu_\star)^-, \omega_{\ell^-}(\mu/\mu_\star)^-\bigr)$$

where κ is the partition with ℓ^--decomposition $\bigl\langle \kappa^-,\kappa^+\bigr\rangle$.

Proof. Proposition 6.5 states that $\bigl\langle \kappa^-,\kappa^+\bigr\rangle$ is the ℓ^--decomposition of an $\bigl(\ell^-+1,\ell(\kappa^-)\bigr)$-large partition and so κ is well-defined. By Remark 6.2, κ is $\bigl(\ell(\kappa^-),\ell(\kappa^+)\bigr)$-large. Since the parts of $\bigl\langle \omega_{\ell^-}(\mu/\mu_\star)^-, \omega_{\ell^-}(\mu/\mu_\star)^+\bigr\rangle$ are partitions by Lemma 4.4, the same holds for

$$K\langle \kappa^-,\kappa^+\rangle + W\bigl\langle \omega_{\ell^-}(\mu/\mu_\star)^-, \omega_{\ell^-}(\mu/\mu_\star)^+\bigr\rangle.$$

By Lemma 9.6 and the following remark, this pair is the ℓ^--decomposition of the partition $\kappa \oplus (W-1)(\kappa^-,\kappa^+) \oplus W\bigl(\omega_{\ell^-}(\mu/\mu_\star)^-, \omega_{\ell^-}(\mu/\mu_\star)^-\bigr)$. This partition is $\bigl(\ell^-+1,\ell(\kappa^+)\bigr)$-large because κ is. □

As some motivation for the definition of ω in the following corollary, we recall from Example 13.9 that to get suitable upper bounds in a stable partition system when $\lambda = (4,1,1)$ and $(\kappa^-,\kappa^+) = \bigl(\varnothing,(4,1,1)\bigr)$ we had to begin with $\lambda \oplus (\kappa^-,\kappa^+)$, rather than λ, because there could be a single exceptional column. The partition κ is well defined by Proposition 6.5.

Corollary 13.24 (Outer Twisted Weight Bound)**.** *Let κ be a strongly c^+-maximal weight of shape $\mu/\mu_\star$, size R and sign $+1$. Fix $\ell^- = \ell(\kappa^-)$. Let ν be a partition and set $\nu^{(M)} = \nu + (M^R)$. Let λ be an $\big(\ell^-, \ell(\kappa^+)\big)$-large partition of $|\mu/\mu_\star||\nu|$. Let κ be the unique partition with ℓ^--decomposition $\langle \kappa^-, \kappa^+\rangle$. Set $E = E_{c^+,(\kappa^-,\kappa^+)}(\nu, \mu/\mu_\star : \lambda)$. If $\nu = \varnothing$ then define $\omega = \varnothing$ and otherwise define*

$$\omega = \begin{cases} \kappa \oplus (B_R(\nu) + ER)\big(\omega_{\ell^-}(\mu/\mu_\star)^-, \omega_{\ell^-}(\mu/\mu_\star)^+\big) \\ \kappa \oplus (B_R(\nu) + ER)\big(\omega_{\ell^-}(\mu/\mu_\star)^-, \omega_{\ell^-}(\mu/\mu_\star)^+\big) \oplus (\nu_R - E - 1)(\kappa^-, \kappa^+) \end{cases}$$

choosing the case according to whether $E \geq \nu_R$ or $E < \nu_R$. Then ω is an $\big(\ell^- + 1, \ell(\kappa^+)\big)$-large partition of

$$\begin{cases} |\lambda| + (E - \nu_R + 1)Rm & \text{if } E \geq \nu_R \\ |\lambda| & \text{if } E < \nu_R. \end{cases}$$

Suppose that $\sigma \trianglerighteq \lambda \oplus M\big(\kappa^-, \kappa^+\big)$ in the ℓ^--twisted dominance order. If s_σ is a constituent of the plethysm $s_{\nu^{(M)}} \circ s_{\mu/\mu_\star}$ then

$$\sigma \trianglelefteq \begin{cases} \omega \oplus \big(-(E - \nu_R + 1) + M\big)(\kappa^-, \kappa^+) & \text{if } E \geq \nu_R \\ \omega \oplus M(\kappa^-, \kappa^+) & \text{if } E < \nu_R \end{cases}$$

for all $M \in \mathbb{N}_0$ such that $M > E - \nu_R$.

Proof. Let $m = |\mu/\mu_\star|$ and let $n = |\nu|$. By Definition 13.15 we have

$$B_R(\nu) + ER = \begin{cases} |\nu| + (E - \nu_R)R & \text{if } E \geq \nu_R \\ |\nu| - (\nu_R - E)R & \text{if } E < \nu_R. \end{cases}$$

Hence, using that $|\kappa^-| + |\kappa^+| = Rm$ and $|\omega_{\ell^-}(\mu/\mu_\star)^-| + |\omega_{\ell^-}(\mu/\mu_\star)^+| = m$, we obtain

$$|\omega| = \begin{cases} Rm + (|\nu| + (E - \nu_R)R)m & \text{if } E \geq \nu_R \\ Rm - (\nu_R - E)Rm + mn + (\nu_R - E - 1)Rm = mn & \text{if } E < \nu_R \end{cases}$$

which, since $|\lambda| = |\nu|Rm$, shows that the size of ω is as claimed. By Lemma 13.23, ω is $\big(\ell(\kappa^-) + 1, \ell(\kappa^+)\big)$-large. Note that if $\nu = \varnothing$ then since $E = -1$, the final case applies and $\omega = \varnothing$. Note also that, by Lemma 13.23,

$$\begin{aligned} \langle \omega^-, \omega^+\rangle &= \kappa + K\langle \kappa^-, \kappa^+\rangle \\ &\quad + (B_R(\nu) + ER)\big\langle \omega_{\ell^-}(\mu/\mu_\star)^-, \omega_{\ell^-}(\mu/\mu_\star)^+\big\rangle \end{aligned} \tag{13.10}$$

where $K = 0$ if $E \geq \nu_R$ and $K = \nu_R - E - 1$ otherwise. We now follow part of the proof of Proposition 10.7. By Lemma 6.12, s_σ is a direct summand of $e_{\sigma^-} h_{\sigma^+}$. Hence, by Proposition 5.6 we have

$$\big|\,\mathrm{PSSYT}\big(\nu + M(1^R), \mu/\mu_\star\big)_{(\sigma^-,\sigma^+)}\big| = \langle e_{\sigma^-} h_{\sigma^+}, s_{\nu + M(1^R)} \circ s_{\mu/\mu_\star}\rangle \geq 1.$$

By hypothesis $\sigma \trianglerighteq \lambda \oplus M\big(\kappa^-, \kappa^+\big)$. Hence, by the definition of the ℓ^--twisted dominance order in Definition 6.6, the largeness assumption on λ and Lemma 9.6, we have

$$\langle \sigma^-, \sigma^+\rangle \trianglerighteq \langle \lambda^-, \lambda^+\rangle + M(\kappa^-, \kappa^+).$$

Suppose that $E \geq \nu_R$. Then applying the weaker second bound in Proposition 13.21 to the hypothesis $\mathrm{PSSYT}_\kappa\bigl(\nu + M(1^R), \mu/\mu_\star\bigr)_{(\sigma^-,\sigma^+)} \neq \varnothing$ we obtain

$$\begin{aligned}(\sigma^-,\sigma^+) &\trianglelefteq (\kappa^-,\kappa^+) + \bigl(B_R(\nu)+ER\bigr)\bigl(\omega_{\ell^-}(\mu/\mu_\star)^-, \omega_{\ell^-}(\mu/\mu_\star)^+\bigr) \\ &\qquad\qquad\qquad\qquad + \bigl(\nu_R - E + M - 1\bigr)(\kappa^-,\kappa^+) \\ &= (\omega^-,\omega^+) + \bigl(\nu_R - E + M - 1\bigr)(\kappa^-,\kappa^+)\end{aligned}$$

for $M > E - \nu_R$. By (13.10) this is equivalent to

$$\sigma \trianglelefteq \omega \oplus \bigl(\nu_R - E + M - 1\bigr)(\kappa^-,\kappa^+),$$

as required. The proof in the remaining case $E < \nu_R$ is entirely analogous, now using Proposition 13.21 to get

$$\begin{aligned}(\sigma^-,\sigma^+) \trianglelefteq (\nu_R - E)(\kappa^-,\kappa^+) + \bigl(B_R(\nu)+ER\bigr)\bigl\langle\omega_{\ell^-}(\mu/\mu_\star)^-, \omega_{\ell^-}(\mu/\mu_\star)^+\bigr\rangle \\ + M(\kappa^-,\kappa^+)\end{aligned}$$

where again, since we are in the case $\nu_R - E \geq 1$, the first two summands are the ℓ^--decomposition of the partition ω. $\square$

Remark 13.25. If $R = 1$ and $\nu \neq \varnothing$ then $E = 0$ and, as we show in §15.1, the conclusion of Corollary 13.24 reduces to $\sigma \trianglelefteq \omega^{(n)}_{\ell^-}(\mu/\mu_\star)$ where $\omega^{(n)}_{\ell^-}(\mu/\mu_\star)$ is as defined in Definition 10.6. The special case of the corollary is therefore logically equivalent to Proposition 10.7. This should be expected, because as seen in Example 13.5 using Lemma 4.17, the unique strongly maximal signed weight of shape $\mu/\mu_\star$ and size 1 is the signed weight of the greatest semistandard signed tableau $t_{\ell^-}(\mu/\mu_\star)$.

The upper bound in Corollary 13.24 is usually far from tight.

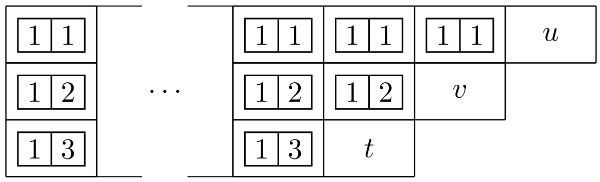
Example 13.26. We finish our first 'unsigned' running example (see Examples 13.8, 13.9, 13.12 and 13.20), with the strongly 1-maximal signed weight $\bigl(\varnothing,(4,1,1)\bigr)$ of shape (2) and size 3, so $\mu/\mu_\star = (2)/\varnothing$. As in Example 13.20 we take $\nu = (2,1) + C(1,1,1)$ and $\lambda = (4,2) + C(4,1,1)$. We saw in this example that $B_R(\nu) = B_3\bigl((2,1) + C(1,1,1)\bigr) = 3$ and $E_{1,(\varnothing,(4,1,1))}\bigl((2,1)+C(1,1,1),(2);(4,2)+C(4,1,1)\bigr) = 2$ for all $C \in \mathbb{N}_0$. Since $\kappa = \kappa^+ = (4,1,1)$ we have $\ell(\kappa^-) = 0$. Since $\omega_0\bigl((2)\bigr)^- = \varnothing$ and $\omega_0\bigl((2)\bigr)^+ = (2)$, the partition ω in Corollary 13.24 is therefore

$$\omega = \begin{cases}(4,1,1) + (3+2.3)(2) \\ (4,1,1) + (3+2.3)(2) + (C-3)(4,1,1)\end{cases}$$

choosing the case according to whether $2 \geq C$ or $2 < C$. Remembering that the 0-twisted dominance order is simply the usual dominance order, the conclusion of the corollary is that if $\sigma \trianglerighteq (4,2) + C(4,1,1) + M(4,1,1)$ and $\mathrm{PSSYT}_\kappa\bigl((2,1)+(C+M)(1,1,1),(2)\bigr)_{(\varnothing,\sigma)} \neq \varnothing$ then

$$\begin{aligned}\sigma &\trianglelefteq \begin{cases}(4,1,1) + 9(2) + (C-3+M)(4,1,1) \\ (4,1,1) + 9(2) + (C-3)(4,1,1) + M(4,1,1)\end{cases} \\ &= (22,1,1) + (C+M-3)(4,1,1) \qquad (13.11)\end{aligned}$$

for all $M > 2 - C$. (The unification of the two cases is expected for the same reason mentioned in Example 13.20 that columns of signed weight $\bigl(\varnothing, (4,1,1)\bigr)$ are typical.) We verify this bound directly when $C = 0$. Recall from after Definition 3.5 that $\mathrm{wt}(t)$ is the positive part of the signed weight of a tableau having only positive integer entries. We saw in Example 13.20 that if $\pi^+ \trianglerighteq (4,2) + M(4,1,1)$ then there is at most one non-typical column in any $T \in \mathrm{PSSYT}_\kappa\bigl((2,1)+(M,M,M),(2)\bigr)_{(\varnothing,\pi^+)}$, and this column is large-exceptional. Hence T has the form



where, by counting 1s as in Example 13.20, we require $4M + 1 + \mathrm{wt}(t)_1 + \mathrm{wt}(u)_1 + \mathrm{wt}(v)_1 \geq 4M + 4$. Similarly, by counting entries in $\{1,2\}$ and $\{1,2,3\}$, we obtain the necessary and sufficient condition $\mathrm{wt}(t) + \mathrm{wt}(u) + \mathrm{wt}(v) \trianglerighteq (3,2,1)$. If there is an exceptional column then $t = \boxed{2\,|\,2}$ and $u = \boxed{1\,|\,1}$ and *either* $v = \boxed{1\,|\,2}$ *or* $v = \boxed{1\,|\,3}$. If $v = \boxed{1\,|\,2}$ then T has weight $(M-1)(4,1,1) + (3,3) + (5,1) = (16,6,2) + (M-3)(4,1,1)$ and if $v = \boxed{1\,|\,3}$ then, very similarly T has weight $(16,5,3) + (M-3)(4,1,1)$. Otherwise there are two cases:

(a) $t = \boxed{1\,|\,3}$ and either $u = \boxed{1\,|\,2}$, $v = \boxed{1\,|\,2}$ or $u = \boxed{1\,|\,1}$, $v = \boxed{2\,|\,2}$;

(b) $t = \boxed{2\,|\,3}$ and $u = \boxed{1\,|\,1}$ and $v = \boxed{1\,|\,2}$;

in which, once again, T has weight $(M-1)(4,1,1) + (4,1,1) + (4,2) = (M-1)(4,1,1) + (3,2,1) + (5,1) = (16,5,3) + (M-3)(4,1,1)$. Thus, as expected from the remark before this example, the upper bound (13.11) is easily met.

