NOTES ON THE WEYL CHARACTER FORMULA

The aim of these notes is to give a self-contained algebraic proof of the Weyl Character Formula. The necessary background results on modules for \( \mathfrak{sl}_2(\mathbb{C}) \) and complex semisimple Lie algebras are outlined in the first two sections. Some technical details are left to the exercises at the end; solutions are provided when the exercise is needed for the proof.

1. REPRESENTATIONS OF \( \mathfrak{sl}_2(\mathbb{C}) \)

Define
\[
  h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}
\]
and note that \( \langle h, e, f \rangle = \mathfrak{sl}_2(\mathbb{C}) \). Let \( u, v \) be the canonical basis of \( E = \mathbb{C}^2 \). Then each \( \text{Sym}^d E \) is irreducible with \( u^d \) spanning the highest-weight space of weight \( d \) and, up to isomorphism, \( \text{Sym}^d E \) is the unique irreducible \( \mathfrak{sl}_2(\mathbb{C}) \)-module with highest weight \( d \). (See Exercises 1.1 and 1.2.) The diagram below shows the actions of \( h, e \) and \( f \) on the canonical basis of \( \text{Sym}^d E \): loops show the action of \( h \), arrows to the right show the action of \( e \), arrow to the left show the action of \( f \).

In particular

(a) the eigenvalues of \( h \) on \( \text{Sym}^d E \) are \( -d, -d+2, \ldots, d-2, d \) and each \( h \)-eigenspace is 1-dimensional,

(b) if \( w \in \text{Sym}^d E \) and \( h \cdot w = (d - 2c)w \) then \( f \cdot e \cdot w = c (d - c + 1)w \).

If \( V \) is an arbitrary \( \mathfrak{sl}_2(\mathbb{C}) \)-module then, by Weyl’s Theorem (see [1, Appendix B] or [3, §6.3]), \( V \) decomposes as a direct sum of irreducible \( \mathfrak{sl}_2(\mathbb{C}) \)-submodules. Let \( V_r = \{ v \in V : h \cdot w = rv \} \) for \( r \in \mathbb{Z} \). Then (a) implies

(c) if \( r \geq 0 \) then the number of irreducible summands of \( V \) with highest weight \( r \) is \( \dim V_r - \dim V_{r+2} \).

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2. PREREQUISITES ON COMPLEX SEMISIMPLE LIE ALGEBRAS

In this section we recall the basic setup of a Cartan subalgebra \( H \) inside a complex semisimple Lie algebra \( L \), a lattice of weights \( \Lambda \subseteq H^*_R \) and a root system \( \Phi \subseteq \Lambda \). The mathematically most interesting parts are that \( H \) is self-centralizing (see Exercise 2.2) and the trick used to construct an \( \mathfrak{sl}_2(C) \)-subalgebra corresponding to each root. For an example of all the theory below, see Exercise 2.7.

**Cartan subalgebras.** We define a *Cartan subalgebra* of \( L \) to be a Lie subalgebra \( H \) of \( L \) maximal subject to the condition that \( \text{ad} h : L \to L \) is diagonalizable for all \( h \in H \). It is an interesting fact (see Exercise 2.1) that any Cartan subalgebra is abelian. We may therefore decompose \( L \) as a direct sum of simultaneous eigenspaces for the elements of \( H \). To each simultaneous eigenspace \( V \) we associate the unique \( \alpha \in H^* \) such that \((\text{ad} h)x = \alpha(h)x \) for all \( h \in H \) and \( x \in V \). For \( \alpha \in H^* \) let

\[ L_\alpha = \{ x \in L : (\text{ad} h)x = \alpha(h)x \text{ for all } h \in H, x \in V \} \]

and let \( \Phi \) be the set of all non-zero \( \alpha \in H^* \) such that \( L_\alpha \neq 0 \). The elements of \( \Phi \) are called *roots* and \( L_\alpha \) is the *root space* corresponding to \( \alpha \in \Phi \) and we have

\[ L = L_0 \oplus \bigoplus_{\alpha \in \Phi} L_\alpha. \]

Note that \( L_0 \) is the centralizer of \( H \) in \( L \). It is an important and non-obvious fact (see Exercise 2.2) that \( L_0 = H \), so \( H \) is self-centralizing: An easy calculation shows that

\[ [L_\alpha, L_\beta] \subseteq L_{\alpha + \beta} \quad \text{for all } \alpha, \beta \in \Phi_0. \]

