MINIMAL AND MAXIMAL CONSTITUENTS OF TWISTED FOULKES CHARACTERS

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Abstract. We prove combinatorial rules that give the minimal and maximal partitions labelling the irreducible constituents of a family of characters for the symmetric group that generalize Foulkes permutation characters. Restated in the language of symmetric functions, our results determine all minimal and maximal partitions that label Schur functions appearing in the plethysms $s_\nu \circ s_{(m)}$. As a corollary we prove two conjectures of Agaoka on the lexicographically least constituents of the plethysms $s_\nu \circ s_{(m)}$ and $s_\nu \circ s_{(1^m)}$.

1. Introduction

Fix $m, n \in \mathbb{N}$ and let $S_m \wr S_n \le S_{mn}$ be the transitive imprimitive wreath product of the symmetric groups S_m and S_n. The *Foulkes character* $\phi^{(m^n)}$ is the permutation character arising from the action of S_{mn} on the cosets of $S_m \wr S_n$. Finding the decomposition of $\phi^{(m^n)}$ into irreducible characters of S_{mn} is a long-standing open problem that spans representation theory and algebraic combinatorics; a solution to this problem would also solve Foulkes' Conjecture (see [6, end §1]). Equivalently, one may ask for the decomposition of $\mathrm{Sym}^n(\mathrm{Sym}^m E)$ into irreducible $\mathrm{GL}(E)$-modules, where E is a finite-dimensional rational vector space, or, taking formal characters, for the decomposition of the plethysm $s_{(n)} \circ s_{(m)}$ as an integral linear combination of Schur functions. The problem of finding a clearly positive combinatorial rule for these coefficients was identified by Stanley in Problem 9 of [24] as one of the key open problems in algebraic combinatorics. We survey the existing results in Section 2 below.

In this paper we study a generalization of Foulkes characters. Let ν be a partition of n. Let $\mathrm{Inf}_{S_n}^{S_m \wr S_n} \chi^\nu$ denote the character of $S_m \wr S_n$ inflated from the irreducible character of χ^ν of S_n using the canonical quotient map $S_m \wr S_n \to S_n$. Let

$$\phi_\nu^{(m^n)} = (\mathrm{Inf}_{S_n}^{S_m \wr S_n} \chi^\nu)\big\uparrow_{S_m \wr S_n}^{S_{mn}}.$$

Date: September 2014.

2010 *Mathematics Subject Classification.* 20C30; Secondary: 20C15, 05E05.

Key words and phrases. twisted Foulkes module, Specht module, module homomorphism, set family, multiset family.

We call these characters *twisted Foulkes characters*. The corresponding polynomial representation of $\mathrm{GL}(E)$ is $\mathrm{Sym}^\nu\big(\mathrm{Sym}^m E\big)$, and the corresponding plethysm is $s_\nu \circ s_{(m)}$.

The two main results of this paper give combinatorial rules that determine the minimal partitions and the maximal partitions in the dominance order that label the irreducible constituents of these characters. As a corollary, we prove two conjectures of Agaoka [1] on the lexicographically least constituents of the plethysms $s_\nu \circ s_{(m)}$ and $s_\nu \circ s_{(1^m)}$.

To state our main results we need the following definitions. Let λ' denote the conjugate partition to a partition λ and let $\unlhd$ denote the dominance order on partitions.

Definition 1.1.
(i) A *set family* $\mathcal{P}$ of *shape* (m^n) is a collection of n distinct m-subsets of $\mathbb{N}$. The *type* of the set family $\mathcal{P}$, if defined, is the partition λ such that the number of sets in $\mathcal{P}$ that contain i is λ'_i.
(ii) Let $\mathcal{P}_1, \ldots \mathcal{P}_c$ be set families. Then $(\mathcal{P}_1, \ldots, \mathcal{P}_c)$ is called a *set family tuple*. The *type* of the set family tuple $(\mathcal{P}_1, \ldots, \mathcal{P}_c)$, if defined, is the partition λ such that the total number of sets in the set families $\mathcal{P}_1, \ldots, \mathcal{P}_c$ that contain i is λ'_i.

Not all set family tuples possess a type, but we shall be primarily concerned with those that do. A set family $\mathcal{P}$ of type λ is *minimal* if there is no set family $\mathcal{R}$ of type $\mu \lhd \lambda$ that has the same shape as $\mathcal{P}$. A set family tuple $(\mathcal{P}_1, \ldots, \mathcal{P}_c)$ of type λ is called *minimal* if there is no set family tuple $(\mathcal{R}_1, \ldots, \mathcal{R}_c)$ of type $\mu \lhd \lambda$ such that each $\mathcal{R}_i$ has the same shape as $\mathcal{P}_i$.

We now make a similar definition replacing sets by multisets.

Definition 1.2.
(i) A *multiset family* $\mathcal{Q}$ of *shape* (m^n) is a collection of n distinct multisets each of cardinality m having elements in $\mathbb{N}$. The *type* of the multiset family $\mathcal{Q}$, if defined, is the partition λ such that λ'_i is the total number of occurrences of i in the multisets contained in $\mathcal{Q}$.
(ii) Let $\mathcal{Q}_1, \ldots \mathcal{Q}_c$ be multiset families. Then $(\mathcal{Q}_1, \ldots, \mathcal{Q}_c)$ is called a *multiset family tuple*. The *type* of the multiset family tuple $(\mathcal{Q}_1, \ldots, \mathcal{Q}_c)$, if defined, is the partition λ such that λ'_i is the total number of occurrences of i in the multisets contained in $\mathcal{Q}_1, \ldots, \mathcal{Q}_c$.

Minimal multiset family tuples are then defined in the same way as minimal set family tuples.

Given a character ψ of S_r and a partition λ of $r \in \mathbb{N}$, we say that χ^λ is a *minimal constituent* of ψ if $\langle \psi, \chi^\lambda \rangle \geq 1$, and λ is minimal in the dominance order on partitions of r with this property. The definition of *maximal constituent* is analogous.

Our two main results are as follows.

Theorem 1.3. *Let ν be a partition of n and let λ be a partition of mn. Set $\kappa = \nu$ if m is even and $\kappa = \nu'$ if m is odd. Let k be the first part of κ. Then χ^λ is a minimal constituent of $\phi_\nu^{(m^n)}$ if and only if there is a minimal set family tuple $(\mathcal{P}_1, \ldots, \mathcal{P}_k)$ of type λ such that each $\mathcal{P}_j$ has shape $(m^{\kappa'_j})$.*

Theorem 1.4. *Let ν be a partition of n with first part ℓ and let λ be a partition of mn. Then χ^λ is a maximal constituent of $\phi_\nu^{(m^n)}$ if and only if there is a minimal multiset family tuple $(\mathcal{Q}_1, \ldots, \mathcal{Q}_\ell)$ of type λ' such that each $\mathcal{Q}_j$ has shape $(m^{\nu'_j})$.*

We pause to give a small example of these theorems. This example is continued in Section 4.3.

Example. By Theorem 1.3 the minimal constituents of $\phi_{(2,1,1)}^{(2^4)}$ are $\chi^{(4,2,1,1)}$ and $\chi^{(3,3,2)}$, corresponding to the minimal set family tuples

$$\big(\big\{\{1,2\},\{1,3\},\{1,4\}\big\},\big\{\{1,2\}\big\}\big) \text{ and } \big(\big\{\{1,2\},\{1,3\},\{2,3\}\big\},\big\{\{1,2\}\big\}\big),$$

respectively. By Theorem 1.4 the maximal constituents of $\phi_{(2,1,1)}^{(2^4)}$ are $\chi^{(6,1,1)}$ and $\chi^{(5,3)}$, corresponding to the minimal multiset family tuples

$$\big(\big\{\{1,1\},\{1,2\},\{1,3\}\big\},\big\{\{1,1\}\big\}\big) \text{ and } \big(\big\{\{1,1\},\{1,2\},\{2,2\}\big\},\big\{\{1,1\}\big\}\big),$$

respectively.

To prove Theorem 1.3 we construct an explicit module affording the character $\phi_\nu^{(m^n)}$, using the plethysm functor from representations of S_n to representations of S_{mn} defined in Section 3.2 below. We then define explicit homomorphisms from Specht modules into this module. These constructions are of independent interest. In Section 8.3 we show that our homomorphisms give irreducible characters appearing in $\phi_\nu^{(m^n)}$ beyond those predicted by our two main theorems.

The maximal constituents of $\phi_\nu^{(m^n)}$ are in bijection with the minimal constituents of $\mathrm{sgn}_{S_{mn}} \times \phi_\nu^{(m^n)}$. To prove Theorem 1.4 we define explicit modules affording these characters and determine their minimal constituents by adapting the arguments used to prove Theorem 1.3.

The outline of this paper is as follows. The common preliminary results we need are collected in Sections 3 and 4. We give a complete proof of Theorem 1.3 when m is even in Section 5, and indicate in Section 6 the modifications required for odd m. By contrast, it is possible to prove both cases of Theorem 1.4 in an almost uniform way: we do this in Section 7. We end in Section 8 with a number of corollaries of the main theorems. In particular, we prove the two conjectures of Agaoka mentioned above by determining the lexicographically least and greatest constituents of the characters $\phi_\nu^{(m^n)}$. We also give a necessary and sufficient condition for $\phi_\nu^{(m^n)}$ to have a unique minimal or maximal constituent, and find an $\mathrm{SL}(E)$-invariant subspace in the polynomial representation corresponding to certain twisted Foulkes characters. Finally we show that $\phi_{(1^n)}^{(2^n)}$ has the interesting property that all its constituents are both minimal and maximal; we use our two main theorems to give a new proof of the decomposition of this character into irreducible characters.