See Example 15.10 for the stable plethysm from the example above. For a further 'signed' example of Corollary 13.24, used in the context of the proof of Theorem 1.2, see Example 14.6, which continues the running example in Examples 13.13 and 13.22.

13.7. **Results for both signs.** We now give the analogous definitions and a combined final result applicable to strongly maximal signed weights of either sign. We have already defined exceptional rows in Definition 13.10. In the following definition $B_R^+(\nu)$ is the same as $B_R(\nu)$ defined in Definition 13.15.

Definition 13.15. (Both signs.) Given a partition ν, $R \in \mathbb{N}$ and a sign ± 1, let $B_R^+(\nu)$ be the number of boxes (i,j) of $[\nu]$ such that either $i > R$ or $\nu'_j < R$ and let $B_R^-(\nu)$ be the number of boxes (i,j) of $[\nu]$ such that either $j > R$ or $\nu_i < R$.

In the following definition $E_{c^+,(\kappa^-,\kappa^+)}(\nu, \mu/\mu_\star : \lambda)$ is as already defined in Definition 13.18 if (κ^-,κ^+) has sign $+1$. For ease of reference we recall from Definition 13.16 that we have defined $A^-(\mu/\mu_\star) = |\omega_{\ell^-}(\mu/\mu_\star)^-|$ and $A^+(\mu/\mu_\star) = |\omega_{\ell^-}(\mu/\mu_\star)^-| + \sum_{i=1}^{c^+} \omega_{\ell^-}(\mu/\mu_\star)^+_i$.

Definition 13.18. (Both signs.) Let (κ^-, κ^+) be a strongly c^+-maximal signed weight of shape $\mu/\mu_\star$ and size R where $R \geq 2$. Let ν be a partition and let λ be a partition of $|\nu||\mu/\mu_\star|$. Fix $\ell^- = \ell(\kappa^-)$. Set $\nu_R^+ = \nu_R$ and $\nu_R^- = \nu'_R$. Define

$$E^- = B_R^\pm(\nu) A^-(\mu/\mu_\star) + \nu_R^\pm |\kappa^-| - |\lambda^-|,$$
$$E^+ = B_R^\pm(\nu) A^+(\mu/\mu_\star) + \nu_R^\pm\big(|\kappa^-| + \textstyle\sum_{i=1}^{c^+} \kappa_i^+\big) - |\lambda^-| - \sum_{i=1}^{c^+} \lambda_i^+$$

where the sign in the four appearances of $\pm$ is the sign of (κ^-, κ^+). Define

$$E_{c^+,(\kappa^-,\kappa^+)}(\nu, \mu/\mu_\star : \lambda) = \max\big(E^- + E^+ + \textstyle\sum_{i=\ell(\kappa^+)+1}^{\ell(\lambda^+)} \lambda_i^+, 0\big).$$

If $R = 1$ we instead define $E_{c^+,(\kappa^-,\kappa^+)}(\nu, \mu/\mu_\star : \lambda) = 0$. Finally we set $E_{c^+,(\kappa^-,\kappa^+)}(\varnothing, \mu/\mu_\star : \varnothing) = -1$.

Again we remind the reader that the partition κ in the following corollary is well-defined by Proposition 6.5.

Corollary 13.24 (Outer Twisted Weight Bound)**. (Both signs.)** *Let (κ^-, κ^+) be a strongly c^+-maximal signed weight of shape $\mu/\mu_\star$ and size R. Fix $\ell^- = \ell(\kappa^-)$. Let ν be a partition. Let λ be an $\big(\ell^-, \ell(\kappa^+)\big)$-large partition. Set $\nu_R^+ = \nu_R$ and $\nu_R^- = \nu'_R$. Let*

$$\nu^{(M)} = \begin{cases} \nu + M(1^R) & \textit{if } (\kappa^-, \kappa^+) \textit{ has sign } +1 \\ \nu \sqcup (R^M) & \textit{if } (\kappa^-, \kappa^+) \textit{ has sign } -1. \end{cases}$$

Set $E = E_{c^+,(\kappa^-,\kappa^+)}(\nu, \mu/\mu_\star : \lambda)$. Suppose that $\pi \trianglerighteq \lambda \oplus M(\kappa^-, \kappa^+)$ in the ℓ^--twisted dominance order and that $T \in \mathrm{PSSYT}_\kappa(\nu^{(M)}, \mu/\mu_\star)_{(\pi^-,\pi^+)}$. Throughout $\pm$ is the sign of (κ^-, κ^+).

(i) *If (κ^-, κ^+) has sign $+1$ then T has at most E exceptional columns and if (κ^-, κ^+) has sign -1 then T has at most E exceptional rows.*

(ii) *Let $M \geq -\nu_R^\pm + E$. If T has d small columns whose top R boxes have signed weights $(\phi_1^-, \phi_1^+), \ldots, (\phi_d^-, \phi_d^+)$ (when (κ^-, κ^+) has sign $+1$) or d small rows whose leftmost R boxes have signed weights $(\phi_1^-, \phi_1^+), \ldots, (\phi_d^-, \phi_d^+)$ (when (κ^-, κ^+) has sign -1) then*

$$\begin{aligned}(\pi^-, \pi^+) \trianglelefteq{} & \big(B_R^\pm(\nu) + ER\big)\big(\omega_{\ell^-}(\mu/\mu_\star)^-, \omega_{\ell^-}(\mu/\mu_\star)^+\big) \\ & + (\phi_1^-, \phi_1^+) + \cdots + (\phi_d^-, \phi_d^+) + \big(\nu_R^\pm - d - E + M\big)(\kappa^-, \kappa^+)\end{aligned}$$

in the ℓ^--signed dominance order for $M \geq -\nu_R^\pm + d + E$ and the weaker bound

$$\begin{aligned}(\pi^-, \pi^+) \trianglelefteq{} & \big(B_R^\pm(\nu) + ER\big)\big(\omega_{\ell^-}(\mu/\mu_\star)^-, \omega_{\ell^-}(\mu/\mu_\star)^+\big) \\ & + \big(\nu_R^\pm - E + M\big)(\kappa^-, \kappa^+)\end{aligned}$$

for $M \geq -\nu_R^\pm + E$ also holds.

(iii) *Let κ be the unique partition with ℓ^--decomposition $\langle \kappa^-, \kappa^+ \rangle$. If $\nu = \varnothing$ then define $\omega = \varnothing$ and otherwise define*

$$\omega = \begin{cases} \kappa \oplus (B_R^\pm(\nu) + ER)\big(\omega_{\ell^-}(\mu/\mu_\star)^-, \omega_{\ell^-}(\mu/\mu_\star)^+\big) \\ \kappa \oplus (B_R^\pm(\nu) + ER)\big(\omega_{\ell^-}(\mu/\mu_\star)^-, \omega_{\ell^-}(\mu/\mu_\star)^+\big) \oplus (\nu_R^\pm - E - 1)(\kappa^-, \kappa^+) \end{cases}$$

choosing the case according to whether $E \geq \nu_R^{\pm}$ *or* $E < \nu_R^{\pm}$. *Then* ω *is an* $\bigl(\ell^- + 1, \ell(\kappa^+)\bigr)$*-large partition of* $|\lambda| + SR|\mu/\mu_\star|$ *where*

$$S = \begin{cases} E - \nu_R^{\pm} + 1 & \text{if } E \geq \nu_R^{\pm} \\ 0 & \text{if } E < \nu_R^{\pm}. \end{cases}$$

Suppose that $\sigma \trianglerighteq \lambda \oplus M\bigl(\kappa^-, \kappa^+\bigr)$ *in the* ℓ^-*-twisted dominance order. If* s_σ *is a constituent of the plethysm* $s_{\nu^{(M)}} \circ s_{\mu/\mu_\star}$ *then*

$$\sigma \trianglelefteq \begin{cases} \omega \oplus \bigl(-(E - \nu_R^{\pm} + 1) + M\bigr)(\kappa^-, \kappa^+) & \text{if } E \geq \nu_R^{\pm} \\ \omega \oplus M(\kappa^-, \kappa^+) & \text{if } E < \nu_R^{\pm} \end{cases}$$

for all $M \in \mathbb{N}_0$ *such that* $M > \nu_R^{\pm} - E$.

Proof. In (iii) the size of ω and that ω is $\bigl(\ell^- + 1, \ell(\kappa^+)\bigr)$-large follow exactly as in Corollary 13.24; note that the sign of (κ^-, κ^+) is irrelevant to this first part of the proof. Part (i) is proved in Corollary 13.19 when (κ^-, κ^+) has sign $+1$; the proof is precisely analogous for sign -1, using the modified definitions above and the obvious modifications of Lemma 13.17 and Remark 13.11. When the sign is $+1$, (ii) is the stronger bound in Proposition 13.21, and (iii) is Corollary 13.24. Again in all three cases the proof is precisely analogous for sign -1. □

14. Proof of Theorem 1.2

In this section we prove Theorem 1.2 using the Signed Weight Lemma (Lemma 7.3). We begin with the second part of the theorem where the stable multiplicity is zero in §14.1. In §14.2 we construct a suitable stable partition system. Then in §14.3 we prove a final preliminary lemma on the length of signed weights, closely analogous to a well known result on the length of vectors in the Type A root system. Then finally in §14.5 we prove Theorem 14.7 which restates Theorem 1.2 with an explicit bound.

14.1. The vanishing case of Theorem 1.2.

Recall from Definition 11.1 that $\mathrm{LZ}\bigl([\lambda, \omega]_\trianglelefteq, (\kappa^-, \kappa^+), (\eta^-, \eta^+)\bigr)$ is defined whenever $(\eta^-, \eta^+) \not\trianglelefteq (\kappa^-, \kappa^+)$, and so in particular, whenever $(\eta^-, \eta^+) \trianglerighteq (\kappa^-, \kappa^+)$. (Here $\trianglerighteq$ is the signed dominance order of Definition 4.1). See Definition 13.18 in §13.7 for the definition of $E_{c^+,(\kappa^-,\kappa^+)}(\nu, \mu/\mu_\star : \lambda)$ in its 'both signs' version.

Proposition 14.1. *Let* (κ^-, κ^+) *be a strongly maximal signed weight of size* R*, shape* $\mu/\mu_\star$ *and sign* ε*. Fix* $\ell^- = \ell(\kappa^-)$*. Let* η^- *and* η^+ *be partitions with* $\ell(\eta^-) \leq \ell^-$*. Let* $\ell^+ = \max\bigl(\ell(\kappa^+), \ell(\eta^+)\bigr)$*. Let* ν *be a partition and let* λ *be an* (ℓ^-, ℓ^+)*-large partition of* $|\nu||\mu/\mu_\star|$*. Set* $\nu^{(M)} = \nu + (M^R)$ *if* (κ^-, κ^+) *has sign* $+1$ *and* $\nu^{(M)} = \nu \sqcup (R^M)$ *if* (κ^-, κ^+) *has sign* -1*. Set* $\nu_R^+ = \nu_R$ *and* $\nu_R^- = \nu'_R$ *and let* $\pm$ *be the sign of* (κ^-, κ^+)*. Set* $E = E_{c^+,(\kappa^-,\kappa^+)}(\nu, \mu/\mu_\star : \lambda)$*. If* $(\kappa^-, \kappa^+) \triangleleft (\eta^-, \eta^+)$ *then*

$$\bigl\langle s_{\nu^{(M)}} \circ s_{\mu/\mu_\star}, s_{\lambda \oplus M(\eta^-,\eta^+)} \bigr\rangle = 0$$

for all

$$M > \begin{cases} \mathrm{LZ}\big([\lambda \oplus (E-\nu_R^{\pm}+1)(\eta^-,\eta^+),\omega]_{\trianglelefteq},(\kappa^-,\kappa^+),(\eta^-,\eta^+)\big)+(E-\nu_R^{\pm}+1) \\ \mathrm{LZ}\big([\lambda,\omega]_{\trianglelefteq},(\kappa^-,\kappa^+),(\eta^-,\eta^+)\big) \end{cases}$$

choosing the case according to whether $E \geq \nu_R^{\pm}$ or $E < \nu_R^{\pm}$, where ω is the relevant partition for the two cases taken from Corollary 13.24.