**Killing form.** The *Killing form* on \( L \) is the bilinear form \( \kappa(x, y) = \text{Tr}(\text{ad} x \circ \text{ad} y) \). By Cartan’s Criterion \( \kappa \) is non-degenerate. It follows from (1) that if \( x \in L_\alpha \) and \( y \in L_\beta \) where \( \alpha, \beta \in \Phi_0 \), then \( \text{ad} x \circ \text{ad} y \) is nilpotent, unless \( \alpha + \beta = 0 \). Therefore if \( \alpha, \beta \in \Phi_0 \) then \( L_\alpha \perp L_\beta \) unless \( \beta = -\alpha \). Hence \( \alpha \) is a root if and only if \( -\alpha \) is a root and the restriction of \( \kappa \) to \( L_\alpha \times L_{-\alpha} \) is non-degenerate. In particular, the restriction of \( \kappa \) to \( H \times H \) is non-degenerate. For each \( \alpha \in \Phi \), let \( t_\alpha \in H \) be the unique element of \( H \) such that

\[ \kappa(t_\alpha, h) = \alpha(h) \quad \text{for all } h \in H. \]

**\( \mathfrak{sl}_2 \) subalgebras.** Choose \( e \in L_\alpha \) and \( f \in L_{-\alpha} \) such that \( \kappa(e, f) \neq 0 \). By the associativity of the Killing form

\[ \kappa(h, [e, f]) = \kappa([h, e], f) = \alpha(h)\kappa(e, f) \quad \text{for all } h \in H. \]
Since $\kappa$ is non-degenerate on $H$, there exists $h \in H$ such that $\alpha(h) = \kappa(t_\alpha, h) \neq 0$. Since $\kappa(e, f) \neq 0$, the previous equation then implies that $[e, f] \neq 0$. Consider the Lie subalgebra

$$\langle e, f, [e, f] \rangle$$

of $L$. Since $[e, f] \in [L_\alpha, L_{-\alpha}] \subseteq H$ we have $[\langle e, f \rangle, e] = \alpha([e, f])e$ and $[[e, f], f] = -\alpha([e, f])f$.

If $\alpha([e, f]) = 0$ then $[e, f]$ is central in $\langle e, f, [e, f] \rangle$. By Exercise 2.3 below $[e, f]$ is nilpotent. But $[e, f] \in H$ and all the elements of $H$ are semisimple. So $[e, f] = 0$, which contradicts the previous paragraph. Therefore $\alpha([e, f]) \neq 0$ and we can scale $e$ so that $\alpha([e, f]) = 2$ and so $\langle e, f, [e, f] \rangle \cong \mathfrak{sl}_2(C)$.

For each $\alpha \in \Phi$ let $\langle e_\alpha, f_\alpha, h_\alpha \rangle$ be a subalgebra of $L$ constructed as above so that

$$[e_\alpha, f_\alpha] = h_\alpha, \quad [h_\alpha, e_\alpha] = 2e_\alpha, \quad [h_\alpha, f_\alpha] = 2f_\alpha.$$ We may suppose that these elements are chosen so that $e_{-\alpha} = f_\alpha$ and $f_{-\alpha} = e_\alpha$ for each $\alpha \in \Phi$.

**Relationship between $t_\alpha$ and $h_\alpha$.** By choice of $t_\alpha$ we have $\kappa(t_\alpha, h) = \alpha(h)$ for all $h \in H$. By associativity of the Killing form we also have

$$\kappa([e_\alpha, f_\alpha], h) = \kappa(e_\alpha, [f_\alpha, h]) = \kappa(e_\alpha, \alpha(h)f_\alpha) = \alpha(h)\kappa(e_\alpha, f_\alpha).$$

Hence

$$\kappa \left( t_\alpha - \frac{[e_\alpha, f_\alpha]}{\kappa(e_\alpha, f_\alpha)}, h \right) = 0 \quad \text{for all } h \in H.$$ Since the restriction of $\kappa$ to $H \times H$ is non-degenerate it follows that

$$t_\alpha = \frac{h_\alpha}{\kappa(e_\alpha, f_\alpha)}.$$ Since $\kappa(t_\alpha, t_\alpha) = \alpha(t_\alpha)$, this implies the useful relations

$$2 = \alpha(h_\alpha) = \kappa(t_\alpha, h_\alpha) = \frac{\kappa(h_\alpha, h_\alpha)}{\kappa(e_\alpha, f_\alpha)} = \kappa(e_\alpha, f_\alpha)\kappa(t_\alpha, t_\alpha).$$

**Transport of the Killing form to $H^*_R$.** We saw earlier that for all $\alpha \in \Phi$ there exists $h \in H$ such that $\alpha(h) \neq 0$. It follows that $\Phi$ spans $H^*$ and there is a unique bilinear form $(\ , \ )$ on $H^*$ such that

$$(\alpha, \beta) = \kappa(t_\alpha, t_\beta) \quad \text{for } \alpha, \beta \in \Phi.$$ By (2) and (3) we have the fundamental formula

$$\frac{2(\alpha, \beta)}{(\beta, \beta)} = \kappa \left( t_\alpha, \frac{2t_\beta}{\kappa(t_\beta, t_\beta)} \right) = \kappa(t_\alpha, h_\beta) = \alpha(h_\beta).$$

