Representations of Symmetric Groups 3

Throughout let F be a field of prime characteristic p. Unless otherwise stated, all modules are defined over F. If λ is a p-regular partition of n, we denote by D^λ the irreducible FS_n-module $S^\lambda_F/S^\lambda_F \cap (S^\lambda_F)^\perp$.

1. Let λ be a partition of n and let s and t be λ-tableaux. Show that

$$e(s)b_t = \langle e(s), e(t)\rangle e(t).$$

2. Let λ be a partition of n.

(a) Show that if λ has at most $p-1$ parts then $e(t)b_t = \alpha e(t)$ for some $\alpha \neq 0$. Hence show that $\mathrm{End}_{FS_n}(S^\lambda) \cong F$.

(b) Generalize (a) by using Question 1 and Lemma 5.5 to show that $\mathrm{End}_{FS_n}(S^\lambda) \cong F$ whenever λ is p-regular.

(In fact this result holds for all partitions when p is odd: see page 7 of the lecture notes or Corollary 13.7 in James' book.)

3. Let μ and ν be partitions of n and suppose that ν is p-regular.

(a) Use Theorem 5.8 to show that if D^ν is a composition factor of S^μ then $\nu \trianglerighteq \mu$.

(b) Show that $S^\nu/\operatorname{rad} S^\nu \cong D^\nu$, so D^ν is the *unique* top composition factor of S^ν.

4. Show that $(S^\lambda)^\star \cong S^{\lambda'} \otimes \mathrm{sgn}$ where λ' denotes the conjugate partition to λ.

5. (a) Let λ be a partition of n. Show that the restriction of $S^\lambda_{\mathbf{C}}$ to the alternating group A_n is reducible if and only if λ is self-conjugate.

(b) Show that the conjugacy class of S_n labelled by the partition μ splits when the conjugacy action is restricted to A_n if and only if the parts of μ are distinct and have odd size.

6. Let $\bigwedge^r U$ denote the rth exterior power of a vector space U. Use the Standard Basis Theorem (Theorem 6.2) to show that $\bigwedge^r S^{(n-1,1)} \cong S^{(n-r,1^r)}$.

7. Let λ be a partition of n and let t be a column standard λ-tableau. Let $\bar{t}$ denote the tableau obtained from t by sorting the rows of t into increasing order. By Sheet 1, Question 4, $\bar{t}$ is row-standard. Show that in $S^\lambda_{\mathbf{Z}}$,

$$e(t) = e(\bar{t}) + x$$

where x is an integral linear combination of standard polytabloids $e(s)$ for tableaux s such that $\{t\} > \{s\}$ and $t \succ s$. (Here $>$ and $\succ$ refers to the orders defined in Definitions 6.3 and 6.9, respectively.)

8. (a) Show that if t is a λ-tableau with two columns of equal length and s is the λ-tableau obtained from t by swapping these columns, then $e(s) = e(t)$.

(b) Is there a proof of this result using only the Garnir relations?

9. Let $n, r \in \mathbf{N}$ with $r \leq n/2$. The *colexicographic order* on r-subsets of $\{1, 2, \ldots, n\}$ is defined by $A \preceq B$ if and only if there exists m such that (i) $m \in A$, $m \notin B$ and (ii) if $j > m$ then $j \in A \cap B$. Show that the order on r-subsets of $\{1, 2, \ldots, n\}$ induced by the order $>$ on $(n-r, r)$-tabloids agrees with the colexicographic order.

10. Given a permutation $g \in S_n$, let $P(g)$ denote the corresponding permutation matrix in $\mathrm{GL}_n(\mathbf{C})$. Prove Brauer's Permutation Lemma, that if g, $h \in S_n$ and $P(g)$ and $P(h)$ are conjugate in $\mathrm{GL}_n(\mathbf{C})$, then g and h are conjugate in S_n.

11. Let $n \geq 3$. Find the Gram matrix of the bilinear form $\langle\ ,\ \rangle$ with respect to the standard basis of $S^{(2,1^{n-2})}$ and verify that Theorem 5.7 holds in this case.

12. Let λ be a partition of n and let t be a λ-tableau. Show that in any decomposition of M_F^λ as a direct sum of indecomposable modules, there is a single summand which contains S^λ and that this module is unique up to isomorphism. [*Hint: The Krull–Schmidt Theorem will be helpful.*]

(This module is known as the *Young module* for λ, and is usually denoted Y^λ.)

13. Let t be a tableau of shape λ where λ is a partition of n. Let $g \in S_n$. Show that $g \notin R(t)C(t)$ if and only if there exist transpositions $h \in C(t)$ and $k \in R(t)$ such that $kgh = g$.

If μ is a partition of n, and F is a field, let $\widetilde{M}^\mu$ to be the quotient of the FS_n-module permutation module spanned by all μ-tableaux by the submodule spanned by

$$\{t + tg : g \in C(t)\}.$$

Thus $\widetilde{M}^\mu$ can be thought of as the module spanned by all *column* equivalence classes of μ-tableaux, but where swapping two numbers in a column introduces a sign of -1.

14. Let λ and μ be partitions of n.

(a) Show that

$$\mathrm{Hom}_{\mathbf{C}S_n}(M^\lambda, \widetilde{M}^\mu)$$

has a basis indexed by zero-one matrices with row sums λ and column sums μ'.

(b) Show that $\widetilde{M}^\mu \cong M^{\mu'} \otimes \mathrm{sgn}$.

(c) It is known that if $\nu \trianglerighteq \lambda$ then M^λ has a submodule isomorphic to S^ν. (This is proved using semistandard homomorphisms in James' lecture notes.) Using this, Question 4, and (a) and (b), show that if $\lambda \trianglelefteq \mu$ then there is a zero-one matrix with rows sums λ and column sums μ'.