We remark that Theorem 2.6 in the authors' earlier paper [22] is the special case of Theorem 1.3 when m is odd and $\nu = (n)$. The authors recently learned of work by Klivans and Reiner [17, Proposition 5.10] which gives a result equivalent to this special case. The proofs in this paper use some similar ideas to [22], but are considerably shorter, and give more general results.

2. Background on plethysms

Let ν be a partition of n. Under the characteristic isomorphism $\phi_\nu^{(m^n)}$ is sent to the plethysm of Schur functions $s_\nu \circ s_{(m)}$ (see [20, I, Appendix A, (6.2)]). The existing results on the characters $\phi_\nu^{(m^n)}$ are limited and have mainly been obtained using the methods of symmetric polynomials. We shall use this language throughout this section. The following plethysms of the form $s_\nu \circ s_{(m)}$ have a known decomposition into Schur functions:

(i) $s_{(1^2)} \circ s_{(m)}$, $s_{(2)} \circ s_{(m)}$, $s_{(n)} \circ s_{(1^2)}$ and $s_{(n)} \circ s_{(2)}$; see Littlewood [19],
(ii) $s_{(3)} \circ s_{(m)}$; see Thrall [25, Theorem 5] or Dent and Siemons [4, Theorem 4.1],
(iii) $s_\nu \circ s_{(m)}$ when ν is a partition of 4; see Theorem 27 of Foulkes [7] for an explicit decomposition in a special case and the remarks on the general case immediately following,
(iv) $s_\nu \circ s_{(m)}$ when ν is a partition of 2, 3 or 4; see Howe [13, Section 3.5 and Remark 3.6(b)]. Howe's statements are usually more convenient than Foulkes'.

There are several further results which, like our two main theorems, give information about constituents of a special form. The Cayley–Sylvester formula states that the multiplicity of $s_{(mn-r,r)}$ in $s_{(n)} \circ s_{(m)}$ is equal to the number of partitions of r whose Young diagram is contained in the Young diagram of (m^n). A striking generalization due to Manivel [21] states that the two-variable symmetric function $(s_{(n^k)} \circ s_{(m+k-1)})(x_1, x_2)$ is symmetric under any permutation of m, n and k. Taking $k = 1$ and swapping m and n gives the Cayley–Sylvester formula, while taking $k = 1$ and swapping k and n gives $(s_{(n)} \circ s_{(m)})(x_1, x_2) = (s_{(1^n)} \circ s_{(m+n-1)})(x_1, x_2)$. In [18] Langley and

Remmel used symmetric functions methods to determine the multiplicities in $s_\nu \circ s_\mu$ of the Schur functions $s_{(mn-r,1^r)}$, $s_{(mn-r-s,s,1^r)}$ and $s_{(mn-r-2t,2^t,1^r)}$, for any partition μ of m. Giannelli [11, Theorem 1.2] later used character-theoretic methods to determine the multiplicities of a much larger class of 'near hook' constituents of $s_{(n)} \circ s_{(m)}$.

For sufficiently small partitions ν and μ, the plethysm $s_\nu \circ s_\mu$ can readily be calculated using any of the computer algebra systems MAGMA [3], GAP [10] or SYMMETRICA [16]. A new algorithm for computing $s_{(n)} \circ s_{(m)}$ was given in [5, Proposition 5.1], and used to verify Foulkes' Conjecture (see [6, end §1]) for all m and n such that $m + n \leq 19$.

Applying the ω involution (see [20, Ch. I, Equation (2.7)]) gives further results for the plethysms $s_\nu \circ s_{(1^m)}$, via the relation

$$\omega(s_\nu \circ s_{(m)}) = \begin{cases} s_{\nu'} \circ s_{(1^m)} & \text{if } m \text{ is odd} \\ s_\nu \circ s_{(1^m)} & \text{if } m \text{ is even,} \end{cases} \tag{1}$$

which follows from [20, Ch. I, Equation (3.8) and §8, Example 1(a)]. This equation is reformulated in terms of modules and characters in Section 3.3.

Finally we note that the lexicographically greatest constituent of $s_\nu \circ s_\mu$ was determined by Iijima in [14, Theorem 4.2], confirming a conjecture of Agaoka [1, Conjecture 1.2]. We give a short proof of the special cases of Iijima's result when $\mu = (m)$ or $\mu = (1^m)$ in Section 8 below.

3. Specht modules and plethysm

In this section we recall a standard construction of Specht modules as modules defined by generators and relations. We then give a functorial interpretation of plethysm for categories of modules for symmetric groups. This leads to an explicit construction of modules affording the characters $\phi_\nu^{(m^n)}$ and $\mathrm{sgn}_{S_{mn}} \times \phi_\nu^{(m^n)}$.

3.1. **Garnir elements.** Let λ be a partition of $r \in \mathbb{N}$. We use the standard definition [15, Definition 4.3] of the rational Specht module S^λ as the $\mathbb{Q}S_r$-submodule of the Young permutation module M^λ spanned by the λ-polytabloids e_t for t a λ-tableau. It is well known that S^λ affords the irreducible character χ^λ.

Following Fulton (see [8, Chapter 7, Section 4]), we define a λ-*column tabloid* to be an equivalence class of λ-tableaux up to column equivalence. We denote the column tabloid corresponding to a tableau t by $|t|$ and represent it by omitting the horizontal lines from the representative t. The symmetric group acts in an obvious way on the set of λ-column tabloids: let $U \cong M^{\lambda'}$ denote the corresponding permutation module for $\mathbb{Q}S_r$. We define $\widetilde{M}^\lambda = \mathrm{sgn}_{S_r} \otimes U$. (This is equivalent to Fulton's definition using orientations.) By a small abuse of notation we shall write $|t|$ for the basis element of $\widetilde{M}^\lambda$ corresponding to the λ-column tabloid t. There is a canonical surjective homomorphism of $\mathbb{Q}S_{mn}$-modules $\widetilde{M}^\lambda \to S^\lambda$ defined by $|t| \mapsto e_t$.

It follows from the corollary on page 101 of [8] and the proof of Theorem 8.4 of [15] that the kernel of the canonical surjection $\widetilde{M}^\lambda \to S^\lambda$ is spanned by all elements of $\widetilde{M}^\lambda$ of the form

$$|t| \sum_{\sigma \in S_{X\cup Y}} \sigma \operatorname{sgn}(\sigma) \tag{2}$$

where t is a λ-tableau, X is a subset of set of all entries in column i of t and Y is a subset of the entries in column $i+1$ of t such that $|X|+|Y| > \lambda'_i$. By Exercise 16 on page 102 of [8], we need only consider the case when Y is a singleton set; note that this result requires that the ground field has characteristic zero. An easy calculation now shows that, if t is any fixed λ-tableau, then the kernel is generated, as a $\mathbb{Q}S_{mn}$-module, by the *t-Garnir elements* $|t|\sum_{\sigma\in S_{X\cup\{y\}}} \sigma \operatorname{sgn}(\sigma)$, where X is the set of entries in column i of t and y is the entry at the top of column $i+1$ of t. (This term is not standard, but will be convenient in this paper.)

3.2. **The plethysm functor P.** Let m, $n \in \mathbb{N}$ and let ν be a partition of n. Let S_{mn} act naturally on the set Ω of size mn. Given a module V for $\mathbb{Q}S_n$ we define

$$P(V) = \big(\mathrm{Inf}_{S_n}^{S_m\wr S_n} V\big)\big\uparrow_{S_m\wr S_n}^{S_{mn}}.$$

Since P is the composition of an inflation and an induction functor, P is an exact functor from the category of $\mathbb{Q}S_n$-modules to the category of $\mathbb{Q}S_{mn}$-modules. By definition $P(S^\nu)$ affords the irreducible character $\phi_\nu^{(m^n)}$.

We now give an explicit model for the modules $P(M^\nu)$, $P(\widetilde{M^\nu})$ and $P(S^\nu)$. These modules have bases defined using tableaux, tabloids and column tabloids with entries taken from the set of m-subsets of the set Ω of size mn; we shall refer to these objects as *set-tableaux*, *set-tabloids* and *column set-tabloids*. Let $S_m\wr S_n \le S_{mn}$ have $\{\Delta_1, \ldots, \Delta_n\}$ as a system of blocks of imprimitivity. As a concrete module isomorphic to $\mathrm{Inf}_{S_n}^{S_m\wr S_n} M^\nu$, we take the rational vector space W with basis the set of set-tabloids of shape ν with entries from $\{\Delta_1, \ldots, \Delta_n\}$. Let W' denote the rational vector space with basis the set of all set-tabloids of shape ν such that the union of all the m-subsets appearing in each set-tabloid is Ω. Then W' is a $\mathbb{Q}S_{mn}$-module of dimension $|S_{mn}|/|S_m\wr S_n| \dim W$, generated by the $\mathbb{Q}(S_m\wr S_n)$-submodule W. Hence $W' \cong W\big\uparrow_{S_m\wr S_n}^{S_{mn}}$ and so $W' \cong P(M^\nu)$. By the functoriality of P the canonical inclusion map $S^\nu \hookrightarrow M^\nu$ induces a canonical inclusion

$$P(S^\nu) \hookrightarrow P(M^\nu).$$

An entirely analogous construction with set-tableaux and column set-tabloids gives modules isomorphic to $P(\mathbb{Q}S_n)$ and $P(\widetilde{M}^\nu)$, respectively, with canonical quotient maps

$$P(\widetilde{M}^\nu) \twoheadrightarrow P(S^\nu).$$

We illustrate this construction in Section 4.3 below.

3.3. **The signed plethysm functor Q.** Let $\widetilde{\mathrm{sgn}}$ denote the unique 1-dimensional module for $S_m \wr S_n$ that restricts to the module $\mathrm{sgn} \otimes \cdots \otimes \mathrm{sgn}$ of the base group $S_m \times \cdots \times S_m$ and on which the complement S_n acts trivially. Given a module V for $\mathbb{Q}S_n$ we define

$$Q(V) = (\widetilde{\mathrm{sgn}} \otimes \mathrm{Inf}_{S_n}^{S_m \wr S_n} V)\big\uparrow_{S_m \wr S_n}^{S_{mn}}.$$

Again Q is an exact functor from the category of $\mathbb{Q}S_n$-modules to the category of $\mathbb{Q}S_{mn}$-modules.