Proof. By Corollary 13.24(iii), if s_σ is a constituent of $s_{\nu^{(M)}} \circ s_{\mu/\mu_\star}$ such that $\sigma \trianglerighteq \lambda \oplus M(\kappa^-,\kappa^+)$ then

$$\sigma \trianglelefteq \begin{cases} \omega \oplus \big(-(E-\nu_R^{\pm}+1)+M\big)(\kappa^-,\kappa^+) & \text{if } E \geq \nu_R^{\pm} \\ \omega \oplus M(\kappa^-,\kappa^+) & \text{if } E < \nu_R^{\pm} \end{cases}$$

for all $M \in \mathbb{N}_0$ such that $M > -\nu_R^{\pm} + E$. Since $(\eta^-,\eta^+) \triangleright (\kappa^-,\kappa^+)$ we may apply this result taking $\sigma = \lambda \oplus M(\eta^-,\eta^+)$. In the second case, when $E < \nu_R^{\pm}$, the proposition then follows by an argument very closely analogous to the proof of Proposition 11.2; the analogue of (11.1) is that if $\sigma \leftrightarrow \big\langle \sigma^-,\sigma^+ \big\rangle$ then

$$\begin{aligned}(\sigma^-,\sigma^+) \trianglelefteq (\kappa^-,\kappa^+) &+ (B_R^{\pm}(\nu)+ER)(\omega_{\ell^-}(\mu/\mu_\star)^-,\omega_{\ell^-}(\mu/\mu_\star)^+) \\ &+ (\nu_R^{\pm}-E-1)(\kappa^-,\kappa^+) + M(\kappa^-,\kappa^+)\end{aligned}$$

and so, substituting $(\lambda^-,\lambda^+) + M(\eta^-,\eta^+)$ for (σ^-,σ^+), as it justified by Lemma 9.6 since $(\eta^-,\eta^+) \trianglerighteq (\kappa^-,\kappa^+)$, the analogue of (11.2) is

$$\begin{aligned}(\lambda^-,\lambda^+) + M(\eta^-,\eta^+) \trianglelefteq (B_R^{\pm}(\nu)+ER)&(\omega_{\ell^-}(\mu/\mu_\star)^-,\omega_{\ell^-}(\mu/\mu_\star)^+) \\ &+ (\nu_R^{\pm}-E)(\kappa^-,\kappa^+) + M(\kappa^-,\kappa^+)\end{aligned}$$

where since $E < \nu_R^{\pm}$, we add $M(\kappa^-,\kappa^+)$ to a well-defined signed weight. The application of the inequalities is then exactly as before, giving a contradiction whenever $M > \mathrm{LZ}\big([\lambda,\omega]_{\trianglelefteq},(\kappa^-,\kappa^+),(\eta^-,\eta^+)\big)$. In the first case, when $E \geq \nu_R^{\pm}$, we note that because of the shift in M, we now require

$$-(E-\nu_R^{\pm}+1)+M \geq \mathrm{LZ}\big([\lambda \oplus (E-\nu_R^{\pm}+1)(\eta^-,\eta^+),\omega]_{\trianglelefteq},(\kappa^-,\kappa^+),(\eta^-,\eta^+)\big)$$

where ω is now defined by the first case in Corollary 13.24. The argument is otherwise the same. □

14.2. **Stable partition system for Theorem 1.2.** The following lemma is the analogue of Lemma 11.3. Again we remind the reader that the statistic statistic $E_{c^+,(\kappa^-,\kappa^+)}(\nu,\mu/\mu_\star : \lambda)$ is defined in Definition 13.18. Here we also use $B_R^{\pm}(\nu)$ from Definition 13.15; each definition is given in its 'both signs' versions in §13.7. The partition κ in the following lemma is well-defined by Proposition 6.5.

Lemma 14.2. *Let (κ^-,κ^+) be a strongly c^+-maximal signed weight of shape $\mu/\mu_\star$ and size R. Fix $\ell^- = \ell(\kappa^-)$. Let ν be a partition. Let λ be an $\big(\ell^-+1,\ell(\kappa^+)\big)$-large partition of $|\nu||\mu/\mu_\star|$. Let $\nu_R^{\pm} = \nu_R$ if (κ^-,κ^+) has sign $+1$ and let $\nu_R^{\pm} = \nu'_R$ if (κ^-,κ^+) has sign -1. Define*

$$\nu^{(M)} = \begin{cases} \nu + M(1^R) & \text{if } (\kappa^-,\kappa^+) \text{ has sign } +1 \\ \nu \sqcup (R^M) & \text{if } (\kappa^-,\kappa^+) \text{ has sign } -1. \end{cases}$$

Set $E = E_{c^+,(\kappa^-,\kappa^+)}(\nu, \mu/\mu_\star : \lambda)$. Let κ be the unique partition with ℓ^--decomposition $\langle \kappa^-, \kappa^+ \rangle$. Define $\omega = \varnothing$ if $\nu = \varnothing$ and otherwise

$$\omega = \begin{cases} \kappa \oplus (B^{\pm}_R(\nu) + ER)\big(\omega_{\ell^-}(\mu/\mu_\star)^-, \omega_{\ell^-}(\mu/\mu_\star)^+\big) \\ \kappa \oplus (B^{\pm}_R(\nu) + ER)\big(\omega_{\ell^-}(\mu/\mu_\star)^-, \omega_{\ell^-}(\mu/\mu_\star)^+\big) \oplus (\nu^{\pm}_R - E - 1)(\kappa^-, \kappa^+) \end{cases}$$

choosing the case according to whether $E \geq \nu^{\pm}_R$ or $E < \nu^{\pm}_R$. If $E \geq \nu^{\pm}_R$ then set $S = -\nu^{\pm}_R + E + 1$ and set $\mathcal{P}^{(M)} = \varnothing$ for $M < S$ and

$$\mathcal{P}^{(M)} = \big[\lambda \oplus M(\kappa^-, \kappa^+), \omega \oplus (M - S)(\kappa^-, \kappa^+)\big]_{\trianglelefteq}$$

if $M \geq S$. If $E < \nu^{\pm}_R$ then set $S = 0$ and set

$$\mathcal{P}^{(M)} = \big[\lambda \oplus M(\kappa^-, \kappa^+), \omega \oplus M(\kappa^-, \kappa^+)\big]_{\trianglelefteq}$$

for all $M \in \mathbb{N}_0$. Then $(\mathcal{P}^{(M)})_{M \in \mathbb{N}_0}$ is a stable partition system with respect to the map $\pi \mapsto \pi \oplus (\kappa^-, \kappa^+)$ and the twisted symmetric functions $g_\pi = e_{\pi^-} h_{\pi^+}$, stable for $M \geq S + K$ where K is the maximum of

- $\mathrm{L}\big([\lambda^- + S\kappa^-, \omega^-]^{\ell^-}_{\blacktriangleleft}, \kappa^-\big)$,
- $\mathrm{L}\big([\lambda^+ + S\kappa^+, \omega^+ + (|\lambda^+| + S|\kappa^+| - |\omega^+|)]_{\trianglelefteq}, \kappa^+\big)$,
- $\big(\omega^+_1 + \omega^+_2 - 2\lambda^+_1 - 2S\kappa^+_1 + 2|\lambda^+| + 2S|\kappa^+| - 2|\omega^+|\big)/(\kappa^+_1 - \kappa^+_2)$,
- $\big(\max(\ell(\lambda^+), \ell(\kappa^+)) + |\omega^-| - |\lambda^-| - S|\kappa^-| - \omega^-_{\ell^-}\big)/\kappa^-_{\ell^-}$.

and zero. Moreover, if $\pi \in \mathcal{P}^{(M)}$ and s_σ is a summand of $e_{\pi^-} h_{\pi^+}$ appearing in the plethysm $s_{\nu^{(M)}} \circ s_{\mu/\mu_\star}$ then $\sigma \in \mathcal{P}^{(M)}$.

Proof. By Corollary 13.24(iii) in its 'both signs' version in §13.7, ω is an $\big(\ell^- + 1, \ell(\kappa^+)\big)$-large partition, of size $|\lambda| + SR|\mu/\mu_\star|$. If $E < \nu^{\pm}_R$ then it is immediate from Corollary 9.20 applied with λ and ω that the partition system $(\mathcal{P}^{(M)})_{M \in \mathbb{N}_0}$ is stable, and since $S = 0$, the bounds above defining K are exactly the bounds defined in the statement of this lemma. If $E \geq \nu^{\pm}_R$ then $S = E - \nu^{\pm}_R + 1$ and we instead apply the corollary to the partitions $\lambda \oplus S(\kappa^-, \kappa^+)$ and ω. Since, by Lemma 9.6 we have $\big(\lambda \oplus S(\kappa^-, \kappa^+)\big)^- = \lambda^- \oplus S\kappa^-$, and so on, the result from Corollary 9.20 is that the partition system $(\mathcal{P}^{(N+S)})_{N \in \mathbb{N}_0}$ is stable for $N \geq K$, where again K is as defined in the statement of this lemma. Since $K \geq 0$, it follows that $(\mathcal{P}^{(M)})_{M \in \mathbb{N}_0}$ is stable for $M \geq S + K$. (The reader may easily check that Definition 7.1 permits any finite number of the sets $\mathcal{P}^{(M)}$ to be empty.) For the 'moreover' part of the result, first note that by Lemma 6.12, $\sigma \trianglerighteq \pi$. By Corollary 13.24(iii) we have $\sigma \trianglelefteq \omega \oplus (M - S)(\kappa^-, \kappa^+)$. Hence $\sigma \in \mathcal{P}^{(M)}$. This completes the proof. □

14.3. **Box moving bound.** Let $\mathcal{V}_{\ell^-} \times \mathcal{V}$ be the abelian group generated by the set $\mathcal{W}_{\ell^-} \times \mathcal{W}$ of signed weights. Identifying $\big((\alpha^-_1, \ldots, \alpha^-_{\ell^-}), (\alpha^+_1, \alpha^+_2, \ldots)\big) \in \mathcal{V}_{\ell^-} \times \mathcal{V}$ with $\big(\alpha^-_1, \ldots, \alpha^-_{\ell^-}, \alpha^+_1, \alpha^+_2, \ldots)$ as in the definition of the ℓ^--signed dominance order on $\mathcal{W}_{\ell^-} \times \mathcal{W}$ (see Definition 4.1) we define $\varepsilon^{(j)} \in \mathcal{V}_{\ell^-} \times \mathcal{V}$ for each $j \in \mathbb{N}$ by

$$\varepsilon^{(j)}_i = \begin{cases} 1 & \text{if } i = j \\ -1 & \text{if } i = j + 1 \\ 0 & \text{otherwise.} \end{cases}$$

For example if $\ell^- = 2$ then $\varepsilon^{(1)} = \big((1,-1),\varnothing\big)$, $\varepsilon^{(2)} = \big((0,1),(-1)\big)$, $\varepsilon^{(3)} = \big((0,0),(1,-1)\big)$, and so on. Observe that the $\varepsilon^{(j)}$ for $j \in \mathbb{N}$ form a $\mathbb{Z}$-basis for the subgroup of $\mathcal{V}_{\ell^-} \times \mathcal{V}$ of elements of sum 0. We say that $(\gamma^-,\gamma^+) \in \mathcal{V}_{\ell^-} \times \mathcal{V}$ is *root-positive* if, under this identification, it is a linear combination of the $\varepsilon^{(j)}$ with non-negative coefficients. Given a root-positive element $(\beta^-,\beta^+) \in \mathcal{V}_{\ell^-} \times \mathcal{V}$ expressed uniquely in the $\varepsilon^{(j)}$ basis as

$$(\beta^-,\beta^+) = \sum_{j\in\mathbb{N}} b_j \varepsilon^{(j)},$$

we define the *root-length* of (β^-,β^+), denoted $||(\beta^-,\beta^+)||$, by

$$||(\beta^-,\beta^+)|| = \sum_j b_j. \tag{14.1}$$

Note the sum in (14.1) is a finite sum of non-negative integers. In practice, it is more often helpful to think of each $\varepsilon^{(j)}$ as defining a single box move as in the remark following Example 6.9.

Lemma 14.3. *Fix $\ell^- \in \mathbb{N}$. Let π, σ and τ be partitions such that $\pi \trianglelefteq \sigma \trianglelefteq \tau$. Then $\tau-\sigma$ is root-positive and $||(\tau^-,\tau^+)-(\sigma^-,\sigma^+)|| \leq ||(\tau^-,\tau^+)-(\pi^-,\pi^+)||$.*

Proof. By Definition 6.6 we have $(\sigma^-,\sigma^+) \trianglelefteq (\tau^-,\tau^+)$ in the ℓ^--signed dominance order on $\mathcal{W}_{\ell^-} \times \mathcal{W}$. By Definition 4.1 this is the usual dominance order on concatenated weights. Hence by a standard result on the dominance order familiar from the Type A root system, which also follows from the remark above about single box moves, $(\tau^- - \sigma^-, \tau^+ - \sigma^+)$ is root-positive. Let

$$(\sigma^- - \pi^-, \sigma^+ - \pi^+) = \textstyle\sum_{j\in\mathbb{N}} b_j \varepsilon^{(j)}$$
$$(\tau^- - \sigma^-, \tau^+ - \sigma^+) = \textstyle\sum_{j\in\mathbb{N}} c_j \varepsilon^{(j)}$$

where $b_j, c_j \geq 0$ for each j. Now $(\tau^- - \pi^-, \tau^+ - \pi^+) = \sum_{j\in\mathbb{N}} (b_j + c_j)\varepsilon^{(j)}$ and since $b_j + c_j \geq c_j$ the remaining claim follows. □

The ℓ^--signed dominance order on the set $\mathcal{W}_{\ell^-} \times \mathcal{W}$ of signed weights used in the following lemma is defined in Definition 4.1.

