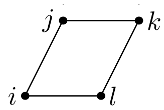
Representations of Symmetric Groups 2

Question 1 should probably be attempted after reading Example 5.2 in James' lecture notes. In Questions 2 and 3 it will be helpful to use Question 2 from Sheet 1. Mackey's Formula for the restriction of an induced character may also be helpful.

1. Let $n \in \mathbf{N}$. Let $M \cong M_{\mathbf{F}_2}^{(n-2,2)}$ be the permutation module of S_n acting on 2-subsets of $\{1, 2, \ldots, n\}$, defined over $\mathbf{F}_2$.

(a) Show that the elements of M are in bijection with graphs on $\{1, 2, \ldots, n\}$.

(b) Show that, under this bijection, $S_{\mathbf{F}_2}^{(n-2,2)}$ is spanned linearly by the graphs shown below, for $1 \leq i \leq j \leq k \leq l \leq n$.



$$\longleftrightarrow \{i, j\} + \{j, k\} + \{k, l\} + \{l, i\}$$

(c) Show that $S_{\mathbf{F}_2}^{(n-2,1,1)}$ is isomorphic to the submodule spanned by all triangles $\{i, j\} + \{j, k\} + \{k, i\}$ for $1 \leq i < j < k \leq n$

(d) Hence prove a generalization of Example 4.5 in the lecture notes: if $n \equiv 3 \mod 4$ and $n \geq 7$ then $S^{(n-2,1,1)} \cong S^{(n-2,2)} + S^{(n)}$. [*Hint: the complete graph on n vertices generates a copy of the trivial module inside M.*]

2. Let π^λ denote the character of the Young permutation module M^λ, defined over $\mathbf{C}$. Show that if $0 \leq r \leq n/2$ then

$$\left\langle \pi^{(n-r,r)}, \pi^{(n-r,r)} \right\rangle = r + 1.$$

3. Let $G \leq S_n$. Let π be the permutation character of S_n acting on the cosets of G. Suppose that G has r_k orbits in its action on the set of k-subsets of $\{1, 2, \ldots, n\}$, where $1 \leq k \leq n$.

(a) Show that $\left\langle \pi, \chi^{(n-1,1)} \right\rangle = r_1 - 1$.

(b) Show that for each k such that $1 \leq k \leq n/2$ there is a unique irreducible character that appears in $\pi^{(n-k,k)}$ but not in $\pi^{(n-k+1,k-1)}$. Show moreover that if this character is denoted $\chi^{(n-r,r)}$, then

$$\pi^{(n-r,r)} = \chi^{(n)} + \chi^{(n-1,1)} + \cdots + \chi^{(n-r,r)}.$$

and

$$\left\langle \pi, \chi^{(n-k,k)} \right\rangle = r_k - r_{k-1}.$$

(c) Use Theorem 4.3 to show that $\chi^{(n-r,r)}$ is the character of the Specht module $S_{\mathbf{C}}^{(n-r,r)}$ and deduce the decomposition of $M_{\mathbf{C}}^{(n-r,r)}$ stated after Example 4.5 in the lecture notes.

4. Let T_n be the character table of S_n, with any order of the rows and columns. Show that $|\det T_n|$ is the product of all parts of all partitions of n. (For example, if $n = 3$ then the partitions are (3), $(2, 1)$ and $(1, 1, 1)$ and $|\det T_3| = 3 \times 2 \times 1^4 = 6$.)

5. Let $n \in \mathbf{N}$ and let F be a field. Determine the matrix of the restriction of $\langle\ ,\ \rangle$ to $S^{(n-1,1)}$ and find $S^{(n-1,1)} \cap (S^{(n-1,1)})^{\perp}$. (The answer will depend on the characteristic of F.)

6. Let F be a field and let λ be a partition of n. Let t be a fixed λ-tableau and let $a_t = \sum_{g \in R(t)} g$ where $R(t)$ is the row-stabiliser group of t.

(a) Show that $M_F^\lambda \cong a_t F S_n$.

(b) Show that $S_F^\lambda \cong a_t b_t F S_n$.

(c) Show that $(a_t b_t)^2 = \gamma a_t b_t$ for some $\gamma \in F$.

7. Let G be a finite group and let F be a field. Given an FG-module V, we define the *dual module* $V^\star$ to have underlying vector space $\mathrm{Hom}_F(V, F)$ and G-action given by

$$v(\varphi g) = (vg^{-1})\varphi \quad \text{for } v \in V,\ \varphi \in V^\star \text{ and } g \in G.$$

(a) Check that $V^\star$ is a well-defined FG-module.

(b) We say that V is *self-dual* if $V \cong V^\star$ as FG-modules. Show that V is self-dual if and only if there is a non-degenerate G-invariant bilinear form on $V \times V$ taking values in F.

(c) Show that if $V = \mathbf{C}$ then V is self-dual if and only if its character takes only real values.

[A bilinear form $\beta : V \times V \to F$ is G-invariant if $\beta(vg, vg) = \beta(v, v)$ for all $v \in V$, $g \in G$.]

8. Let t be a tableau of shape λ where λ is a partition of n. Let $g \in S_n$. Show that $g \notin R(t)C(t)$ if and only if there exist transpositions $h \in C(t)$ and $k \in R(t)$ such that $kgh = g$.