We define $\psi_\nu^{(m^n)}$ to be the character of $Q(S^\nu)$. The twisted Foulkes characters $\phi_\nu^{(m^n)}$ are related to the characters $\psi_\nu^{(m^n)}$ via a sign-twist. We have

$$\mathrm{sgn}_{S_{mn}} \otimes P(V) = \Big(\mathrm{sgn}\big\downarrow_{S_m \wr S_n}^{S_{mn}} \otimes \mathrm{Inf}_{S_n} V\Big)\big\uparrow_{S_m \wr S_n}^{S_{mn}}.$$

The restriction of $\mathrm{sgn}_{S_{mn}}$ to $S_m \wr S_n$ is $\widetilde{\mathrm{sgn}}$ if m is even and $\widetilde{\mathrm{sgn}} \otimes \mathrm{Inf}_{S_n}^{S_m \wr S_n} \mathrm{sgn}_{S_n}$ if m is odd. Therefore

$$\mathrm{sgn}_{S_{mn}} \otimes P(V) \cong \begin{cases} Q(V) & \text{if } m \text{ is even} \\ Q(\mathrm{sgn}_{S_n} \otimes V) & \text{if } m \text{ is odd.} \end{cases} \tag{3}$$

Using the isomorphism

$$\mathrm{sgn}_{S_n} \otimes S^\nu \cong (S^{\nu'})^* \tag{4}$$

(see, for example, [15, Theorem 6.7]), and that Specht modules are self-dual over the rationals (see [15, Theorem 4.12]), we obtain the reformulation of Equation (1) for characters:

$$\mathrm{sgn}_{S_{mn}} \times \phi_\nu^{(m^n)} = \begin{cases} \psi_\nu^{(m^n)} & \text{if } m \text{ is even} \\ \psi_{\nu'}^{(m^n)} & \text{if } m \text{ is odd.} \end{cases} \tag{5}$$

3.4. **Connection with Schur functors.** We remark very briefly on an alternative definition of these functors. Let Δ^λ be the Schur functor corresponding to the partition λ (see [9, page 76] or [23, page 273]). Let E be a rational vector space of dimension at least mn. If F is the functor defined in [12, Section 6.1] from polynomial representations of $\mathrm{GL}(E)$ of degree r to representations of S_r then, by [20, I, Appendix A, (6.2)], $F\big(\Delta^\nu(\mathrm{Sym}^m E)\big) = P(S^\nu)$, corresponding to the plethysm $s_\nu \circ s_{(m)}$, and $F\big(\Delta^\nu(\bigwedge^m E)\big) = Q(S^\nu)$, corresponding to the plethysm $s_\nu \circ s_{(1^m)}$. We use this interpretation of P and Q in Section 8.4 below.

4. Further preliminary results and an example

4.1. **Closed set families.** Let A and B be m-subsets of $\mathbb{N}$. Let a_r be the rth smallest element of A, and let b_r be the rth smallest element of B. We say that B *majorizes* A, and write $A \preceq B$, if $a_r \leq b_r$ for all r. We say that a set family $\mathcal{P}$ of shape (m^n) is *closed* if whenever $B \in \mathcal{P}$ and A is an m-subset of $\mathbb{N}$ such that $A \preceq B$, then $A \in \mathcal{P}$. We say that a set family tuple $(\mathcal{P}_1, \ldots, \mathcal{P}_k)$ is *closed* if $\mathcal{P}_j$ is closed for each j. It is easily seen that closed set families and closed set family tuples have well-defined types.

If $(\mathcal{P}_1, \ldots, \mathcal{P}_k)$ is a minimal set family tuple then it is closed. For if not there is a set family $\mathcal{P}_j$, a set $A \in \mathcal{P}_j$ and an element $i+1 \in A$, such that the set $B = A\backslash\{i+1\} \cup \{i\}$ is not in $\mathcal{P}_j$. A new set family tuple can be formed by replacing A by B in $\mathcal{P}_j$ and this process repeated until a closed set family tuple $(\mathcal{P}'_1, \ldots, \mathcal{P}'_k)$ is obtained: this set family tuple has a well-defined type. By construction, $\mathcal{P}'_j$ has the same shape as $\mathcal{P}_j$ for each j, and the type of $(\mathcal{P}'_1, \ldots, \mathcal{P}'_k)$ is strictly dominated by the type of $(\mathcal{P}_1, \ldots, \mathcal{P}_k)$, contradicting minimality. This argument also shows that if $(\mathcal{P}_1, \ldots, \mathcal{P}_k)$ is a minimal set family tuple then each set family $\mathcal{P}_j$ is minimal.

Closed multiset family tuples are defined analogously and the same argument shows that minimal multiset family tuples are closed.

4.2. **Symbols.** When defining maps from S^λ or from $\widetilde{M}^\lambda$, it will be convenient to think of S_{mn} as the symmetric group on the set Ω^λ whose elements are the formal *symbols* i_j for i and j such that $1 \leq i \leq \lambda_1$ and $1 \leq j \leq \lambda'_i$. We say that i_j has *number* i and *index* j. Let t_λ be the λ-tableau such that column i of t_λ has entries $i_1, \ldots, i_{\lambda'_i}$ when read from top to bottom. Let $C(t_\lambda)$ be the column stabilising subgroup of t_λ; note that $C(t_\lambda)$ permutes the indices of the symbols in Ω^λ, while leaving the numbers fixed. Let $b_{t_\lambda} = \sum_{\sigma \in C(t_\lambda)} \sigma \operatorname{sgn}(\sigma)$.

4.3. **Example.** This example illustrates the definitions so far, and many of the ideas in the proofs of Theorem 1.3 and Theorem 1.4 to follow. Let $m = 2$, let $\nu = (2,1,1)$ and let $\mathcal{P} = (\{\{1,2\},\{1,3\},\{1,4\}\},\{\{1,2\}\})$ be the minimal set family tuple of type $\lambda = (4,2,1,1)$ seen in the introduction. We identify S_8 with the symmetric group on the set $\Omega^{(4,2,1,1)} = \{1_1, 1_2, 1_3, 1_4, 2_1, 2_2, 3_1, 4_1\}$ and choose $S_2 \wr S_4 \leq S_8$ to have blocks of imprimitivity $\{1_1, 2_1\}$, $\{1_2, 3_1\}$, $\{1_3, 4_1\}$, $\{1_4, 2_2\}$. Let T be the set-tableau

$$\begin{array}{|c|c|}\hline \{1_1, 2_1\} & \{1_4, 2_2\} \\ \hline \{1_2, 3_1\} \\ \cline{1-1} \{1_3, 4_1\} \\ \cline{1-1} \end{array}\;.$$

The column set-tabloid $|T|$ generates $\mathrm{Inf}_{S_4}^{S_2 \wr S_4} \widetilde{M}^\nu$ as a $\mathbb{Q}(S_2 \wr S_4)$-module and $P(\widetilde{M}^\nu)$ as a $\mathbb{Q}S_8$-module. For example

$$|T|\,(1_1, 1_2) = \left|\begin{matrix} \{1_2, 2_1\} \\ \{1_1, 3_1\} \\ \{1_3, 4_1\} \end{matrix}\right| \begin{matrix} \{1_4, 2_2\} \\ \\ \\ \end{matrix}\Bigg| = - \left|\begin{matrix} \{1_1, 3_1\} \\ \{1_2, 2_1\} \\ \{1_3, 4_1\} \end{matrix}\right| \begin{matrix} \{1_4, 2_2\} \\ \\ \\ \end{matrix}\Bigg| .$$

There is a unique homomorphism of $\mathbb{Q}S_8$-modules $\widetilde{M}^{(4,2,1,1)} \to P(\widetilde{M}^{(2,1,1)})$ sending $|t_{(4,2,1,1)}|$ to $|T|b_{t_{(4,2,1,1)}}$. We shall see in the proof of Proposition 5.2 below that the kernel of this homomorphism contains all the $t_{(4,2,1,1)}$-Garnir elements. Hence there is a well-defined homomorphism of $\mathbb{Q}S_8$-modules $S^{(4,2,1,1)} \to P(\widetilde{M}^{(2,1,1)})$ defined by $e_{t_{(4,2,1,1)}} \mapsto |T|b_{t_{(4,2,1,1)}}$. After composition with the canonical surjection $P(\widetilde{M}^{(2,1,1)}) \to P(S^{(2,1,1)})$ the image of $e_{t_{(4,2,1,1)}}$ is $e_T b_{t_\lambda} \in P(S^{(2,1,1)}) \subseteq P(M^{(2,1,1)})$. As we argue in Lemma 5.3 below, the coefficient of the tabloid $\{T\}$ in $e_T b_{t_\lambda}$ is 1, and so this map is non-zero. Hence $\langle \phi^{(2^4)}_{(2,1,1)}, \chi^{(4,2,1,1)} \rangle \geq 1$.

Observe that if T is a set-tableau having an entry containing symbols i_j and i_k with $j \neq k$ then $|T|(i_j, i_k) = 1$, whereas $e_{t_\lambda}(i_j, i_k) = -1$. Thus the entries of T must come from set families. (This remark is made precise in the proof of Proposition 5.5 below.) We shall see in Section 7 that the maximal constituents of $\phi^{(2^4)}_{(2,1,1)}$ are determined by homomorphisms into $Q(S^{(2,1,1)}) \cong \mathrm{sgn}_{S_8} \otimes P(S^{(2,1,1)})$. In this setting, thanks to the sign-twist, the two signs agree. This gives one indication why set-tableaux with entries given by multiset families, rather than set families, are relevant to maximal constituents.