Lemma 14.4. *Let (ϕ_i^-,ϕ_i^+) for $1 \leq i \leq d$ and (κ^-,κ^+) be signed weights of the same size such that $(\phi_i^-,\phi_i^+) \lhd (\kappa^-,\kappa^+)$ in the ℓ^--signed dominance order for each i. Then $d(\kappa^-,\kappa^+) - \sum_{i=1}^d (\phi_i^-,\phi_i^+)$ is root-positive and*

$$\Big|\Big|d(\kappa^-,\kappa^+) - \sum_{i=1}^d (\phi_i^-,\phi_i^+)\Big|\Big| \geq d.$$

Proof. By Lemma 14.3, $(\kappa^-,\kappa^+) - (\phi^-,\phi^+)_i$ is root-positive. Write $(\kappa^- - \phi_i^-, \kappa^+ - \phi_i^+) = \sum_{j\in\mathbb{N}} b_{ij}\varepsilon^{(j)}$ where $b_{ij} \in \mathbb{N}_0$ for all i and j. For each i, at least one coefficient b_{ij} is non-zero, and so

$$d(\kappa^-,\kappa^+) - \sum_{i=1}^d (\phi_i^-,\phi_i^+) = \sum_{j\in\mathbb{N}} \sum_{i=1}^d b_{ij}\varepsilon^{(j)}$$

where the sum of all the coefficients is at least d. □

14.4. **Small columns and rows.** Let (κ^-,κ^+) be a strongly c^+-maximal signed weight of shape $\mu/\mu_\star$ and size R. We use '/' to distinguish the cases when (κ^-,κ^+) has sign $+1/-1$. Recall from Definition 13.10 that in each non-exceptional column/row of a plethystic semistandard signed tableau T of inner shape $\mu/\mu_\star$ *either* the top/leftmost R boxes in the column/row form the plethystic semistandard signed tableau $T_{(\kappa^-,\kappa^+)}$, *or* the column/row is small having signed weight (ϕ^-,ϕ^+) such that $(\phi^-,\phi^+) \lhd (\kappa^-,\kappa^+)$ in the $\ell(\kappa^-)$-signed dominance order on set $\mathcal{W}_{\ell^-}\times\mathcal{W}$ defined in Definition 4.1. The bound $E_{c^+,(\kappa^-,\kappa^+)}(\nu,\mu/\mu_\star : \lambda)$ in the following lemma is defined in Definition 13.18 and the statistic $B^{\pm}_R(\nu)$ is defined in Definition 13.15, each in their 'both signs' version in §13.7.

Lemma 14.5. *Let (κ^-,κ^+) be a strongly c^+-maximal signed weight of shape $\mu/\mu_\star$ and size R. Let ν be a partition. Fix $\ell^- = \ell(\kappa^-)$. Let λ be a $\bigl(\ell^-,\ell(\kappa^+)\bigr)$-large partition of $|\mu/\mu_\star||\nu|$. Let $E = E_{c^+,(\kappa^-,\kappa^+)}(\nu,\mu/\mu_\star : \lambda)$. Let $\nu^{\pm}_R = \nu_R$ if (κ^-,κ^+) has sign $+1$ and let $\nu^{\pm}_R = \nu'_R$ if (κ^-,κ^+) has sign -1. Let $M \in \mathbb{N}_0$ and set*

$$\nu^{(M)} = \begin{cases} \nu + M(1^R) & \text{if } (\kappa^-,\kappa^+) \text{ has sign } +1 \\ \nu \sqcup (R^M) & \text{if } (\kappa^-,\kappa^+) \text{ has sign } -1. \end{cases}$$

Let $T \in \mathrm{PSSYT}_\kappa(\nu^{(M)},\mu/\mu_\star)_{(\pi^-,\pi^+)}$ be a (κ^-,κ^+)-adapted plethystic semistandard signed tableau where $(\pi^-,\pi^+) \trianglerighteq (\lambda^-,\lambda^+) + M(\kappa^-,\kappa^+)$ in the ℓ^--signed dominance order on $\mathcal{W}_{\ell^-}\times\mathcal{W}$. Set

$$D = \bigl|\bigl|\bigl(B^{\pm}_R(\nu)+ER\bigr)\bigl(\omega_{\ell^-}(\mu/\mu_\star)^-,\omega_{\ell^-}(\mu/\mu_\star)^+\bigr)+(\nu^{\pm}_R-E)(\kappa^-,\kappa^+)-(\lambda^-,\lambda^+)\bigr|\bigr|$$

where in this equation the root-length is of a root-positive element. If $M \in \mathbb{N}_0$ and $M \geq D + E - \nu^{\pm}_R$ then T has at most D small columns/rows.

Proof. For ease of notation set $(\omega^-,\omega^+) = \bigl(\omega_{\ell^-}(\mu/\mu_\star)^-,\omega_{\ell^-}(\mu/\mu_\star)^+\bigr)$. Suppose that T has exactly d small columns/rows. The bound in the 'both signs' version of Corollary 13.24(ii) states that, in the ℓ^--signed dominance order, $(\pi^-,\pi^+) \trianglelefteq (\sigma^-,\sigma^+)$ where

$$(\sigma^-,\sigma^+) = \bigl(B^{\pm}_R(\nu)+ER\bigr)(\omega^-,\omega^+) + \sum_{i=1}^{d}(\phi^-_i,\phi^+) + (\nu^{\pm}_R - E + M - d)(\kappa^-,\kappa^+)$$

for $M \geq -\nu^{\pm}_R + d + E$ and the weaker bound in the same result is that $(\pi^-,\pi^+) \trianglelefteq (\tau^-,\tau^+)$ where

$$(\tau^-,\tau^+) = \bigl(B^{\pm}_R(\nu) + ER\bigr)(\omega^-,\omega^+) + (\nu^{\pm}_R - E + M)(\kappa^-,\kappa^+)$$

for $M \geq E - \nu^{\pm}_R$. Applying Lemma 14.3 to $(\pi^-,\pi^+) \trianglelefteq (\sigma^-,\sigma^+)(\tau^-,\tau^+)$ we obtain

$$\bigl|\bigl|(\tau^-,\tau^+) - (\pi^-,\pi^+)\bigr|\bigr| \geq \bigl|\bigl|d(\kappa^-,\kappa^+) - \sum_{i=1}^{d}\phi_i\bigr|\bigr| \geq d. \tag{14.2}$$

where the final inequality uses Lemma 14.4. On the other hand, applying Lemma 14.3 to the chain of inequalities

$$(\lambda^-,\lambda^+) + M(\kappa^-,\kappa^+) \trianglelefteq (\pi^-,\pi^+) \trianglelefteq (\tau^-,\tau^+)$$

valid for $M \in \mathbb{N}_0$ with $M \geq -\nu_R^{\pm} + d + E$ we get

$$\begin{aligned}
&\big|\big|(\tau^-, \tau^+) - (\pi^-, \pi^+)\big|\big| \\
&\quad \leq \big|\big|\big(B_R^{\pm}(\nu) + ER\big)(\omega^-, \omega^+) + (\nu_R^{\pm} - E + M)(\kappa^-, \kappa^+) \\
&\qquad\qquad\qquad\qquad\qquad - \big((\lambda^-, \lambda^+) + M(\kappa^-, \kappa^+)\big)\big|\big| \\
&\quad = \big|\big|\big(B_R^{\pm}(\nu) + ER\big)(\omega^-, \omega^+) + (\nu_R^{\pm} - E)(\kappa^-, \kappa^+) - (\lambda^-, \lambda^+)\big|\big|. \qquad (14.3)
\end{aligned}$$

By the same lemma, the right-hand side is the root-length of a root-positive element. Combining (14.2) and (14.3) we obtain the required bound $d \leq D$, valid for all $M \in \mathbb{N}_0$ with $M \geq D + E - \nu_R^{\pm}$. □

Example 14.6. Continuing Examples 13.13 and 13.22 we take the strongly 1-maximal signed weight $\big((2,2),(3,1)\big)$, defined by the maximal plethystic semistandard tableau family $\big\{\boxed{\mathbf{1}|\mathbf{2}|1|1}, \boxed{\mathbf{1}|\mathbf{2}|1|2}\big\}$ of size 2 and sign $+1$, and consider the plethysm coefficients $\langle s_{(2,1)+M(1,1)} \circ s_{(4)}, s_{\lambda \oplus M((2,2),(3,1))}\rangle$ for $M \in \mathbb{N}_0$. The partition κ with 2-decomposition $\big\langle(2,2),(3,1)\big\rangle$ is $(5,3)$. Let $E = E_{1,((2,2),(3,1))}\big((2,1),(4) : \lambda\big)$. Note that $B_R(\nu) = B_2\big((2,1)\big) = 1$. If $E \geq \nu_2 = 1$ then the first case for ω in Corollary 13.24 and Lemma 14.2 applies and so the partition ω is

$$\omega = (5,3) \oplus (1+2E)\big((1,1),(2)\big) = (7+4E, 3, 2^{1+2E}).$$

If instead $E = 0$ then the second case for ω in these results applies, but now $\nu_R^{\pm} - E - 1 = 1 - 0 - 1 = 0$, and so the second case defines exactly the same partition. Similarly, in either case, the statistic S in Lemma 14.2 is $-\nu_2 + E + 1 = -1 + E + 1 = E$ and so the stable partition system from this lemma is

$$\mathcal{P}^{(M)} = \big[\lambda \oplus M\big((2,2),(3,1)\big), \omega \oplus (M - E)\big((2,2),(3,1)\big)\big]_{\trianglelefteq}$$

for $M \geq E$. As a small check, note that λ is a partition of 12 and ω is a partition of $12+8E$, and so the sizes of the partitions defining the interval for the 2-twisted dominance order agree, as they must. In the remainder of this example we take $\lambda = (8,3,1)$, the first of the cases in the earlier examples; note that λ is $\big(\ell(\kappa^-) + 1, \ell(\kappa^+)\big) = (3,2)$-large as required by Lemma 14.2. We show all the ideas in the proof of Theorem 14.7 by checking the conditions for the Signed Weight Lemma (Lemma 7.3).

The stable partition system for $\lambda = (8,3,1)$ *explicitly.* We saw in Example 13.22 that $E = 1$, and so $\omega = (11,3,2^3) = (8,2,2) \oplus \big((2,2),(3,1)\big)$. It is routine to check using the definition of the 2-twisted dominance order in Definition 6.6 that the partition system $(\mathcal{P}^{(M)})_{M \in \mathbb{N}_0}$ is

$$\mathcal{P}^{(M)} = \big\{(8,3,1),(9,2,1),(7,3,2),(8,2,2)\big\} \oplus M\big((2,2),(3,1)\big)$$

for $M \geq 1$, where the notation indicates that $M\big((2,2),(3,1)\big)$ is adjoined to all four partitions in the given set. The bounds in Lemma 14.2 are -2, -1, -3 and -1 and so $K = \max(-2,-1,-3,-1,0) = 0$. The shift S in this lemma is $-\nu_R^{\pm} + E + 1 = -1 + 1 + 1 = 1$. Therefore, Lemma 14.2 states that $(\mathcal{P}^{(M)})_{M \in \mathbb{N}_0}$ is stable for $M \geq 1$. Here the bound is tight, since, by the definition in Lemma 14.2, we have $\mathcal{P}^{(0)} = \varnothing$.

Condition (ii) *in the Signed Weight Lemma for* $\lambda = (8,3,1)$. We start with (ii) because the calculations are helpful for (i). We saw in Example 13.13 that

$$\left|\mathrm{PSSYT}_{((2,2),(3,1)}\big((2,1) + M(1,1),(4)\big)_{((3+2M,2+2M),(6+3M,1+M))}\right| = 4$$

for all $M \geq 1$, giving condition (ii) in the Signed Weight Lemma (Lemma 7.3) for the partitions obtained by adjoining to $(8,3,1) \leftrightarrow \big\langle(3,2),(6,1)\big\rangle$. It is routine to check by similar arguments using the 2-decompositions $\big\langle(3,2),(7)\big\rangle$, $\big\langle(3,3),(5,1)\big\rangle$ and $\big\langle(3,3),(6)\big\rangle$ of the three larger partitions in $\mathcal{P}^{(0)}$ that the corresponding sets of plethystic semistandard signed tableaux for these partitions have sizes 1, 1 and 0 for all $M \geq 0$. However, rather than use this ad-hoc argument, we take the opportunity to motivate the relevant part of the proof of Theorem 14.7. Let $\langle\pi^-,\pi^+\rangle$ be the 2-decomposition of one of the four partitions in $\mathcal{P}^{(M)}$. Then the map defined by inserting the plethystic semistandard signed tableau shown in the margin as a new typical first column into a plethystic semistandard tableau in $\mathrm{PSSYT}_{((2,2),(3,1))}\big((2,1) + M(1,1),(4)\big)_{(\pi^-,\pi^+)}$ is surjective, and so bijective, if and only if every plethystic semistandard signed tableaux in $\mathrm{PSSYT}_{((2,2),(3,1))}\big((2,1) + (M+1)(2,2), (4)\big)_{(\pi^-+(3,1),\pi^++(2))}$ has at least one typical column, i.e. one equal to the tableau in the margin. Since $E = 1$, each such T has at most one exceptional column. By Lemma 14.5, with $(\lambda^-,\lambda^+) = \big((3,2),(6,1)\big)$, T has at most

1	**2**	1	1
1	**2**	1	2

$$\begin{aligned}\big|\big|(1+2.1)\big((1,1),(2)\big) + (1-1)\big((2,2),(3,1)\big) - \big((3,2),(6,1)\big)\big|\big| \\ &= \big|\big|\big((3,3),(6)\big) - \big((3,2),(6,1)\big)\big|\big| \\ &= \big|\big|\big((0,1),(0,-1)\big)\big|\big| \\ &= \big|\big|\varepsilon^{(2)} + \varepsilon^{(3)}\big|\big| \\ &= 2\end{aligned}$$

small columns. By Lemma 13.14, every column that is not small or exceptional is typical. Since $E = 1$, there is at most one exceptional column, and so the insertion map is surjective for $M \geq 3$. In fact, as seen in Example 13.13 when $(\pi^-,\pi^+) = (8,3,1) + M\big((1,1),(2)\big)$, and as follows from the ad-hoc calculation earlier in this paragraph, the insertion map is surjective for all $M \geq 1$.