Note also that $\alpha(h_\beta)$ is an eigenvalue of $h_\beta$ in the finite-dimensional $\mathfrak{sl}(\beta)$-module $L$. It follows that $(\ , \ )$ takes real values on the roots and from the
equation $\kappa(h, k) = \sum_{\alpha \in \Phi} \alpha(h) \alpha(k)$ for $h, k \in H$, we see that it is a real-valued inner-product on $H^*_R = \langle \alpha : \alpha \in \Phi \rangle_R$. Exercise 2.4 shows that the angles between the roots are determined by (4). (In fact if $L$ is a simple Lie algebra then $\Phi$ is a connected root system and $(\ , \ )$ is completely determined by (4) and $(\alpha, \alpha)$ for any single root $\alpha \in \Phi$.)

**Angled brackets notation.** It will be convenient to define

$$\langle \lambda, \mu \rangle = \frac{2(\lambda, \mu)}{(\mu, \mu)}$$

for $\lambda, \mu \in H^*_R$. Note that the form $(\ , \ )$ is linear only in its first component. This notation will often be used when $\mu \in \Phi$, in which case (4) implies that $\langle \lambda, \beta \rangle = \lambda(h_{\beta})$.

**Fundamental dominant weights.** Recall that $\{\alpha_1, \ldots, \alpha_\ell\}$ is a base for $\Phi$ if every element of $\Phi$ can be written uniquely as either a non-negative or non-positive integral linear combination of the $\alpha_i$. (For a proof that every root system has a basis, see [1, Theorem 11.10] or [3, Theorem 10.1].) Fix, once and for all, a base $\{\alpha_1, \ldots, \alpha_\ell\}$ for $\Phi$ and let $\Phi^+$ be the set of positive roots with respect to this basis. There exist unique $\omega_1, \ldots, \omega_\ell \in H^*_R$ such that, for all $i, j \in \{1, \ldots, \ell\}$,

$$\omega_i(h_{\alpha_j}) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Let $\Lambda = \langle \omega_1, \ldots, \omega_\ell \rangle_Z \subseteq H^*$.

**Weight space decomposition.** The elements of $H$ act semisimply in any finite-dimensional $L$-module (see [3, Corollary 6.3]). By Section 1, the eigenvalues of each $h_{\alpha_j}$ are integral. Hence if $V$ is a finite-dimensional $L$-module then

$$V \downarrow_H = \bigoplus_{\lambda \in \Lambda} V_\lambda$$

where

$$V_\lambda = \{ v \in V : h \cdot v = \lambda(h)v \text{ for all } h \in H \}.$$ 

(The root spaces defined earlier are weight spaces for the action of $L$ on itself by the adjoint representation.) We shall say that an element of $V$ lying in some non-zero $V_\lambda$ is a weight vector. Starting with any weight vector, and then repeatedly applying the elements $e_\alpha$ for $\alpha \in \Phi^+$, it follows that $V$ contains a weight vector $v$ such that $e_\alpha \cdot v = 0$ for all $\alpha \in \Phi^+$. Such a vector is said to be a highest-weight vector with respect to the base $\{\alpha_1, \ldots, \alpha_\ell\}$. By Exercise 2.6, the submodule of $V$ generated by a highest weight vector is irreducible.
3. Freudenthal’s Formula

Let $V$ be an irreducible $L$-module of highest weight $\mu \in \Lambda$. Let $n_\nu = \dim V_\nu$ for each $\nu \in \Lambda$. The aim of this section is to prove Freudenthal’s Formula, that if $\lambda \in \Lambda$ then

$$(||\mu + \delta||^2 - ||\lambda + \delta||^2)n_\lambda = 2 \sum_{\alpha \in \Phi^+} \sum_{m=1}^{\infty} n_{\lambda+m\alpha}(\lambda + m\alpha, \alpha)$$

where $\delta = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$. The key idea in this proof (which is based on [5, VIII.2]) is to calculate the scalar by which a central element in the universal enveloping algebra $U(L)$ acts on $V$, using the theory of $\mathfrak{sl}_2(\mathbb{C})$-modules in Section 1. The following lemma gives a construction of such central elements.

**Lemma 3.1.** Suppose that $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ are bases of $L$ such that

$$\kappa(x_i, y_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

Then $\sum_{i=1}^{n} x_i y_i$ is in the centre of $U(L)$.