5. Proof of Theorem 1.3 for m even

Fix an even number m. Let ν be a partition of n with first part k. The proof of Theorem 1.3 for even m has two steps. In the first we construct an explicit homomorphism $S^\lambda \to P(S^\nu)$ for each closed set family tuple of type λ. We then use these homomorphisms to show that the minimal constituents of the character $\phi^{(m^n)}_\nu$ are as claimed in the theorem. We must begin with one more definition.

Let $(\mathcal{P}_1, \ldots, \mathcal{P}_k)$ be a set family tuple of type λ such that $\mathcal{P}_j$ has shape $(m^{\nu'_j})$ for each j. Let $\mathcal{A}(\mathcal{P}_1, \ldots, \mathcal{P}_k)$ be the set of all ordered pairs (j, B) such that $1 \leq j \leq k$ and $B \in \mathcal{P}_j$. We totally order $\mathcal{A}(\mathcal{P}_1, \ldots, \mathcal{P}_k)$ so that $(i, A) \leq (j, B)$ if and only if $i < j$ or $i = j$ and $A \leq B$, where the final inequality refers to the lexicographic order on sets.

Definition 5.1.

(i) The *column set-tableau* corresponding to $(\mathcal{P}_1, \ldots, \mathcal{P}_k)$ is the unique set-tableau T of shape ν such that if $\mathcal{P}_j = \{A_1, \ldots, A_{\nu'_j}\}$ then the entries in column j of T are obtained by appending indices to the numbers in the sets $A_1, \ldots, A_{\nu'_j}$, listing the sets in lexicographic order and choosing indices in the order specified by the order on $\mathcal{A}(\mathcal{P}_1, \ldots, \mathcal{P}_k)$.

(ii) The *column set-tabloid* corresponding to $(\mathcal{P}_1, \ldots, \mathcal{P}_k)$ is $|T| \in P(\widetilde{M}^\nu)$, where T is the column set-tableau corresponding to $(\mathcal{P}_1, \ldots, \mathcal{P}_k)$.

Observe that the union of the entries in the column set-tableau corresponding to $(\mathcal{P}_1, \ldots, \mathcal{P}_k)$ is the set Ω^λ. For example, the set-tableau T in Section 4.3 is the column set-tableau corresponding to the set family tuple $\big(\{\{1,2\},\{1,3\},\{1,4\}\},\{\{1,2\}\}\big)$.

Let $(\mathcal{P}_1, \ldots, \mathcal{P}_k)$ be a closed set family tuple of type λ such that $\mathcal{P}_j$ has shape $(m^{\nu'_j})$ for each j. Let $|T| \in P(\widetilde{M}^\nu)$ be the column set-tabloid corresponding to $(\mathcal{P}_1, \ldots, \mathcal{P}_k)$. Let t_λ be the λ-tableau defined in Section 4.2, and let

$$f_{(\mathcal{P}_1,\ldots,\mathcal{P}_k)} : \widetilde{M}^\lambda \to P(\widetilde{M}^\nu)$$

be the unique $\mathbb{Q}S_{mn}$-homomorphism such that

$$|t_\lambda| f_{(\mathcal{P}_1,\ldots,\mathcal{P}_k)} = |T| b_{t_\lambda}.$$

Proposition 5.2. *The kernel of $f_{(\mathcal{P}_1,\ldots,\mathcal{P}_k)}$ contains every t_λ-Garnir element.*

Proof. Let $1 \leq i < \lambda_1$ and let $X = \{i_1, \ldots, i_{\lambda'_i}\}$ be the set of entries in column i of t_λ. We have

$$|t_\lambda| G_{X \cup \{(i+1)_1\}} f_{(\mathcal{P}_1,\ldots,\mathcal{P}_k)} = |T| \sum_{\tau \in C(t_\lambda)} \tau G_{X \cup \{(i+1)_1\}} \operatorname{sgn}(\tau).$$

To prove that the right-hand side is zero we shall construct an involution on $C(t_\lambda)$, denoted $\tau \mapsto \tau^\star$, with the following two properties:

(a) if $\tau = \tau^\star$ then $|T|\tau G_{X \cup \{(i+1)_1\}} = 0$,

(b) if $\tau \neq \tau^\star$ then $|T|\big(\tau \operatorname{sgn}(\tau) + \tau^\star \operatorname{sgn}(\tau^\star)\big) G_{X \cup \{(i+1)_1\}} = 0$.

Let $\tau \in C(t_\lambda)$. Consider $|T|\tau$. If there exists a symbol $i_x \in X$ such that there is an entry in $|T|\tau$ containing both i_x and $(i+1)_1$, then we have $|T|\tau(1-(i_x,(i+1)_1)) = 0$. Taking coset representatives for $\langle (i_x,(i+1)_1)\rangle \leq S_{X \cup \{(i+1)_1\}}$, it follows that $|T|\tau G_{X \cup \{(i+1)_1\}} = 0$. Hence if we define $\tau^\star = \tau$ in this case then (a) holds.

Now suppose that no entry in $|T|\tau$ contains both $(i+1)_1$ and a symbol $i_x \in X$. Let the entry of $|T|\tau$ containing $(i+1)_1$ be

$$B_{(i+1)_1} = \{c(1)_{b(1)}, \ldots, c(m-1)_{b(m-1)}, (i+1)_1\}.$$

Suppose that $B_{(i+1)_1}$ lies in column j of $|T|\tau$. This column is defined using the set family $\mathcal{P}_j$. Since $\mathcal{P}_j$ is closed, there exists unique symbols $c(1)_{a(1)}, \ldots, c(m-1)_{a(m-1)}$ and i_u such that the multiset

$$A_{(i+1)_1} = \{c(1)_{a(1)}, \ldots, c(m-1)_{a(m-1)}, i_u\}$$

is also an entry in column j of $|T|\tau$. Define

$$\pi = (c(1)_{a(1)}, c(1)_{b(1)}) \ldots (c(m-1)_{a(m-1)}, c(m-1)_{b(m-1)}) \in C(t_\lambda)$$

and define $\tau^\star = \tau\pi$. Since the column set-tabloids $|T|\tau$ and $|T|\tau^\star$ differ only in indices attached to numbers other than i and $i+1$, we have $\tau^{\star\star} = \tau$. Since m is even we have $\operatorname{sgn}(\tau) = -\operatorname{sgn}(\tau^\star)$ and since $\pi(i_u, (i+1)_1)$ swaps two entries in column j of $|T|\tau$ we have

$$|T|\tau^\star(i_u, (i+1)_1) = |T|\tau\pi(i_u, (i+1)_1) = -|T|\tau.$$

Using this relation to eliminate $\tau^\star$ we obtain

$$\big(|T|\tau \operatorname{sgn}(\tau) + |T|\tau^\star \operatorname{sgn}(\tau^\star)\big)\big(1 - (i_u, (i+1)_1)\big) = 0.$$

Hence $\big(|T|\tau \operatorname{sgn}(\tau) + |T|\tau^\star \operatorname{sgn}(\tau^\star)\big)G_{X\cup\{(i+1)_1\}} = 0$, as required in (b). □

It now follows from Section 3.1 that $f_{(\mathcal{P}_1,\ldots,\mathcal{P}_k)}$ induces a homomorphism $S^\lambda \to P(\widetilde{M}^\nu)$. Let

$$\bar{f}_{(\mathcal{P}_1,\ldots,\mathcal{P}_k)} : S^\lambda \to P(S^\nu)$$

denote the composition of this homomorphism with the canonical quotient map $P(\widetilde{M}^\nu) \to P(S^\nu)$. Thus $\bar{f}_{(\mathcal{P}_1,\ldots,\mathcal{P}_k)}$ is defined on the generator e_{t_λ} of S^λ by

$$e_{t_\lambda}\bar{f}_{(\mathcal{P}_1,\ldots,\mathcal{P}_k)} = e_T b_{t_\lambda}.$$

Lemma 5.3. *The homomorphism $\bar{f}_{(\mathcal{P}_1,\ldots,\mathcal{P}_k)} : S^\lambda \to P(S^\nu)$ is non-zero.*

Proof. Since b_{t_λ} permutes the indices attached to numbers, while leaving the numbers fixed, it is clear that the coefficient of the set-tabloid $\{T\}$ in $\{T\}b_{t_\lambda}$ is 1. This is also the coefficient of $\{T\}$ in $e_T b_{t_\lambda}$. □

We summarize the results proved so far in the following corollary. We show in Section 8.3 that this corollary gives constituents of $\phi_\nu^{(m^n)}$ beyond those predicted by Theorem 1.3.

Corollary 5.4. *Let m be even and let ν be a partition of n with first part k. If there is a closed set family tuple $(\mathcal{P}_1, \ldots, \mathcal{P}_k)$ of type λ such that $\mathcal{P}_i$ has shape $(m^{\nu'_i})$ for each i, then $\langle \phi_\nu^{(m^n)}, \chi^\lambda \rangle \geq 1$.*

Proof. This follows immediately from Proposition 5.2 and Lemma 5.3 □

The final ingredient in the proof of Theorem 1.3 in the case when m is even is a result that goes in the opposite direction to Corollary 5.4.