As a further illustration of this bound, we remark that if instead $\lambda = (6,3,3)$ then $E = 2$, as seen in Example 13.22, and the bound on the number of small columns from Lemma 14.5 is now $\big|\big|(1+2.2)\big((1,1),(2)\big) + (1-2)\big((2,2),(3,1)\big) - \big((3,3),(4,1,1)\big)\big|\big| = ||(0,0,3,-2,-1)|| = 4$. In Example 13.13 we saw that in fact there is at most one small column in each plethystic semistandard tableau of outer shape $(2,1) + M(1,1)$, inner shape (3) and signed weight $\big((3+2M,3+2M),(4+3M,1+M,1)\big)$, so in this case, as is typical, the bound is not very strong.

Condition (i) *in the Signed Weight Lemma for* $\lambda = (8,3,1)$. As promised by the final claim in Lemma 14.2, if $\sigma \trianglerighteq (8,3,1) \oplus M\big((2,2),(3,1)\big)$ and s_σ

appears in $s_{(2,1)+M(1,1)} \circ s_{(4)}$ then σ is one of the four partitions in $\mathcal{P}^{(M)}$; in fact, since there are no plethystic semistandard signed tableaux of signed weight $\bigl((3,3),(6)\bigr)+M\bigl((1,1),(2)\bigr)$ equal to the 2-decomposition of the upper bound $(8,2,2)\oplus M\bigl((2,2),(3,1)\bigr)$, only the first three partitions listed in $\mathcal{P}^{(M)}$ appear.

Conclusion. Using computer algebra one may obtain the constant values of $\langle s_{(2,1)+M(1,1)} \circ s_{(4)}, s_{\sigma\oplus M((2,2),(3,1))}\rangle$ for $\sigma \in \mathcal{P}^{(0)}$; they are 2, 1, 1 and 0, attained for $M \geq 1$ when $\sigma = \lambda = (8,3,1)$ and $M \geq 0$ when $\sigma = (9,2,1),(7,3,2)$ or $(8,2,2)$. We shall see below in Example 14.8 that the bound from Theorem 14.7 is $M \geq 2$ for $(8,3,1)$.

We mention that since the insertion map inserts a new column of height R, it is a new first column if and only if $\ell(\nu) \leq R$, and otherwise it must become a new column $\nu_{R+1}+1$. This is the main feature of the general positive sign case not seen in the previous example; in the negative sign case we instead insert a new row, and a similar remark applies.

14.5. **Proof of Theorem 1.2.** The 'moreover' part of Theorem 1.2 has already been proved in Proposition 14.1. The next theorem proves the main part of Theorem 1.2 with an explicit stability bound. Note that by Remark 3.2 there is no loss of generality in the 'largeness' hypotheses in the theorem. The greatest signed weight $\bigl(\omega_{\ell^-}(\mu/\mu_\star)^-, \omega_{\ell^-}(\mu/\mu_\star)^+\bigr)$ is defined in Definition 4.3 and strongly c^+-maximal signed weights are defined in Definition 4.10. The L bounds are defined in Definition 9.2. (Remark 9.1 explains the small difference in notation for the intervals in the first two bounds.) The statistics $B_R(\nu)$ and $E_{c^+,(\kappa^-,\kappa^+)}(\nu,\mu/\mu_\star : \lambda)$ are defined in Definition 13.18 in its 'both signs' version in §13.7. The 'shift' S below was first seen in Example 13.9 and then in the definition of ω in its continuation in Example 13.26.

Theorem 14.7. *Let (κ^-,κ^+) be a strongly c^+-maximal signed weight of a semistandard tableau family of shape $\mu/\mu_\star$ and size R. Fix $\ell^- = \ell(\kappa^-)$. Let ν be a partition and let λ be a $\bigl(\ell^-,\ell(\kappa^+)\bigr)$-large partition of $|\nu||\mu/\mu_\star|$. Set $\nu_R^+ = \nu_R$ and $\nu_R^- = \nu'_R$ and let $\pm$ denote the sign of κ. Set $\nu^{(M)} = \nu \sqcup (R^M)$ if (κ^-,κ^+) has sign -1 and $\nu^{(M)} = \nu + (M^R)$ if (κ^-,κ^+) has sign $+1$. Set $E = E_{c^+,(\kappa^-,\kappa^+)}(\nu,\mu/\mu_\star : \lambda)$ and*

$$S = \begin{cases} E - \nu_R^{\pm} + 1 & \text{if } E \geq \nu_R^{\pm} \\ 0 & \text{if } E < \nu_R^{\pm}. \end{cases}$$

Set

$$D = \Bigl|\Bigl|\bigl(B_R^{\pm}(\nu)+ER\bigr)\bigl(\omega_{\ell^-}(\mu/\mu_\star)^-,\omega_{\ell^-}(\mu/\mu_\star)^+\bigr)+(\nu_R^{\pm}-E)(\kappa^-,\kappa^+)-(\lambda^-,\lambda^+)\Bigr|\Bigr|.$$

Let κ be the unique partition with ℓ^--decomposition $\langle\kappa^-,\kappa^+\rangle$. Define $\omega = \varnothing$ if $\nu = \varnothing$ and otherwise

$$\omega = \begin{cases} \kappa \oplus (B_R^{\pm}(\nu) + ER)\bigl(\omega_{\ell^-}(\mu/\mu_\star)^-,\omega_{\ell^-}(\mu/\mu_\star)^+\bigr) \\ \kappa \oplus (B_R^{\pm}(\nu) + ER)\bigl(\omega_{\ell^-}(\mu/\mu_\star)^-,\omega_{\ell^-}(\mu/\mu_\star)^+\bigr) \oplus (\nu_R^{\pm} - E - 1)(\kappa^-,\kappa^+) \end{cases}$$

choosing the case according to whether $E \geq \nu^{\pm}_R$ *or* $E < \nu^{\pm}_R$*. Let* L *be the maximum of*

- $S + \mathrm{L}\bigl([\lambda^- + S\kappa^-, \omega^-]^{\ell^-}_{\blacktriangleleft}, \kappa^-\bigr)$,
- $S + \mathrm{L}\bigl([\lambda^+ + S\kappa^+, \omega^+ + (|\lambda^+| + S|\kappa^+| - |\omega^+|)]_{\unlhd}, \kappa^+\bigr)$,
- $S + \bigl(\omega^+_1 + \omega^+_2 - 2\lambda^+_1 - 2S\kappa^+_1 + 2|\lambda^+| + 2S|\kappa^+| - 2|\omega^+|\bigr)/(\kappa^+_1 - \kappa^+_2)$,
- $S + \bigl(\max(\ell(\lambda^+), \ell(\kappa^+)) + |\omega^-| - |\lambda^-| - S|\kappa^-| - \omega^-_{\ell^-}\bigr)/\kappa^-_{\ell(\kappa^-)}$
- $E + D - \nu^{\pm}_R + \nu^{\pm}_{R+1}$,

omitting the third if $\kappa^+_1 = \kappa^+_2$ *and the fourth if* $\kappa^- = \varnothing$*. Then*

$$\bigl\langle s_{\nu^{(M)}} \circ s_{\mu/\mu_\star}, s_{\lambda \oplus M(\kappa^-,\kappa^+)} \bigr\rangle$$

is constant for $M \geq L$*. Moreover if* $\lambda \oplus S(\kappa^-, \kappa^+) \not\trianglelefteq \omega$ *in the* ℓ^-*-twisted dominance order then the plethysm coefficient is* 0 *for all* $M \in \mathbb{N}_0$*.*

Proof. We apply the Signed Weight Lemma (Lemma 7.3) to the stable partition system

$$\mathcal{P}^{(M)} = \bigl[\lambda \oplus M(\kappa^-, \kappa^+), \omega \oplus (M - S)(\kappa^-, \kappa^+)\bigr]_{\unlhd}$$

defined in Lemma 14.2. The intervals are, as ever, for the ℓ^--twisted dominance order. By hypothesis λ is $\bigl(\ell^-, \ell(\kappa^+)\bigr)$-large. By Corollary 13.24, ω is an $\bigl(\ell(\kappa^-) + 1, \ell(\kappa^+)\bigr)$-large partition of $|\lambda| + SR|\mu/\mu_\star|$. Again by this corollary, if σ is a partition of $|\lambda| + MR|\mu/\mu_\star|$, such that $\sigma \trianglerighteq \lambda \oplus M(\kappa^-, \kappa^+)$ such that s_σ is a constituent of the plethysm $s_{\nu^{(M)}} \circ s_{\mu/\mu_\star}$ then $\sigma \trianglelefteq \omega \oplus (M - S)(\kappa^-, \kappa^+)$. But by Lemma 9.6 and the 'if' direction of Lemma 9.7, if $\lambda \oplus S(\kappa^-, \kappa^+) \not\trianglelefteq \omega$ then, writing

$$\lambda \oplus M(\kappa^-, \kappa^+) = \lambda \oplus S(\kappa^-, \kappa^+) \oplus (M - S)(\kappa^-, \kappa^+),$$

we have

$$\lambda \oplus M(\kappa^-, \kappa^+) \not\trianglelefteq \omega \oplus (M - S)(\kappa^-, \kappa^+).$$

Hence if $\lambda \oplus S(\kappa^-, \kappa^+) \not\trianglelefteq \omega$ then $\langle s_{\lambda \oplus (\kappa^-,\kappa^+)}, s_{\nu^{(M)}} \circ s_{\mu/\mu_\star} \rangle = 0$ for all $M \geq S$. This proves the final claim in the theorem. Moreover, we may now assume that, for all $M \geq S$, the twisted interval $\mathcal{P}^{(M)}$ is non-empty.

Condition (i) *in the Signed Weight Lemma.* By Lemma 14.2 the stable partition system $\mathcal{P}^{(M)}$ satisfies condition (i) of the Signed Weight Lemma (Lemma 7.3) for the plethysms $s_{\nu^{(M)}} \circ s_{\mu/\mu_\star}$.

Condition (ii) *in the Signed Weight Lemma.* Let $M \in \mathbb{N}_0$ and let $\pi \in \mathcal{P}^{(M)}$. Recall that $\mathrm{PYT}(\nu, \mu/\mu_\star)$ denotes the set of plethystic signed tableaux of shape ν having entries from the set $\mathrm{YT}(\mu/\mu_\star)$ of signed tableaux of shape $\mu/\mu_\star$. Let $\rho = (1^R)$ if (κ^-, κ^+) has sign $+1$ and let $\rho = (R)$ if (κ^-, κ^+) has sign -1. Recall from Definition 13.1 that $T_{(\kappa^-,\kappa^+)}$ is the unique plethystic semistandard signed tableau of size R, outer shape ρ, inner shape $\mu/\mu_\star$ and signed weight (κ^-, κ^+). By Remark 13.3, it remains semistandard in the (κ^-, κ^+)-adapted colexicographic order. Define

$$\mathcal{H} : \mathrm{PSSYT}_\kappa(\nu^{(M)}, \mu/\mu_\star) \to \mathrm{PYT}(\nu^{(M+1)}, \mu/\mu_\star)$$

on $T \in \mathrm{PSSYT}_\kappa(\nu^{(M)}, \mu/\mu_\star)$ by inserting $T_{(\kappa^-,\kappa^+)}$ as a new column immediately after column ν_R of T when (κ^-, κ^+) has sign $+1$ and as a new row immediately after row ν'_R of T when (κ^-, κ^+) has sign -1. Since $T_{(\kappa^-,\kappa^+)}$ has semistandard entries, all $\mu/\mu_\star$-tableau entries in the image are semistandard.