Proposition 5.5. *Let m be even and let ν be a partition of n with first part k. If χ^μ is a constituent of $\phi_\nu^{(m^n)}$ then there is a set family tuple $(\mathcal{R}_1, \ldots, \mathcal{R}_k)$ of type μ such that $\mathcal{R}_j$ has shape $(m^{\nu'_j})$ for each j.*

Proof. Let ζ be the character of $P(\widetilde{M}^\nu)$. We have $\langle \zeta, \chi^\mu \rangle \geq \langle \phi_\nu^{(m^n)}, \chi^\mu \rangle \geq 1$. Hence there is a non-zero $\mathbb{Q}S_{mn}$-module homomorphism $f : S^\mu \to P(\widetilde{M}^\nu)$. Identify S_{mn} with the symmetric group on the symbols Ω^μ. Let T be a set-tableau such that the coefficient of $|T|$ in $e_{t_\mu} f$ is non-zero. Let i_j and $i_{j'}$ be symbols appearing in t_μ. If there is an entry in T containing both i_j and $i_{j'}$ then we have $|T|(i_j, i_{j'}) = |T|$, whereas $e_{t_\mu}(i_j, i_{j'}) = -e_{t_\mu}$, a contradiction. Now suppose that there is a column of T containing entries $\{c(1)_{a(1)}, \ldots, c(m)_{a(m)}\}$ and $\{c(1)_{b(1)}, \ldots, c(m)_{b(m)}\}$ that are equal up to the indices attached to numbers. Let

$$\rho = (c(1)_{a(1)}, c(1)_{b(1)}) \ldots (c(m)_{a(m)}, c(m)_{b(m)}).$$

Since ρ swaps two entries in the same column of $|T|$, we have $|T|\rho = -|T|$. But since ρ is even, $e_{t_\mu}\rho = e_{t_\mu}$, so again we have a contradiction. It follows that removing the indices attached to the numbers in column j of $|T|$ gives a set family $\mathcal{R}_j$ of shape $(m^{\nu'_j})$. Since the union of the entries in $|T|$ is Ω^μ, the set family tuple $(\mathcal{R}_1, \ldots, \mathcal{R}_k)$ has type μ, as required. $\square$

We are now ready to prove Theorem 1.3 for even values of m. Suppose that $(\mathcal{P}_1, \ldots, \mathcal{P}_k)$ is a minimal set family tuple of type λ such that each $\mathcal{P}_j$ has shape $(m^{\nu'_j})$. We saw in Section 4.1 that any minimal set family tuple is closed. Hence, by Corollary 5.4, $\langle \phi_\nu^{(m^n)}, \chi^\lambda \rangle \geq 1$. If μ is a partition of mn such that $\lambda \trianglerighteq \mu$ and $\langle \phi_\nu^{(m^n)}, \chi^\mu \rangle \geq 1$ then, by Proposition 5.5, there is a set family tuple $(\mathcal{R}_1, \ldots, \mathcal{R}_k)$ of type μ such that $\mathcal{R}_j$ has shape $(m^{\nu'_j})$ for each j. But $(\mathcal{P}_1, \ldots, \mathcal{P}_k)$ is minimal, so we must have $\lambda = \mu$. Hence χ^λ is a minimal constituent of $\phi_\nu^{(m^n)}$.

Conversely suppose that χ^λ is a minimal constituent of $\phi_\nu^{(m^n)}$. By Proposition 5.5 there is a set family tuple $(\mathcal{R}_1, \ldots, \mathcal{R}_k)$ of type λ such that $\mathcal{R}_j$ has shape $(m^{\nu'_j})$ for each j. Hence there is a minimal set family tuple $(\mathcal{P}_1, \ldots, \mathcal{P}_k)$ of type μ where $\lambda \trianglerighteq \mu$ such that $\mathcal{P}_i$ has shape $(m^{\nu'_j})$ for each j. Once again we have $\langle \phi_\nu^{(m^n)}, \chi^\mu \rangle \geq 1$. But χ^λ is a minimal constituent of $\phi_\nu^{(m^n)}$ so we must have $\lambda = \mu$. Hence $(\mathcal{R}_1, \ldots, \mathcal{R}_k)$ is a minimal set family tuple. This completes the proof.

6. Proof of Theorem 1.3 for m odd

Theorem 1.3 can be proved for odd values of m by modifying the proof in the case of m even, following the same logical structure of Section 5. We give the required changes in detailed outline.

Let ν be a partition of n with precisely k parts and let $(\mathcal{P}_1, \ldots, \mathcal{P}_k)$ be a set family tuple of type λ such that $\mathcal{P}_j$ has shape (m^{ν_j}) for each j. Define the totally ordered set $\mathcal{A}(\mathcal{P}_1, \ldots, \mathcal{P}_k)$ of pairs (j, X) with $1 \leq j \leq k$ and $X \in \mathcal{P}_j$ as before. We define the *row set-tableau* T and the *set-tabloid* $\{T\}$ corresponding to $(\mathcal{P}_1, \ldots, \mathcal{P}_k)$ by analogy with Definition 5.1. Thus T and $\{T\}$ have shape ν, the entries in row j of T and $\{T\}$ are determined by the order on $\mathcal{A}(\mathcal{P}_1, \ldots, \mathcal{P}_k)$, and the union of all the entries in T or $\{T\}$ is Ω^λ.

Let $(\mathcal{P}_1, \ldots, \mathcal{P}_k)$ be a closed set family tuple of type λ as above and let $\{T\} \in P(M^\nu)$ be the corresponding set-tabloid. We define

$$g_{(\mathcal{P}_1, \ldots, \mathcal{P}_k)} : \widetilde{M}^\lambda \to P(M^\nu)$$

by $|t_\lambda| g_{(\mathcal{P}_1, \ldots, \mathcal{P}_k)} = \{T\} b_{t_\lambda}$. We now follow Section 5, making the following changes.

(1) *Proposition 5.2.* The proof of the analogue of Proposition 5.2 goes through almost unchanged. Now swapping two entries in the same row of a set tabloid $\{T\}$ leaves the sign unchanged, but the permutation π is even. The pattern of cancellation in $\big(\{T\}\tau \operatorname{sgn}(\tau) + \{T\}\tau^\star \operatorname{sgn}(\tau^\star)\big) G_{X \cup \{(i+1)_1\}}$ is therefore the same.

(2) *Definition of homomorphisms into* $P(S^\nu)$. Let $\{u\} \in M^\nu$ be a fixed tabloid. By [15, Equation (6.8)], there is a surjective $\mathbb{Q}S_n$-homomorphism $M^\nu \to \operatorname{sgn}_{S_n} \otimes S^{\nu'}$ defined on the generator $\{u\}$ by $\{u\} \mapsto w \otimes e_{u'}$, where u' is the tableau conjugate to u and w generates sgn_{S_n}. Applying P gives a canonical quotient map $P(M^\nu) \to P(\operatorname{sgn}_{S_n} \otimes S^{\nu'})$. Composing the map induced by $g_{(\mathcal{P}_1, \ldots, \mathcal{P}_k)}$ on S^λ with this surjection gives a homomorphism $\bar{g}_{(\mathcal{P}_1, \ldots, \mathcal{P}_k)} : S^\lambda \to P(\operatorname{sgn}_{S_n} \otimes S^{\nu'})$ sending e_{t_λ} to $(w \otimes e_{T'}) b_{t_\lambda}$, where T' is the conjugate set-tableau to T. The isomorphisms $\operatorname{sgn}_{S_n} \otimes S^{\nu'} \cong (S^\nu)^* \cong S^\nu$ seen in Equation (4) and the following remark in Section 3.3 now identify the codomain of $\bar{g}_{(\mathcal{P}_1, \ldots, \mathcal{P}_k)}$ with $P(S^\nu)$.

(3) *Lemma 5.3.* Thinking of the codomain of $\bar{g}_{(\mathcal{P}_1, \ldots, \mathcal{P}_k)}$ as a submodule of $P(\operatorname{sgn}_{S_n} \otimes M^{\nu'})$ it follows by looking at the coefficient of $\{T'\}$ in $e_{t_\lambda} \bar{g}_{(\mathcal{P}_1, \ldots, \mathcal{P}_k)}$ that $\bar{g}_{(\mathcal{P}_1, \ldots, \mathcal{P}_k)}$ is non-zero.

(4) *Corollary 5.4.* The analogous result holds with the same proof.

(5) *Proposition 5.5.* The use of characters at the start of the proof can be avoided as follows: given a non-zero homomorphism $f : S^\mu \to P(S^\nu)$, composing with the map induced by the canonical inclusion $S^\nu \to M^\nu$ gives a non-zero homomorphism $f : S^\mu \to P(M^\nu)$. Then take a set-tabloid $\{T\}$ with non-zero coefficient in the image of e_{t_μ}, as before. The proof goes through changing columns to rows. Observe

that swapping two entries in a row of $\{T\}$ leaves $\{T\}$ unchanged but the permutation ρ is now odd, so $e_{t_\mu}\rho = -e_{t_\mu}$.

The end of the proof goes through essentially unchanged.

7. Proof of Theorem 1.4

In this section we prove the following theorem which determines the minimal constituents of the characters $\psi_\nu^{(m^n)}$ defined in Section 3.3.

Theorem 7.1. *Let ν be a partition of n and λ be a partition of mn. Set $\kappa = \nu$ if m is even and $\kappa = \nu'$ if m is odd. Let k be the first part of κ. Then χ^λ is a minimal constituent of $\psi_\nu^{(m^n)}$ if and only if there is a minimal multiset family tuple $(\mathcal{Q}_1, \ldots, \mathcal{Q}_k)$ of type λ such that each $\mathcal{Q}_j$ has shape $(m^{\kappa'_j})$.*

Theorem 1.4 then follows at once by Equations (4) and (5) in Section 3.3.

The proof of Theorem 7.1 again follows the same structure as that of Theorem 1.3, although this time we are usually able to treat the even and odd cases together. We give full details since there are several places where the change from sets to multisets means that new ideas are required.

Let ν, κ and k be as in Theorem 7.1. Let $(\mathcal{Q}_1, \ldots, \mathcal{Q}_k)$ be a closed multiset family tuple of type λ such that $\mathcal{Q}_j$ has shape $(m^{\kappa'_j})$ for each j. We define the *column multiset-tableau* T and *column multiset-tabloid* $|T|$ corresponding to $(\mathcal{Q}_1, \ldots, \mathcal{Q}_k)$ by replacing sets with multisets in Definition 5.1. Note that T and $|T|$ both have shape κ. Let v span $\mathrm{sgn}_{S_{mn}}$. Let

$$h_{(\mathcal{Q}_1,\ldots,\mathcal{Q}_k)} : \widetilde{M}^\lambda \to \mathrm{sgn}_{S_{mn}} \otimes P(\widetilde{M}^\kappa)$$

be the unique $\mathbb{Q}S_{mn}$-homomorphism such that

$$|t_\lambda| h_{(\mathcal{Q}_1,\ldots,\mathcal{Q}_k)} = (v \otimes |T|) b_{t_\lambda}.$$

Proposition 7.2. *The kernel of $h_{(\mathcal{Q}_1,\ldots,\mathcal{Q}_k)}$ contains every t_λ-Garnir element.*

Proof. As before, let $1 \leq i < \lambda_1$ and let $X = \{i_1, \ldots, i_{\lambda'_i}\}$ be the set of entries in column i of t_λ. We have

$$|t_\lambda| G_{X \cup \{(i+1)_1\}} h_{(\mathcal{Q}_1,\ldots,\mathcal{Q}_k)} = \sum_{\tau \in C(t_\lambda)} (v \otimes |T|\tau) G_{X \cup \{(i+1)_1\}} \,\mathrm{sgn}(\tau).$$

To show the right-hand side is zero, it suffices to construct an involution on $C(t_\lambda)$, denoted $\tau \mapsto \tau^\star$, with the following two properties:

(a) if $\tau = \tau^\star$ then $(v \otimes |T|\tau) G_{X \cup \{(i+1)_1\}} = 0$,
(b) if $\tau \neq \tau^\star$ then $(v \otimes |T|\tau + v \otimes |T|\tau^*) G_{X \cup \{(i+1)_1\}} = 0$.

Let $\tau \in C(t_\lambda)$. Consider $|T|\tau$. Suppose that $|T|\tau$ has a column with two entries both entirely contained in $X \cup \{(i+1)_1\}$. Let these entries be $\{i_{d(1)}, i_{d(2)}, \ldots, i_{d(m)}\}$ and $\{(i+1)_1, i_{e(2)}, \ldots, i_{e(m)}\}$. Set

$$\vartheta = (i_{d(1)}, (i+1)_1)(i_{d(2)}, i_{e(2)}) \cdots (i_{d(m)}, i_{e(m)}) \in S_{X\cup\{(i+1)_1\}}.$$

We have $|T|\tau\vartheta = -|T|\tau$ since ϑ swaps two entries in the same column of $|T|\tau$. Since $v\,\mathrm{sgn}(\vartheta)\vartheta = v$, we get

$$(v \otimes |T|\tau)(1 + \mathrm{sgn}(\vartheta)\vartheta) = v \otimes |T|\tau + v \otimes |T|\tau\vartheta = 0.$$

Taking coset representatives for $\langle\vartheta\rangle \leq S_{X\cup\{(i+1)_1\}}$, it follows that $(v \otimes |T|\tau)G_{X\cup\{(i+1)_1\}} = 0$. Hence if we define $\tau^* = \tau$ in this case then (a) holds.

We now assume that each column of $|T|\tau$ has at most one entry contained in $X \cup \{(i+1)_1\}$. Let the entry of $|T|\tau$ containing $(i+1)_1$ be

$$B_{(i+1)_1} = \{i_{e(1)}, \ldots, i_{e(s)}, (i+1)_1, c(1)_{b(1)}, \ldots, c(m-s-1)_{b(m-s-1)}\},$$

where $s \in \mathbb{N}_0$ and $c(1), \ldots, c(m-s-1) \neq i$. Suppose that $B_{(i+1)_1}$ lies in column j of $|T|\tau$. This column is defined using the multiset family $\mathcal{Q}_j$. Since $\mathcal{Q}_j$ is closed, there exist unique symbols $i_{d(1)}, \ldots, i_{d(s)}, i_{d(s+1)}, c(1)_{a(1)}, \ldots, c(m-s-1)_{a(m-s-1)}$ such that the multiset

$$A_{(i+1)_1} = \{i_{d(1)}, \ldots, i_{d(s)}, i_{d(s+1)}, c(1)_{a(1)}, \ldots, c(m-s-1)_{a(m-s-1)}\}$$

is also an entry in column j of $|T|\tau$. Define

$$\vartheta = (i_{d(1)}, i_{e(1)}) \cdots (i_{d(s)}, i_{e(s)})(i_{d(s+1)}, (i+1)_1) \in S_{X\cup\{(i+1)_1\}},$$

and

$$\pi = (c(1)_{a(1)}, c(1)_{b(1)}) \cdots (c(m-s-1)_{a(m-s-1)}, c(m-s-1)_{b(m-s-1)}) \in C(t_\lambda).$$

Our assumption ensures that π is not the identity. Set $\tau^* = \tau\pi$. Since the column set-tabloids $|T|\tau$ and $|T|\tau^\star$ differ only in indices attached to numbers other than i and $i+1$, we have $\tau^{\star\star} = \tau$. Since $\pi\vartheta$ swaps two entries in column j of $|T|$ we have $|T|\tau\pi\vartheta = -|T|\tau$. Hence

$$(v \otimes |T|\tau + v \otimes |T|\tau^*)(1 + \mathrm{sgn}(\vartheta)\vartheta) = \\ v \otimes |T|\tau + v \otimes |T|\tau\vartheta + v \otimes |T|\tau\pi + v \otimes |T|\tau\pi\vartheta = 0.$$

It follows that $(v \otimes |T|\tau + v \otimes |T|\tau^*)G_{X\cup\{(i+1)_1\}} = 0$, as required in (b). $\square$

Therefore $h_{(\mathcal{Q}_1,\ldots,\mathcal{Q}_k)}$ induces a homomorphism $S^\lambda \to \mathrm{sgn}_{S_{mn}} \otimes P(\widetilde{M}^\kappa)$, sending e_{t_λ} to $(v \otimes |T|)b_{t_\lambda}$. Let $\bar{h}_{(\mathcal{Q}_1,\ldots,\mathcal{Q}_k)} : S^\lambda \to \mathrm{sgn}_{S_{mn}} \otimes P(S^\kappa)$ denote the composition of this homomorphism with the canonical quotient map $\mathrm{sgn}_{S_{mn}} \otimes P(\widetilde{M}^\kappa) \to \mathrm{sgn}_{S_{mn}} \otimes P(S^\kappa)$. Thus $e_{t_\lambda}\bar{h}_{(\mathcal{Q}_1,\ldots,\mathcal{Q}_k)} = (v \otimes e_T)b_{t_\lambda}$.

We now obtain the analogues of Lemma 5.3, Corollary 5.4 and Proposition 5.5.

Lemma 7.3. *The homomorphism* $\bar{h}_{(\mathcal{Q}_1,\ldots,\mathcal{Q}_k)} : S^\lambda \to \mathrm{sgn}_{S_{mn}} \otimes P(S^\kappa)$ *is non-zero.*

Proof. The coefficients of $v \otimes \{T\}$ in $(v \otimes e_T)b_{t_\lambda}$ and $(v \otimes \{T\})b_{t_\lambda}$ agree. Since $(v \otimes \{T\})\sigma \,\mathrm{sgn}(\sigma) = v \otimes \{T\}\sigma$, this coefficient is the order of the subgroup of $C(t_\lambda)$ that permutes amongst themselves the indices appearing in each entry of T. In particular this coefficient is non-zero. □

Corollary 7.4.

(i) *If there is a closed multiset family tuple* $(\mathcal{Q}_1, \ldots, \mathcal{Q}_k)$ *of type* λ *such that* $\mathcal{Q}_i$ *has shape* $(m^{\kappa'_i})$ *for each* i*, then* $\langle \psi_\nu^{(m^n)}, \chi^\lambda \rangle \geq 1$.

(ii) *If there is a closed multiset family tuple* $(\mathcal{Q}_1, \ldots, \mathcal{Q}_\ell)$ *of type* λ *such that* $\mathcal{Q}_i$ *has shape* $(m^{\nu'_i})$ *for each* i*, then* $\langle \phi_\nu^{(m^n)}, \chi^{\lambda'} \rangle \geq 1$.

Proof. If m is even then, by Equation (3) in Section 3.3, the codomain of $\bar{h}_{(\mathcal{Q}_1,\ldots,\mathcal{Q}_k)}$ is isomorphic to $Q(S^\nu)$. If m is odd then the codomain of $\bar{h}_{(\mathcal{Q}_1,\ldots,\mathcal{Q}_k)}$ is $\mathrm{sgn}_{S_{mn}} \otimes P(S^{\nu'})$, and by Equations (3) and (4), we have isomorphisms $\mathrm{sgn}_{S_{mn}} \otimes P(S^{\nu'}) \cong Q(\mathrm{sgn}_{S_n} \otimes S^{\nu'}) \cong Q\big((S^\nu)^\star\big) \cong Q(S^\nu)$. Since $Q(S^\nu)$ affords the character $\psi_\nu^{(m^n)}$, part (i) now follows from Lemma 7.3. Part (ii) then follows from part (i) using Equation (5) in Section 3.3. □

Proposition 7.5. *If* χ^μ *is a constituent of* $\psi_\nu^{(m^n)}$ *then there is a multiset family tuple* $(\mathcal{R}_1, \ldots, \mathcal{R}_k)$ *of type* μ *such that* $\mathcal{R}_j$ *has shape* $(m^{\kappa'_j})$ *for each* j.