Suppose that $M \geq E - \nu^\pm_R + \nu^\pm_{R+1}$. (The reason for adding $\nu^\pm_{R+1}$ to the bound from Lemma 14.5 will be seen shortly.) By Corollary 13.24(i), T has at most E exceptional columns/rows. By Lemma 14.5, using that $\lambda \oplus S(\kappa^-, \kappa^+) \trianglelefteq \omega$, the bound D is well-defined (i.e. we take the root-length of a root-positive element) and T has at most D small columns/rows. By Definition 13.10, a column/row is either exceptional, typical or small. Since there $M + \nu^\pm_R$ columns/rows of T of height at least R, there are at least $M + \nu^\pm_R - D - E$ typical columns, in which the top/leftmost R entries form the plethystic semistandard signed tableau $T_{(\kappa^-,\kappa^+)}$. (This requires our use of the (κ^-, κ^+)-adapted colexicographic order to order the inner $\mu/\mu_\star$-tableau entries of T: see Figure 13.2.) Therefore if $M + \nu^\pm_R - D - E \geq \nu^\pm_{R+1}$, the map $\mathcal{H}$ inserts $T_{(\kappa^-,\kappa^+)}$ as a new column/row immediately to the right/below an identical column/row. (Note that this condition implies $M \geq -\nu^\pm_R + D + E$, and since D is an upper bound for the number of small columns, the hypothesis on M in Corollary 13.24 is satisfied.) Hence $\mathcal{H}$ is a well-defined bijection for $M \geq E + D - \nu^\pm_R + \nu^\pm_{R+1}$. □

Example 14.8. In the final part of the running example in Examples 13.13, 13.22 and 14.6 using the strongly 1-maximal signed weight $\big((2,2),(3,1)\big)$ we saw that $\langle s_{(2,1)+M(1,1)} \circ s_{(4)}, s_{\lambda \oplus M((2,2),(3,1))} \rangle$ is ultimately constant for each $\lambda \in \big\{(8,3,1),(9,2,1),(7,3,2),(8,2,2)\big\}$. To illustrate Theorem 14.7 we find an explicit bound for $(8,3,1)$. In this context we have $R = 2$, $B_2\big((2,1)\big) = 1$, $E = 1$ and $\big(\omega_2\big((4)\big)^+, \omega_2\big((4)\big)^-\big) = ((1,1),(2))$ and we saw that the bound D on the number of small columns is 2. The fifth bound in Theorem 14.7 is therefore $1 + 2 - 1 + 0 = 2$. We saw earlier that the other bounds are respectively $-2, -1, -3$ and -1, and so the overall bound is 2. We also saw that when $\lambda = (8,3,1)$ the constant value was attained for $M = 1$, so in this case the bound from Theorem 14.7 is not sharp. We remark that if instead $\lambda = (8,2,2) \leftrightarrow \big\langle (3,3),(6) \big\rangle$ then $E = 0$, $S = 0$ and $\omega = (5,3) \oplus \big((1,1),(2,2)\big) = (7,3,2) \leftrightarrow \big\langle (3,3),(5,1) \big\rangle$ and so we are in the final case of the theorem where $\lambda \not\trianglelefteq \omega$ in the 2^--twisted dominance order (in fact $\lambda \trianglerighteq \omega$), and so the plethysm coefficient is 0 for all $M \in \mathbb{N}_0$.

14.6. **The case $\nu = \varnothing$ of Theorem 14.7.** This is a notable and simple corollary of the theorem. The partitions $\kappa^{(M)}$ in the following corollary are well defined by Proposition 6.5.

Corollary 14.9. *Let (κ^-, κ^+) be a strongly c^+-maximal signed weight of shape $\mu/\mu_\star$ and size R. Fix $\ell^- = \ell(\kappa^-)$ and let $\kappa^{(M)}$ be the unique partition with ℓ^--decomposition $M\langle \kappa^-, \kappa^+ \rangle$. If (κ^-, κ^+) has sign -1 then*

$$\langle s_{(R^M)} \circ s_{\mu/\mu_\star}, s_{\kappa^{(M)}} \rangle = 1$$

for all $M \in \mathbb{N}_0$ and if (κ^-, κ^+) has sign $+1$ then the same holds replacing (R^M) with (M^R).

Proof. We apply Theorem 14.7 with (κ^-, κ^+) and $\nu = \lambda = \varnothing$. Thus $\nu^{(M)} = (R^M)$ if (κ^-, κ^+) has sign -1 and $\nu^{(M)} = (M^R)$ if (κ^-, κ^+) has sign $+1$.

Moreover $\lambda^{(M)} = \varnothing \oplus M(\kappa^-, \kappa^+) = \kappa^{(M)}$. Therefore the theorem states that the plethysm coefficients in the corollary are constant for all M at least the bound in the theorem. Since $\nu = \varnothing$ we have $E = -1$ by Definition 13.18, and so the case $E < \nu^{\pm}_R$ applies and we have $S = 0$ and $\omega = \varnothing$. It is now easily seen that the first three bounds specified in Theorem 14.7 are 0; the fourth is $\ell(\kappa^+)/\kappa^-_{\ell^-}$ and the fifth is 0 since D is the root-length of the zero weight. The fourth bound comes from Lemma 14.2. Inspection of the proof shows that in this case the stable partition system is $\mathcal{P}^{(M)} = \{\kappa^{(M)}\}$, which is stable for $M \geq 0$, and so this bound can be dropped. Therefore the constant value is attained for $M = 0$, and since $s_\varnothing \circ s_{\mu/\mu_\star} = 1$ (the unit element in the ring of symmetric functions) and $\kappa^{(0)} = \varnothing$, the constant value is 1, as claimed. □

For example, we saw in Example 4.18(i) that $\big((1^d), (m-d)\big)$ is a strongly 1-maximal signed weight of shape (m) and sign $(-1)^d$. The unique partition $\kappa^{(M)}$ with d-decomposition $M\big\langle(1^d), (m-d)\big\rangle$ is $(d^M) + (M(m-d))$ and so Corollary 14.9 implies that if d is odd then

$$\big\langle s_{(1^M)} \circ s_{(m)}, s_{(d^M)+M(m-d)}\big\rangle = 1$$

for all $M \geq 0$ and the same holds replacing (1^M) with (M) if d is even. The analogous stability result, which we believe is even less obvious, obtained from the case $R = 2$ of Example 4.18(ii) is that if d is odd then

$$\big\langle s_{(2^M)} \circ s_{(m)}, s_{(d^{2M})+M(2m-2d-1,1)}\big\rangle = 1$$

for all $M \geq 0$, and the same holds replacing (2^M) with (M, M) if d is even. For an example of the corollary in the case of skew partitions see Example 15.2.

15. Applications of Theorem 1.2

15.1. **Theorem 1.2 for singleton strongly maximal signed weights.** We saw in Lemma 4.17 that the signed weight $(\omega_{\ell^-}(\mu/\mu_\star)^-, \omega_{\ell^-}(\mu/\mu_\star)^+)$ of the greatest semistandard signed tableau $t_{\ell^-}(\mu/\mu_\star)$ in Definitions 4.2 and 4.3 is a strongly c^+-maximal signed weight where $c^+ = \ell(\omega_{\ell^-}(\mu/\mu_\star)^+)$ is the greatest positive entry appearing in $t_{\ell^-}(\mu/\mu_\star)$. In this subsection we give the special case of Theorem 14.7 for such strongly maximal weights, which we call *singleton*. The L bounds below are defined in Definition 9.2; see Remark 9.1 for the reason for the difference in the notation for intervals in the first two bounds below.

Corollary 15.1. *Let ν be a partition of n, let $\mu/\mu_\star$ be a skew partition, and let λ be a partition of $|\nu||\mu/\mu_\star|$. Fix $\ell^- \in \mathbb{N}_0$. Set $(\kappa^-, \kappa^+) = \big(\omega_{\ell^-}(\mu/\mu_\star)^-$, $\omega_{\ell^-}(\mu/\mu_\star)^+\big)$. Let $\nu^{(M)} = \nu \sqcup (1^M)$ if $|\kappa^-|$ is odd and $\nu^{(M)} = \nu + (M)$ if $|\kappa^-|$ is even. Then*

$$\big\langle s_{\nu^{(M)}} \circ s_{\mu/\mu_\star}, s_{\lambda\oplus M(\kappa^-,\kappa^+)}\big\rangle$$

is constant for all $M \geq L$ where L is the maximum of

- $\mathrm{L}\big([\lambda^-, n\kappa^-]^{(\ell^-)}_{\trianglelefteq}, \kappa^-\big)$,

- $\mathrm{L}\big([\lambda^+, n\kappa^+ + (|\lambda^+| - n|\kappa^+|)]_{\unlhd}, \kappa^+\big)$,
- $\big(n\kappa_1^+ + n\kappa_2^+ - 2\lambda_1^+ + 2|\lambda^+| - 2n|\kappa^+|\big)/(\kappa_1^+ - \kappa_2^+)$,
- $\big(\max(\ell(\lambda^+), \ell(\kappa^+)) + n|\kappa^-| - |\lambda^-| - n\kappa^-_{\ell(\kappa^-)}\big)/\kappa^-_{\ell(\kappa^-)}$,
- $\big|\big|n(\kappa^-, \kappa^+) - (\lambda^-, \lambda^+)\big|\big| - \nu_1^{\pm} + \nu_2^{\pm}$,

omitting the third if $\kappa_1^+ = \kappa_2^+$ *and the fourth if* $\ell^- = 0$ *and so* $\kappa^- = \varnothing$. *Moreover if* $\lambda \not\trianglelefteq \omega^{(n)}(\mu/\mu_\star)$ *then the plethysm coefficient is* 0 *for all* $M \in \mathbb{N}_0$.

Proof. By Lemma 4.17, $\big(\kappa^-, \kappa^+\big)$ is the strongly $\ell\big(\omega_{\ell^-}(\mu/\mu_\star)^+\big)$-maximal signed weight of the singleton tableau family $\{t_{\ell^-}(\mu/\mu_\star)\}$ of shape $\mu/\mu_\star$. (Note this holds even if $t_{\ell^-}(\mu/\mu_\star)$ has only negative entries, in which case the positive part of the signed weight is $\varnothing$.) Since $t_{\ell^-}(\mu/\mu_\star)$ has $|\kappa^-|$ negative entries, its sign is $(-1)^{|\kappa^-|}$. Since the tableau family has size $R = 1$, Definition 13.18 states that $E = 0$, unless $\nu = \varnothing$ when $E = -1$. Either way, the case $E < \nu_R^{\pm}$ of Theorem 14.7 applies. The partition κ in Theorem 14.7 is the unique partition with ℓ^--decomposition $\big\langle \omega_{\ell^-}(\mu/\mu_\star)^-, \omega_{\ell^-}(\mu/\mu_\star)^+\big\rangle$. (This ℓ^--decomposition is well-defined by Lemma 6.4.) We have

$$B_R^{\pm}(\nu) = n - a(\nu^{\pm}).$$

If $\nu = \varnothing$ then the upper bound partition ω in Theorem 14.7 is $\varnothing$, and otherwise it is

$$\kappa \oplus \big(n - a(\nu^{\pm})\big)(\kappa^-, \kappa^+) \oplus \big(a(\nu^{\pm}) - 0 - 1\big)(\kappa^-, \kappa^+) = \kappa \oplus (n-1)(\kappa^-, \kappa^+)$$

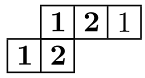
with ℓ^--decomposition $n\big\langle\kappa^-, \kappa^+\big\rangle = n\big\langle \omega_{\ell^-}(\mu/\mu_\star)^-, \omega_{\ell^-}(\mu/\mu_\star)^+\big\rangle$. Thus in all cases ω is the partition $\omega_{\ell^-}^{(n)}(\mu/\mu_\star)$ defined in Definition 10.6. Similarly, we have

$$\begin{aligned} D &= \big|\big|\big(n - a(\nu^{\pm})\big)(\kappa^-, \kappa^+) + \big(a(\nu^{\pm}) - 0\big)(\kappa^-, \kappa^+) - (\lambda^-, \lambda^+)\big|\big| \\ &= \big|\big|n(\kappa^-, \kappa^+) - (\lambda^-, \lambda^+)\big|\big|. \end{aligned}$$

Since we are in the case $E < \nu_R^{\pm}$, we have $S = 0$. It is now very easily seen that the five bounds in Theorem 14.7 simplify as claimed. The corollary, including the final claim that if $\lambda \not\trianglelefteq \omega^{(n)}(\mu/\mu_\star)$ then the plethysm coefficient is 0 for all $M \in \mathbb{N}_0$, now follows from this theorem. $\square$

Note that the condition for the plethysm coefficient to vanish is the same as the one in Remark 13.25. The following example illustrates the skew partition case of Corollary 15.1.

Example 15.2. Take $\ell^- = 2$ and $\mu/\mu_\star = (4,2)/(1)$. The tableau $t_2\big((4,2)/(1)\big)$ is as shown in the margin and correspondingly



$$\big(\omega_2\big((4,2)/(1)\big)^-, \omega_2\big((4,2)/(1)\big)^+\big) = \big((2,2),(1)\big).$$

By Lemma 4.17, this is a strongly 1-maximal signed weight of shape $(4,2)/(1)$, size 1 and sign $+1$. Let ν be a partition of n. By Corollary 15.1 the plethysm coefficients

$$\big\langle s_{\nu+(M)} \circ s_{(4,2)/(1)}, s_{\lambda \oplus M((2,2),(1))}\big\rangle$$

are constant for M at least the bound in this corollary, and the constant value is 0 unless $\lambda \trianglelefteq \omega^{(n)}\big((4,2)/(1)\big) = (n+2, 2^{2n-1}) \leftrightarrow \big\langle(2n,2n),(n)\big\rangle$. Note that $s_{(4,2)/(1)} = s_{(4,1)} + s_{(3,2)}$ is not a single Schur function, so, as in Example 4.21, this result needs the generality of skew partitions. Taking $\nu = (1,1)$, the table below shows values for the inner product for varying partitions λ of 12 (shown decreasing in the 2-twisted dominance order), together with the bound from the corollary.