Proof. Arguing as in the proof of Proposition 5.5 if m is even and as in Remark (5) in Section 6 if m is odd, there a non-zero $\mathbb{Q}S_{mn}$-module homomorphism $f : S^\mu \to \mathrm{sgn}_{S_{mn}} \otimes P(\widetilde{M^\kappa})$. Let T be a set-tableau such that the coefficient of $v \otimes |T|$ in $e_{t_\mu} f$ is non-zero. Suppose that there is a column of T containing entries $\{c(1)_{a(1)}, \ldots, c(m)_{a(m)}\}$ and $\{c(1)_{b(1)}, \ldots, c(m)_{b(m)}\}$ that are equal up the indices attached to numbers. Let

$$\rho = (c(1)_{a(1)}, c(1)_{b(1)}) \ldots (c(m)_{a(m)}, c(m)_{b(m)}).$$

Then $e_{t_\mu}\rho = \mathrm{sgn}(\rho) e_{t_\mu}$, whereas

$$(v \otimes |T|)\rho = \mathrm{sgn}(\rho) v \otimes (-|T|) = -\,\mathrm{sgn}(\rho)(v \otimes |T|),$$

since ρ swaps two entries in a column of T. It follows that removing the indices attached to the numbers in column j of $|T|$ gives a multiset family of shape $(m^{\kappa'_j})$. The multiset family tuple obtained has type μ since the union of the entries in $|T|$ is Ω^μ. □

The proof of Theorem 7.1 is completed in exactly the same manner as that of Theorem 1.3.

8. Corollaries

In this section we present a number of corollaries of Theorems 1.3 and 1.4. These include a description of the lexicographically least partitions labelling an irreducible constituent of $\phi_\nu^{(m^n)}$ or $\psi_\nu^{(m^n)}$, confirming two conjectures of Agaoka [1].

8.1. **The conjectures of Agaoka.** Let ν be a partition of n and set $\kappa = \nu$ if m is even and $\kappa = \nu'$ if m is odd. Let k be the first part of κ. It follows from Theorem 1.3 that the lexicographically least partition λ labelling an irreducible constituent of $\phi_\nu^{(m^n)}$ is the lexicographically least type of a set family tuple $(\mathcal{P}_1, \ldots, \mathcal{P}_k)$ such that each $\mathcal{P}_j$ has shape $(m^{\kappa'_j})$. We draw an analogous conclusion from Theorem 7.1 regarding $\psi_\nu^{(m^n)}$. We therefore have the following corollary, which was conjectured by Agaoka in [1, Conjecture 2.1].

Corollary 8.1. *The lexicographically least partition labelling an irreducible constituent of $\phi_\nu^{(m^n)}$ (respectively $\psi_\nu^{(m^n)}$) is obtained by taking the join of the lexicographically least partitions labelling an irreducible constituent of each $\phi_{(\kappa'_j)}^{(m^{\kappa'_j})}$ (respectively $\psi_{(\kappa'_j)}^{(m^{\kappa'_j})}$).*

The lexicographically least set families are given by the colexicographic order on m-subsets on $\mathbb{N}$. This order is defined on distinct m-sets A and B by $A < B$ if and only if $\max(A\backslash B) < \max(B\backslash A)$. Given an m-subset B of $\mathbb{N}$, let $B^{\leq}$ denote the initial segment of the colexicographic order ending at B; that is, $B^{\leq} = \{A \subseteq \mathbb{N} : |A| = m, A \leq B\}$. If A is an m-subset of $\mathbb{N}$, and r is minimal such that $r \in A$ and $r+1 \notin A$, then the successor to A in the colexicographic is the set $B = \{1, \ldots, s\} \cup \{r+1\} \cup (A\backslash\{1, \ldots, r\})$ where s is chosen so that $|B| = m$. Thus the colexicographic order minimizes the size of the largest element in $B\backslash A$. It follows that if B is an m-subset of $\mathbb{N}$ then $B^{\leq}$ is the lexicographically least set family of its shape.

An explicit construction of the lexicographically least set family of shape (m^n) follows from the basic results on the colexicographic order in [2, Chapter 5]. Pick p_1 such that $\binom{p_1}{m} \leq n < \binom{p_1+1}{m}$ and let $\mathcal{P}^{(1)}$ be the set of all m-subsets of $\{1, 2, \ldots, p_1\}$. Then, for each $i \in \{2, \ldots, m\}$ such that $n > \sum_{j=1}^{i-1} \binom{p_j}{m+1-j}$, pick p_i such that

$$\binom{p_i}{m+1-i} \leq n - \sum_{j=1}^{i-1} \binom{p_j}{m+1-j} < \binom{p_i+1}{m+1-i}$$

and let $\mathcal{P}^{(i)}$ be the union of $\mathcal{P}^{(i-1)}$ with the set of all sets of the form $X \cup \{p_{i-1}+1, \ldots p_1+1\}$ where X is a $(m+1-i)$-subset of $\{1, 2, \ldots p_i\}$. The process terminates with $p_1 > p_2 > \cdots > p_r > 0$ such that $n = \sum_{j=1}^{r} \binom{p_j}{m+1-j}$. The final set family $\mathcal{P}^{(r)}$ has shape (m^n).

The construction of the lexicographically least multiset family of shape (m^n) is entirely analogous. Let $\left(\!\binom{q}{m}\!\right)$ denote the number $\binom{q+m-1}{m}$ of multisets of cardinality m with elements taken from $\{1,\ldots,q\}$. We adapt the above construction and express n as $\sum_{j=1}^{s}\left(\!\binom{q_j}{m+1-j}\!\right)$ for $q_1 \geq q_2 \geq \cdots \geq q_s > 0$, with weak inequalities since repetitions are allowed.

Corollary 8.2. *Set $\kappa = (1^n)$ if n is even and $\kappa = (n)$ if m is odd.*

(i) *Let $p_1,\ldots,p_r$ be as just defined. The lexicographically least partition labelling an irreducible constituent of $\phi_\kappa^{(m^n)}$ is*

$$\big((p_1+1)^{a_1}, p_1^{b_1-a_1}, (p_2+1)^{a_2}, p_2^{b_2-a_2}, \ldots, (p_{r-1}+1)^{a_{r-1}}, p_{r-1}^{b_{r-1}-a_{r-1}}, p_r^{b_r}\big),$$

where $a_i = n - \sum_{j=1}^{i}\binom{p_j}{m+1-j}$ and $b_i = \binom{p_i-1}{m-i}$ for each $i \in \{1,\ldots,r\}$.

(ii) *Let $q_1,\ldots,q_s$ be as just defined. The lexicographically least partition labelling an irreducible constituent of $\psi_\kappa^{(m^n)}$ is*

$$\big((q_1+1)^{c_1}, q_1^{d_1-c_1}, (q_2+1)^{c_2}, q_2^{d_2-c_2}, \ldots, (q_{s-1}+1)^{c_{s-1}}, q_{s-1}^{d_{s-1}-c_{s-1}}, q_s^{d_s}\big),$$

where $c_i = n - \sum_{j=1}^{i}\left(\!\binom{q_j}{m+1-j}\!\right)$ and $d_i = \left(\!\binom{q_i+1}{m-i}\!\right)$ for each $i \in \{1,\ldots,s\}$.

Proof. Let λ be the partition in (i). We note that it is possible that $p_i = p_{i+1}+1$ for one or more indices i; in this case $b_i - a_i$ may be negative, and

$$(\ldots, p_i^{b_i-a_i}, (p_{i+1}+1)^{a_{i+1}}, \ldots)$$

should be interpreted as $(\ldots, p_i^{b_i-a_i+a_{i+1}}, \ldots)$. By Theorem 1.3, it is sufficient to prove that λ is the type of the lexicographically least set family of shape (m^n), as constructed above. Of course this also shows that λ is a well-defined partition.

Let $1 \leq x \leq p_1+1$. Note that if $x \leq p_j$ then x is contained in exactly b_j sets in $\mathcal{P}^{(j)}\backslash\mathcal{P}^{(j-1)}$. It follows that if $p_{j+1}+1 < x \leq p_j$ then x lies in $b_1+\cdots+b_j$ sets in $\mathcal{P}^{(j)}$ and in no other sets in $\mathcal{P}^{(r)}$. This is the number of parts of λ not less than x. If $x = p_i+1$ then $x \leq p_{i-1}$ and so x lies in $b_1+\cdots+b_{i-1}$ sets in $\mathcal{P}^{(i-1)}$ and also in all a_i sets in $\mathcal{P}^{(r)}\backslash\mathcal{P}^{(i)}$. Hence the total multiplicity of x is $b_1+\cdots+b_{i-1}+a_i$, which is again the number of parts of λ not less than x.

The proof of (ii) is similar, replacing sets with multisets, and noting that if $x \in \{1,\ldots,q_1\}$ then the number of multisubsets of $\{1,\ldots,q_1\}$ of cardinality m that contain x with multiplicity at least ℓ is $\left(\!\binom{q_1}{m-\ell}\!\right)$, and so the number of occurrences of x in all multisubsets of $\{1,\ldots,q_1\}$ of cardinality m is given by $\sum_{\ell=1}^{m}\left(\!\binom{q_1}{m-\ell}\!\right) = \left(\!\binom{q_1+1}{m-1}\!\right)$. We note that it is possible that $q_i = q_{i+1}$ for one or more indices and, in this case, it will be necessary to rearrange the parts in the expression given in (ii) to ensure that it is weakly decreasing. $\square$

This result was conjectured by Agaoka in [1, Conjecture 4.2].