λ	0	1	2	3	4	5	bound
$(4,2,1^4)$	0	0	0	0	0	0	0
$(3,3,2,2)$	1	1	1	1	1	1	1
$(4,4,2)$	1	12	19	22	22	22	5
$(8,1,1)$	1	9	17	17	17	17	5
$(8,2)$	0	7	15	16	16	16	6

In the first case $(4,2,1,1,1) \leftrightarrow \big\langle (5,2),(2)\big\rangle$ is incomparable with $(4,2,2,2) \leftrightarrow \big\langle (4,4),(2)\big\rangle$ in the 2-twisted dominance order, and so the constant multiplicity is 0. In each remaining case, the fifth bound, $\big|\big|2((2,2),(1)) - (\lambda^-,\lambda^+)\big|\big|$ is the largest. For instance $s_{(1+M,1)} \circ \big(s_{(4,1)} + s_{(3,2)}\big), s_{(4+M,4,2^{2M+1})}\big\rangle = 22$ for all $M \geq 3$. Similar calculations by computer algebra using the bound from Corollary 15.1 show that

$$\langle s_{(1+M,1)} \circ s_{(4,1)}, s_{(4+M,4,2^{2M+1})}\rangle = 0 \quad \text{for all } M \geq 0$$
$$\langle s_{(1+M,1)} \circ s_{(3,2)}, s_{(4+M,4,2^{2M+1})}\rangle = 7 \quad \text{for all } M \geq 3;$$

this illustrates the failure of the plethysm product to be distributive over addition in its second component.

Remark 15.3. If $\mu_\star = \varnothing$ then, by (6.1), we have $\big(\omega_{\ell^-}(\mu)^-, \omega_{\ell^-}(\mu)^+\big) = (\mu^-,\mu^+)$ and we may replace κ^- with μ^- and κ^+ with μ^+ in all the expressions in Corollary 15.1.

We use this remark in §15.2 and §15.3 below. Combining Corollary 14.9 with Remark 15.3, it follows that, for any fixed ℓ^--decomposition (μ^-,μ^+), if $|\mu^-|$ is odd then

$$\langle s_{(1^M)} \circ s_\mu, s_{M(\mu^-,\mu^+)}\rangle = 1$$

for all $M \in \mathbb{N}_0$; if $|\mu^-|$ is even then the same holds replacing (1^M) with M.

For ease of reference in the following corollary for the case $\kappa^- = \varnothing$, we recall that the greatest tableau $t_{\ell^-}(\mu/\mu_\star)$ is defined in Definition 4.2, the L bound in Definition 9.2 and the root-length $||\alpha||$ in (14.1).

Corollary 15.4. *Let ν be a partition of n, let $\mu/\mu_\star$ be a skew partition, and let λ be a partition of $|\nu||\mu/\mu_\star|$. Let κ be the positive part of the signed weight of the greatest tableau $t_0(\mu/\mu_\star)$. Then*

$$\langle s_{\nu+(M)} \circ s_{\mu/\mu_\star}, s_{\lambda+M\kappa}\rangle$$

is constant for all $M \in \mathbb{N}_0$ such that $M \geq L$, where L is the maximum of $\mathrm{L}\big([\lambda, n\kappa]_\unlhd, \kappa\big)$ *and* $||n\kappa - \lambda|| - \nu_1 + \nu_2$.

Proof. Apply Corollary 15.1 taking $\ell^- = 0$. Thus $\big(\omega_0(\mu/\mu_\star)^-, \omega_0(\mu/\mu_\star)^+\big) = (\varnothing, \kappa)$ and the first bound is ignored. The second is $\mathrm{L}\big([\lambda, n\kappa]_\unlhd, \kappa\big)$, the third is the case $k = 1$ in Definition 9.2, and so is implied by the second. The fourth bound is again one that should be ignored, and the fifth becomes $||n\kappa - \lambda|| - \nu_1 + \nu_2$. $\square$

15.2. **Explicit bounds for hook stability.** By Lemma 4.17, if $1 \leq d \leq m$ then $\big((d), (m-d)\big)$ is a strongly 1-maximal signed weight of shape $(m - d + 1, 1^{d-1})$, corresponding to the singleton tableau family $\big\{t_1\big((m-d+1, 1^{d-1})\big)\big\}$. For instance $t_1\big((3,1,1,1)\big)$ is as shown in the margin.

1	1	1
1		
1		
1		

Proposition 15.5. *Let ν be a partition of n and let $1 \leq d \leq m$. Let $\nu^{(M)} = \nu + (M)$ if d is even and let $\nu^{(M)} = \nu \sqcup (1^M)$ if d is odd. If λ is a partition of mn with 1-decomposition $\big\langle (\ell(\lambda)), \lambda^+ \big\rangle$ then*

$$\langle s_{\nu^{(M)}} \circ s_{(m-d+1, 1^{d-1})}, s_{\lambda + M(m-d) \sqcup (1^{dM})} \rangle$$

is constant for all $M \geq L$ where L is the maximum of
- $\big(|\lambda^+| - 2\lambda_1^+\big)/(m-d)$,
- $\big(2|\lambda^+| - 2\lambda_1^+ - n(m-d)\big)/(m-d)$,
- $\big|\big|\big((nd), (n(m-d))\big) - \big((\ell(\lambda), \lambda^+\big)\big|\big| - \nu_1 + \nu_2$.

Moreover if $\lambda \not\trianglelefteq (1^{nd}) + (n(m-d))$ in the 1-twisted dominance order then the plethysm coefficient is 0 for all $M \in \mathbb{N}_0$.

Proof. We take the singleton strongly maximal weight $(\kappa^-, \kappa^+) = \big((d), (m-d)\big)$ in Corollary 15.1, together with $\mu/\mu_\star = (m-d+1, 1^{d-1})$ and $\ell^- = 1$. By Definition 6.1, the 1-decomposition of the partition λ is $\big\langle (\ell(\lambda)), (\lambda_1 - 1, \ldots, \lambda_b - 1)\big\rangle$, where b is maximal such that $\lambda_b \geq 2$. Hence, by Definition 9.2, the first bound in Theorem 14.7 is $(nd - \ell(\lambda) - nd)/d$, which is non-positive. (Note the case where $\ell(\lambda^-) \leq \ell^-$ applies since λ^- has at most one part.) Similarly since $\ell(\kappa^+) = 1$ the second bound is $(n(m-d) + |\lambda^+| - n(m-d) - 2\lambda_1^+)/(m-d)$ which simplifies to the first bound above. (Again this ignores a potentially stronger bound if $\ell(\lambda^+) \leq 1$.) The third and fifth bounds in the corollary simplify to the final two bounds above. The fourth bound is $\big(\max(\ell(\lambda^+), 1) + nd - \ell(\lambda) - nd\big)/d = \big(\max(\ell(\lambda^+), 1) - \ell(\lambda)\big)/d$ which, since λ^+ has length $b \leq \ell(\lambda)$, is non-positive. Since the partition $\omega^{(n)}\big((m-d+1, 1^{d-1})$ with 1-decomposition $n\big\langle \kappa^-, \kappa^+ \big\rangle = n\big\langle (d), (m-d) \big\rangle$ is $(1^{nd}) + \big(n(m-d)\big)$, the result now follows from Corollary 15.1. $\square$

For example, taking $\nu = (2,1)$, $m = 4$ and $d = 2$ so $\mu = (3,1)$ we find that

$$\langle s_{(2+M,1)} \circ s_{(3,1)}, s_{\lambda + (2M) \sqcup (1^{2M})} \rangle$$

is ultimately constant, and zero unless $\lambda \trianglelefteq (7, 1^5)$ in the 1-twisted dominance order. (This is equivalent to the condition $\ell(\lambda) \leq 6$.) The case $\lambda = (4,3,3,2)$, for which the sequence of plethysm coefficients is 2, 16, 31, 33, 33, ... is illustrative. Here $\lambda^+ = (3,2,2,1)$ and so the bounds from Proposition 15.5 are $(8-6)/(4-2) = 1$, $\big(2 \times 8 - 2 \times 3 - 3(4-2)\big)/(4-2) = 4/2 = 2$ and finally $11 - 2 + 1 = 10$, since

$$\big|\big|\big((6),(6)\big) - \big((4),(3,2,2,1)\big)\big|\big| = \big|\big|\big((2),(3,-2-2,-1)\big)\big|\big| = 2+5+3+1 = 11.$$

Therefore $\langle s_{(2+M,1)} \circ s_{(3,1)}, s_{(4+2M,3,3,2,1^{2M})} \rangle = 33$ for all $M \geq 10$.

15.3. **Explicit bounds for Law–Okitani stability.** Using Corollary 15.1 and Remark 15.3, and very similar arguments to the proof of Proposition 15.5, we can give the first explicit bounds for the stability result discussed in §1.7 due to Law and Okitani [12], and a sufficient condition for the stable plethysm coefficient to be zero. We exclude the case $d = 0$ because it is a special case of Corollary 15.4, and the case $d = m$ because it reduces to the case $d = 0$ by applying the ω involution.

Proposition 15.6. *Let ν be a partition of n and let $1 \le d < m$. Let $\nu^{(M)} = \nu + (M)$ if d is even and let $\nu^{(M)} = \nu \sqcup (1^M)$ if d is odd. If λ is a partition of mn with d-decomposition $\langle \lambda^-, \lambda^+ \rangle$ then*

$$\langle s_{\nu^{(M)}} \circ s_{(m)}, s_{\lambda + M(m-d) \sqcup (d^M)} \rangle$$

is constant for all $M \ge L$ where L is the maximum of
- $n(d-1) - |\lambda^-|$,
- $\big(|\lambda^+| - 2\lambda^+_1\big)/(m-d)$,
- $\big(2|\lambda^+| - 2\lambda^+_1 - n(m-d)\big)/(m-d)$,
- $\max(1, \ell(\lambda^+)) + n(d-1) - |\lambda^-|$,
- $\big|\big|\big((n^d), (n(m-d))\big) - (\lambda^-, \lambda^+)\big|\big| - \nu_1 + \nu_2$.

Moreover if $\lambda \not\trianglelefteq (d^n) + (n(m-d))$ in the d-twisted dominance order then the plethysm coefficient is 0 for all $M \in \mathbb{N}_0$.

Proof. By Example 4.18(i), $\big((1^d), (m-d)\big)$ is a strongly 1-maximal signed weight of shape (m), size 1 and sign $(-1)^d$. We take this as (κ^-, κ^+) in Corollary 15.1, together with $\mu/\mu_\star = (m)$ and $\ell^- = d$. Observe that $\kappa^-_k - \kappa^-_{k+1}$ is non-zero only when $k = d$. Hence, by Definition 9.2, the first bound in Theorem 14.7 is $(nd - |\lambda^-| - n)/1$, which simplify to the first bound above. (Note that the case where $\ell(\lambda^-) \le \ell^-$ applies.) Similarly since $\ell(\kappa^+) = 1$ the second bound is $(n(m-d) + |\lambda^+| - n(m-d) - 2\lambda^+_1)/(m-d)$ which again simplifies as shown. (Again, as in the earlier proof of Proposition 15.5, this ignores a potentially stronger bound if $\ell(\lambda^+) \le 1$.) The third, fourth and fifth bounds are routine specializations of the bounds in the corollary. Since the partition $\omega^{(n)}\big((m)\big)$ with d-decomposition $n\big\langle \kappa^-, \kappa^+ \big\rangle = n\big\langle (1^d), (m-d) \big\rangle$ is $(d^n) + n(m-d)$, the result now follows from Corollary 15.1. □

Example 15.7. We take $m = 4$, $d = 3$ and $\nu = (2,1)$. The table below shows values of $\langle s_{(2,1^{M+1})} \circ s_{(4)}, s_{\lambda + (M) \sqcup (3^M)} \rangle$ for small values of M for varying partitions λ of 12, shown decreasing in the 2-twisted dominance order, together with the bound from the corollary.

λ	0	1	2	3	4	5	6	bound
$(5,3,3,1)$	0	0	0	0	0	0	0	0
$(6,3,3)$	0	0	0	0	0	0	0	0
$(7,3,2)$	1	1	1	1	1	1	1	0
$(7,4,1)$	1	2	2	2	2	2	2	3
$(7,5)$	1	4	5	6	6	6	6	7
$(6,6)$	0	2	5	6	7	7	7	8

In the first case $(5,3,3,1) \leftrightarrow \big\langle (4,3,3),(2)\big\rangle$ is greater than the upper bound $(6,3,3) \leftrightarrow \big\langle (3,3,3),(3)\big\rangle$ in the 3-twisted dominance order, and so the constant multiplicity is 0. For $(7,5)$ the constant multiplicity is indeed 6, as can be checked by using computer algebra to compute the next three values, or using the generalized Cayley–Sylvester formula in (5.3). Thus $\langle s_{(2,1^{M+1})} \circ s_{(4)}, s_{(7+M,5,3^M)}\rangle = 6$ for $M \geq 3$. It is worth noting that we can obtain further information about the *same* plethysm $s_{(2,1^{M+1})} \circ s_{(4)}$ by instead taking $d = 1$ in Proposition 15.6, now using the strongly 1-maximal signed weight $\big((1),(3)\big)$. For instance the proposition implies that $\langle s_{(2,1^{M+1})} \circ s_{(4)}, s_{(7+3M,5,1^M)}\rangle = 6$ for $M \geq 4$; in fact the constant value is attained for $M \geq 3$. Similarly $\langle s_{(2,1^{M+1})} \circ s_{(4)}, s_{(6+3M,6,1^M)}\rangle = 8$ for $M \geq 5$; now the constant value is attained for $M \geq 4$. These results and bounds may be verified using the Magma code mentioned in the introduction.