Agaoka also conjectured the form of the lexicographically greatest Schur function appearing in $s_\nu \circ s_\mu$ in [1, Conjecture 1.2]. This was proved by was Iijima in [14, Theorem 4.2]. Our results provide an alternative proof in the cases $\mu = (n)$ and (1^n). Suppose that ν has exactly k parts and largest part ℓ. By Theorem 1.4, the lexicographically greatest constituent of $\phi_\nu^{(m^n)}$ is $\chi^{((m-1)n+\nu_1,\nu_2,\ldots,\nu_k)}$, corresponding to the closed multiset family tuple with lexicographically greatest conjugate type, namely $(\mathcal{Q}_1,\ldots,\mathcal{Q}_\ell)$ where

$$\mathcal{Q}_i = \big\{\{1,\ldots,1,1\},\{1,\ldots,1,2\},\ldots,\{1,\ldots,1,\nu'_i\}\big\}$$

for each i. Similarly, by Theorem 1.3 and Equation (5), the lexicographically greatest constituent of $\psi_\nu^{(m^n)}$ is $\chi^{(n^{m-1},\nu_1,\nu_2,\ldots,\nu_k)}$, corresponding to the set family tuple $(\mathcal{P}_1,\ldots,\mathcal{P}_\ell)$ where

$$\mathcal{P}_i = \big\{\{1,2,\ldots,m-1,m\},\{1,2,\ldots,m-1,m+1\},\ldots,\{1,2,\ldots,m-1,m+\nu'_i\}\big\}$$

for each i.

8.2. **Unique minimal or maximal constituents.** It is natural to ask when $\phi_\nu^{(m^n)}$ has a unique minimal or maximal constituent. This is easily answered using our results.

Corollary 8.3. *Let ν be a partition of n. If $m = 1$ then $\phi_\nu^{(m^n)} = \chi^\nu$. If $m > 1$ then $\phi_\nu^{(m^n)}$ has χ^λ as a unique minimal constituent if and only if*

(i) *m is even, $\nu = (n)$ and $\lambda = (m^n)$;*
(ii) *m is even, $\nu = (n-r,r)$ and $\lambda = \big((m+1)^r, m^{n-2r}, (m-1)^r\big)$ where $1 \le r \le n/2$;*
(iii) *m is odd, $\nu = (1^n)$ and $\lambda = (m^n)$;*
(iv) *m is odd, $\nu = (2^r, 1^{n-2r})$ and $\lambda = \big((m+1)^r, m^{n-2r}, (m-1)^r\big)$ where $1 \le r \le n/2$.*

Proof. Suppose that $m > 1$ and $r \ge 3$. Let $\mathcal{P}$ be the lexicographically least set family of shape (m^r). Let $X = \{1,\ldots,m-1\}$ and let

$$\mathcal{R} = \big\{X \cup \{m\}, X \cup \{m+1\}, \ldots, X \cup \{m+r-1\}\big\}.$$

It is easily seen that $\mathcal{R}$ is a minimal set family and that $\mathcal{P}$ and $\mathcal{R}$ have different types. If $r \le 2$ then there is a unique closed set family of shape (r^2). It now follows from Theorem 1.3 that if m is even then $\phi_\nu^{(m^n)}$ has a unique minimal constituent, of the type claimed in (i) and (ii), if and only if $\nu'_1 \le 2$. The proof is similar when m is odd. □

Corollary 8.4. *Let ν be a partition of n. If $m > 1$ then $\phi_\nu^{(m^n)}$ has a unique maximal constituent if and only if ν has at most two rows. The unique maximal constituent of $\phi_{(n)}^{(m^n)}$ is $\chi^{(mn)}$ and the unique maximal constituent of $\phi_{(n-r,r)}^{(m^n)}$ is $\chi^{(mn-r,r)}$.*

Proof. Let $\mathcal{P}$ be the lexicographically least multiset family of shape (m^r), let $\mathcal{R} = \big\{\{1,1,\ldots,1,1\},\{1,1,\ldots,1,2\},\ldots,\{1,1,\ldots,1,r\}\big\}$, and argue as in Corollary 8.3, replacing Theorem 1.3 with Theorem 1.4. $\square$

8.3. **Further constituents.** We remark that since there are closed set families and closed multiset families that are not minimal, Corollary 5.4 is not implied by Theorem 1.3 and neither is Corollary 7.4 implied by Theorem 1.4. For example, let $\mathcal{P}_1$ denote those 2-sets majorized by $\{2,4\}$ and let $\mathcal{P}_2$ be those majorized by $\{1,5\}$. Let $\mathcal{R}_1$ be obtained from $\mathcal{P}_1$ by replacing $\{2,4\}$ with $\{1,5\}$, and let $\mathcal{R}_2$ be obtained from $\mathcal{P}_2$ by replacing $\{1,5\}$ with $\{2,3\}$. Then the set family tuple $(\mathcal{P}_1,\mathcal{P}_2)$ is closed but not minimal since $(\mathcal{R}_1,\mathcal{R}_2)$ has strictly smaller type.

8.4. **Rectangular partitions.** As in Section 3.4, let Δ^λ be the Schur functor corresponding to the partition λ. Let a, $b \in \mathbb{N}$. By Section 3.4, $\chi^{(a^b)}$ is a constituent of $\phi_\nu^{(m^n)}$ if and only if $\Delta^{(a^b)}(E)$ appears in $\Delta^\nu\big(\mathrm{Sym}^m E\big)$, where E is a rational vector space of dimension at least b. If E has dimension exactly b then $\Delta^{(a^b)}(E) \cong (\bigwedge^b E)^{\otimes a}$ and so $\Delta^{(a^b)}(E)$ affords the polynomial representation $g \mapsto \det(g)^a$ of $\mathrm{GL}(E)$. It follows that there is a non-zero $\mathrm{SL}(E)$-invariant subspace of $\Delta^\nu\big(\mathrm{Sym}^m E\big)$. This observation motivates the following result.

Corollary 8.5. *Let $a \in \mathbb{N}$ be such that $a \geq m$.*

(i) *If m is odd let ν denote the partition $(\binom{a}{m},\ldots,\binom{a}{m})$ where there are exactly k parts and, if m is even, let ν denote the conjugate of this partition. Set $n = k\binom{a}{m}$ and $b = k\binom{a-1}{m-1}$. Then*
$$\langle \phi_\nu^{(m^n)}, \chi^{(a^b)} \rangle \geq 1.$$

(ii) *Let ν denote $(\left(\!\binom{a}{m}\!\right),\ldots,\left(\!\binom{a}{m}\!\right))$ where there are exactly k parts. Set $n = k\left(\!\binom{a}{m}\!\right)$ and $b = k\left(\!\binom{a+1}{m-1}\!\right)$. Then*
$$\langle \phi_\nu^{(m^n)}, \chi^{(b^a)} \rangle \geq 1.$$

Proof. Consider the set family tuple $(\mathcal{P},\ldots,\mathcal{P})$ where $\mathcal{P}$ consists of all m-subsets of $\{1,\ldots,a\}$. The shape of $\mathcal{P}$ is $(m^{\binom{a}{m}})$ and the type of the set family tuple is $(a^{k\binom{a-1}{m-1}})$. Since $\mathcal{P}$ is clearly closed, the first statement in the corollary now follows from Corollary 5.4, and its analogue for m odd. Replacing $\mathcal{P}$ with the set of all multisets of cardinality m with entries taken from $\{1,\ldots,a\}$, the counting argument in the proof of Corollary 8.2 shows that we obtain a multiset family tuple of type $(a^{k\left(\!\binom{a+1}{m-1}\!\right)})$. The second statement now follows similarly from Corollary 7.4(ii). $\square$

For example, $\phi_{(4,4)}^{(3^8)}$ contains $\chi^{(4^6)}$; the corresponding set family is $\mathcal{P} = \big\{\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\}\big\}$. In fact, by Section 8.1, every closed set family tuple or closed multiset family tuple whose type is a rectangular partition arises from the construction in Corollary 8.5. We note that in general there are further constituents of $\phi_\nu^{(m^n)}$ labelled by rectangular partitions that are not given by this construction. For example, $\chi^{(4^2)}$ appears in $\phi_{(2,2)}^{(2^4)}$.

8.5. **The decomposition of $\phi_{(1^n)}^{(2^n)}$.** Let $\vartheta_n = \phi_{(1^n)}^{(2^n)}$. Remarkably every constituent of ϑ_n is both minimal and maximal. We end by proving this as part of the following corollary, which gives a new proof of the decomposition of $\phi_{(1^n)}^{(2^n)}$. A notable feature of this proof is that each constituent is determined by an explicitly defined homomorphism. For an earlier proof of Corollary 8.6 using symmetric functions see [20, I. 8, Exercise 6(d)].

Given a partition α of n with distinct parts $(\alpha_1, \ldots, \alpha_r)$, let $2[\alpha]$ denote the partition λ of $2n$ such that the leading diagonal hook-lengths of λ are $2\alpha_1, \ldots, 2\alpha_r$ and $\lambda_i = \alpha_i + i$ for $1 \leq i \leq r$.

Corollary 8.6. *For any $n \in \mathbb{N}$ we have*

$$\vartheta_n = \sum_\alpha \chi^{2[\alpha]}$$

where the sum is over all such partitions α of n with distinct parts.

Proof. By Theorem 1.3, the minimal constituents of ϑ_n are given by the types of the minimal set families $\mathcal{P}$ of shape (m^n). By Theorem 1.4, the maximal constituents of ϑ_n are given by the conjugates of the types of the minimal multiset families $\mathcal{Q}$ of shape (m^n). The closed set families of shape (2^n) are

$$\bigcup_{i=1}^{r} \big\{\{i, i+1\}, \ldots, \{i, i+\alpha_i\}\big\}$$

for any $\alpha_1 > \alpha_2 > \cdots > \alpha_r$ with $\sum_{i=1}^r \alpha_i = n$. Such a set family has type $2[\alpha]$. All such partitions $2[\alpha]$ of $2n$ are incomparable in the dominance order and therefore all label minimal constituents of ϑ_n. However, $2[\alpha]'$ is the type of the closed multiset family

$$\bigcup_{i=1}^{r} \big\{\{i, i\}, \ldots, \{i, i+\alpha_i - 1\}\big\}$$

and hence every minimal constituent is also maximal. We conclude that ϑ_n has no further constituents. $\square$

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