15.4. **The positive non-skew case of Theorem 1.2.** In this section we specialize Theorem 14.7 in two ways at once by assuming that $\kappa^- = \varnothing$ and $\mu_\star = \varnothing$. (Taken separately, these specializations do not lead to simplifications significant enough to be worth recording.) We begin by giving the special case of Definition 4.8 and Definition 4.10 since the latter simplifies greatly in this case. Recall that $\max \mathcal{T}$ denotes the maximum integer entry of a family of tableaux with integer entries.

Definition 15.8. Let μ be a non-empty partition and let $R \in \mathbb{N}$. A family $\mathcal{M}$ of R distinct semistandard μ-tableaux with entries from $\mathbb{N}$ of weight κ is *maximal* if κ is maximal in the dominance order amongst all such families. It is *strongly* c-maximal if whenever ϕ is the weight of a maximal family $\mathcal{T}$ such that $\max \mathcal{T} \leq \max \mathcal{M}$ then either $\mathcal{T} = \mathcal{M}$ or $\sum_{i=1}^{c} \phi_i < \sum_{i=1}^{c} \kappa_i$.

It is clear that κ is a strongly c-maximal weight if and only if $(\varnothing, \kappa)$ is a strongly c-maximal signed weight in the sense of Definition 4.10. We give examples in §15.5. In the following corollary, the L bound is defined in Definition 9.2. The unsigned analogue of the LZ bound in Definition 11.1 is defined by specializing this definition: given partitions λ and ω of the same size, and partitions η and κ, we define $\mathrm{LZ}\big([\lambda,\omega]_{\trianglelefteq}, \kappa, \eta\big)$ to to be the minimum of the quantities

- $\dfrac{\sum_{i=1}^{k} \omega_i - \sum_{i=1}^{k} \lambda_i}{\sum_{i=1}^{k} \eta_i - \sum_{i=1}^{k} \kappa_i}$.

taking those k for which the denominator is strictly positive. Below we have $\kappa \lhd \eta$ so the minimum is well-defined.

Corollary 15.9. *Let ν be a partition of n, let μ be a partition of m and let λ be a partition of mn. Let κ be a strongly c-maximal weight of shape μ and size R. Set $E = 0$ if $R = 1$ and otherwise set E to the maximum of*

$$B_R(\nu) \sum_{i=1}^{c} \mu_i + \nu_R \sum_{i=1}^{c} \kappa_i - \sum_{i=1}^{c} \lambda_i + \sum_{i=\ell(\kappa)+1}^{\ell(\lambda)} \lambda_i$$

and zero. Set $D = \big|\big|(B_R(\nu)+ER)\mu+(\nu_R-E)\kappa-\lambda\big|\big|$. *Let* L *be the maximum of*

$$\begin{cases} E-\nu_R+\mathrm{L}\big([\lambda+(E-\nu_R)\kappa,(B_R(\nu)+ER)\mu]_{\trianglelefteq},\kappa\big) & \text{if } E \geq \nu_R \\ \mathrm{L}\big([\lambda,(B_R(\nu)+ER)\mu+(\nu_R-E)\kappa]_{\trianglelefteq},\kappa\big) & \text{if } E < \nu_R \end{cases}$$

and $D+E-\nu_R+\nu_{R+1}$. *Then* $\langle s_{\nu+(M^R)} \circ s_\mu, s_{\lambda+M\kappa}\rangle$ *is constant for* $M \geq L$. *Moreover if* η *is a partition of* MR *such that* $\kappa \lhd \eta$ *then* $\langle s_{\nu+(M^R)} \circ s_\mu, s_{\lambda+M\eta}\rangle = 0$ *for all*

$$M > \begin{cases} E-\nu_R+\mathrm{LZ}\big([\lambda+(E-\nu_R)\eta,(B_R(\nu)+ER)\mu]_{\trianglelefteq},\kappa,\eta\big) & \text{if } E \geq \nu_R \\ \mathrm{LZ}\big([\lambda,(B_R(\nu)+ER)\mu+(\nu_R-E)\kappa]_{\trianglelefteq},\kappa,\eta\big) & \text{if } E < \nu_R. \end{cases}$$

Proof. We apply Theorem 14.7 with $\kappa^- = \varnothing$ and $\kappa^+ = \kappa$. Thus $\ell^- = 0$ and $\big(\omega_{\ell^-}(\mu/\mu_\star)^-, \omega_{\ell^-}(\mu/\mu_\star)^+\big) = (\varnothing,\mu)$ by (6.1) and the following remark. The sign of $(\varnothing,\kappa^+)$ is $+1$ so the definitions given in the main part of §13 apply and $\nu^+ = \nu$ in the statement of the theorem. It is easily seen from Definition 13.18 that $E_{c,(\varnothing,\kappa)}(\nu,\mu:\lambda)$ is E as stated in the corollary and similarly from Theorem 14.7 using that $\omega_0(\mu)^- = \varnothing$ and

$$\omega = \begin{cases} (B_R(\nu)+ER)\mu+\kappa & \text{if } E \geq \nu_R \\ (B_R(\nu)+ER)\mu+(\nu_R-E)\kappa & \text{if } E < \nu_R \end{cases}$$

that D is as stated. Since $|\lambda| + SR|\mu| = |\lambda^+| + S|\kappa^+| = |\omega| = |\omega^+|$, where the second equality uses Corollary 13.24(iii) and that $\mathrm{L}\big([\varnothing,\varnothing]_{\trianglelefteq},\varnothing\big) = 0$, the bounds defining L in this theorem simplify to S, $S+\mathrm{L}\big([\lambda+S\kappa,\omega]_{\trianglelefteq},\kappa\big)$, $S+(\omega_1+\omega_2-2\lambda_1-2S\kappa_1)/(\kappa_1-\kappa_2)$, 0, and $D+E-\nu_R+\nu_{R+1}$, respectively. If $\kappa_1 = \kappa_2$ then the third quantity should be disregarded; otherwise it is one of the lower bounds appearing in Definition 9.2 defining $L\big([\lambda+S\kappa,\omega],\kappa\big)$. Finally we slightly simplify $\mathrm{L}\big([\lambda+S\kappa,\omega],\kappa\big)$ using the lemma that $\mathrm{L}\big([\alpha+\kappa,\beta+\kappa]_{\trianglelefteq},\kappa\big) = L\big([\alpha,\beta]_{\trianglelefteq},\kappa\big)-1$ to show that L is as claimed in the statement of the corollary. The proof of the 'moreover' part is very similar, using the bounds from Proposition 14.1 with the same specializations and the same simplification of the LZ bound. □

We note that, again using Theorem 14.7, the plethysm coefficient is zero if $\lambda + S(\kappa^-,\kappa^+) \ntrianglelefteq \omega$, where $S = E-\nu_R+1$ if $E \geq \nu_R$ and $S = 0$ otherwise, and ω is as defined in the proof above.

15.5. **Examples of Corollary 15.9.** In Examples 4.9 and 4.12, we saw that $(4,1,1)$ and $(3,3)$ are the two strongly maximal weights of shape (2) and size 3 and that the corresponding semistandard tableau families are

$$\big\{\boxed{1\,|\,1},\ \boxed{1\,|\,2},\ \boxed{1\,|\,3}\big\},\ \big\{\boxed{1\,|\,1},\ \boxed{1\,|\,2},\ \boxed{2\,|\,2}\big\}$$

respectively.

Example 15.10. In the running example using the strongly 1-maximal weight $(4,1,1)$ of shape (2), size 3 and sign $+1$ completed in Example 13.26 we saw that if $\nu = (2,1)+C(1,1,1)$ and $\lambda = (4,2)+C(4,1,1)$ then $E = 2$;

this can now be computed more simply using the formula in Corollary 15.9. The quantity D in this corollary is

$$\big|\big|(3+2.3)(2)+(C-2)(4,1,1)-\big((4,2)+C(4,1,1)\big)\big|\big| = \big|\big|(6,-4,-2)\big|\big| = 8,$$

independent of the value of C. Exploiting similar cancellation we have, for $C=0$ or $C=1$,

$$\begin{aligned}\mathrm{L}\big(\big[(4,2)+C(4,1,1)+(2-C)(4,1,1),(3+2.3)(2)\big]_{\unlhd},(4,1,1)\big)\\ = \mathrm{L}\big(\big[(12,4,2),(18)\big]_{\unlhd},(4,1,1)\big) = 0.\end{aligned}$$

Now taking $C=0$, Corollary 15.9 implies that

$$\langle s_{(2,1)+M(1,1,1)} \circ s_{(2)}, s_{(4,2)+M(4,1,1)} \rangle$$

is constant for $M \geq 10$; the two bounds are respectively 2 and 10. In fact it follows from the enumeration of plethystic semistandard tableaux in the running example that the plethysm coefficient is constant for $M \geq 2$; the constant value is 2.

It is routine to give a similar example using the strongly 2-maximal weight $(3,3)$. This gives a special case of Proposition 15.11 below. The proof is a good example of how stronger bounds than the generic bounds in our main theorems can be obtained by ad-hoc reasoning. Note that the assumption $\ell(\mu) \leq \ell$ is without loss of generality, as this condition is necessary for there to be a semistandard μ-tableau with entries from $\{1,\ldots,\ell\}$.

Proposition 15.11. *Fix $\ell \in \mathbb{N}$ and let μ be a partition with $\ell(\mu) \leq \ell$. Let R be the number of semistandard tableaux of shape μ with entries from $\{1,\ldots,\ell\}$. Set $q = R|\mu|/\ell$. Then for any partitions ν and λ with $\ell(\nu) < R$ and $\ell(\lambda) \leq \ell$,*

$$\langle s_{\nu+M(1^R)} \circ s_\mu, s_{\lambda+M(q^\ell)} \rangle$$

is constant for $M \geq 0$.

Proof. Let $\mathcal{T}$ be the strongly maximal tableau family consisting of all semistandard μ-tableaux with entries from $\{1,\ldots,\ell\}$. By Lemma 4.20 this corresponds to the strongly ℓ-maximal weight (q^ℓ) of shape μ and size R. Since $\ell(\nu) < R$ we have $B_R(\nu) = |\nu|$ and since $\ell(\lambda) \leq \ell$, the exceptional column bound E in the corollary is $|\nu||\mu| - |\lambda| = 0$. The bounds in Corollary 15.9 are therefore

$$\mathrm{L}\big([\lambda, |\nu|\mu]_{\unlhd}, (q^\ell)\big) = (|\nu||\mu| - |\lambda| - |\nu|\mu_\ell)/q - |\nu|\mu_\ell/q < 0$$

and $\big|\big|\,|\nu|\mu - \lambda\,\big|\big|$. Inspection of the proof of Theorem 14.7 shows that the second bound is needed to ensure that the insertion map $\mathcal{H}$, defined by inserting a new column of height R into a plethystic semistandard tableau with entries from the tableau family $\mathcal{T}$, is surjective. But since $\mathcal{T}$ contains *all* tableaux of shape μ, *any* column of height R in a plethystic semistandard signed tableau having μ-tableau entries from $\{1,\ldots,\ell\}$ is of this special form. Therefore the plethysm coefficient is immediately constant. ☐

Example 15.12. Take $\mu = (2,1)$ and $\ell = 3$ and the strongly maximal semistandard tableau family of all $(2,1)$-tableaux with entries from $\{1,2,3\}$ relevant to the famous eightfold way adjoint representation of $\mathrm{SU}_3(\mathbb{C})$ (see [10, page 179]), containing the 8 tableaux shown below

$$\begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array},\ \begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array},\ \begin{array}{|c|c|}\hline 1&1\\\hline 3\\\cline{1-1}\end{array},\ \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array},\ \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array},\ \begin{array}{|c|c|}\hline 1&3\\\hline 3\\\cline{1-1}\end{array},\ \begin{array}{|c|c|}\hline 2&2\\\hline 3\\\cline{1-1}\end{array},\ \begin{array}{|c|c|}\hline 2&3\\\hline 3\\\cline{1-1}\end{array}.$$

The corresponding 3-strongly maximal weight is $(8,8,8)$. Applying Proposition 15.11 we find that

$$\langle s_{\nu+M(1^8)} \circ s_{(2,1)}, s_{\lambda+M(8,8,8)} \rangle$$

is constant for $M \geq 0$, whenever ν and λ are partitions with $\ell(\nu) < 8$ and $\ell(\lambda) \leq 3$. For example, taking $\nu = (2)$ and $\lambda = (3,2,1)$, the stable value of the plethysm coefficient is 1.

Many further examples of non-obvious stability results can be given using the strongly maximal weights found in Example 4.22 and Table 4.23 in §4.5.

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