**Proof.** See Exercise 3.3. \hfill \Box

Let $\alpha, \beta \in \Phi$. By (1) we have $\kappa(e_\alpha, f_\beta) = 0$ whenever $\alpha \neq \beta$ and by (3) we have $\kappa(e_\alpha, f_\alpha) = 2/\kappa(t_\alpha, t_\alpha) = 2/(\alpha, \alpha)$ and $\kappa(t_\alpha, h_\alpha) = 2$ for all $\alpha \in \Phi$. Lemma 3.1 therefore implies that

$$\Gamma = \sum_{\alpha \in \Phi^+} \frac{(\alpha, \alpha)}{2} f_\alpha e_\alpha + \frac{1}{2} \sum_{j=1}^{\ell} t_{\alpha_j} h_{\alpha_j}$$

is in the centre of $U(L)$. We may assume that if $\alpha \in \Phi^+$ then $e_{-\alpha} = f_\alpha$ and $f_{-\alpha} = e_\alpha$. Hence $f_{-\alpha} e_{-\alpha} = e_\alpha f_\alpha = h_\alpha + f_\alpha e_\alpha$ and

$$\Gamma = \sum_{\alpha \in \Phi^+} \frac{(\alpha, \alpha)}{2} h_\alpha + \sum_{\alpha \in \Phi^+} (\alpha, \alpha) f_\alpha e_\alpha + \frac{1}{2} \sum_{j=1}^{\ell} t_{\alpha_j} h_{\alpha_j}.$$

The action of each of the three summands of $\Gamma$ preserves the weight spaces $V_\lambda$. The next three lemmas determine the traces of these summands on each $V_\lambda$. The first explains the appearance of $\delta$ in Freudenthal’s Formula.

**Lemma 3.2.** If $\lambda \in \Lambda$ and $v \in V_\lambda$ then

$$\sum_{\alpha \in \Phi^+} \frac{(\alpha, \alpha)}{2} h_\alpha \cdot v = (\lambda, 2\delta)v.$$

**Proof.** Using (4) we get

$$\sum_{\alpha \in \Phi^+} \frac{(\alpha, \alpha)}{2} \lambda(h_\alpha) = \sum_{\alpha \in \Phi^+} \frac{(\alpha, \alpha)}{2} \frac{2(\lambda, \alpha)}{2(\alpha, \alpha)} = \sum_{\alpha \in \Phi^+} (\lambda, \alpha) = (\lambda, 2\delta)$$

as required. \hfill \Box
Lemma 3.3. If \( \alpha \in \Phi \) and \( \lambda \in \Lambda \) then
\[
(\alpha, \alpha) \text{Tr}_{\lambda}(f_\alpha e_\alpha) = 2 \sum_{m=1}^{\infty} n_{\lambda+ma}(\lambda + ma, \alpha).
\]

Proof. Since \( \frac{2(\lambda + ma, \alpha)}{(\alpha, \alpha)} = \langle \lambda + ma, \alpha \rangle \), it is equivalent to prove that
\[
\text{Tr}_{\lambda}(f_\alpha e_\alpha) = \sum_{m=1}^{\infty} n_{\lambda+ma}\langle \lambda + ma, \alpha \rangle.
\]

Let \( W = \bigoplus_{c \in \mathbb{Z}} V_{\lambda + ca} \). Note that \( W \) is a direct sum of weight spaces for the action of \( H \), and that \( W \) is an \( \mathfrak{sl}(\alpha) \)-submodule of \( V \). We may write
\[
W = U^{(1)} \oplus \cdots \oplus U^{(d)}
\]
where each \( U^{(i)} \) is an irreducible \( \mathfrak{sl}(\alpha) \)-module.

Assume first of all that \( \lambda(h_\alpha) \geq 0 \). Suppose that \( U^{(i)} \lambda \neq 0 \). Choose \( m \) maximal such that \( U^{(i)} \lambda + m\alpha \neq 0 \). Then \( U^{(i)} \lambda \) has highest weight \( (\lambda + m\alpha)(h_\alpha) \) as an \( \mathfrak{sl}(\alpha) \)-module and by (b) in Section 1, the scalar by which \( f_\alpha e_\alpha \) acts on a vector in \( U^{(i)} \lambda \) is
\[
m((\lambda + m\alpha)(h_\alpha) - m + 1) = m(\lambda(h_\alpha) + m + 1).
\]
It follows from (c) in Section 1 that the number of summands \( U^{(i)} \) with highest weight \( (\lambda + m\alpha)(h_\alpha) \) as an \( \mathfrak{sl}(\alpha) \)-module is \( n_{\lambda+ma} - n_{\lambda+(m+1)a} \). Hence
\[
\text{Tr}_{\lambda}(f_\alpha e_\alpha) = \sum_{m=0}^{\infty} (n_{\lambda+ma} - n_{\lambda+(m+1)a}) m(\lambda(h_\alpha) + m + 1)
= \sum_{m=1}^{\infty} n_{\lambda+ma} \left( m(\lambda(h_\alpha) + m + 1) - (m - 1)(\lambda(h_\alpha) + m) \right)
= \sum_{m=1}^{\infty} n_{\lambda+ma}(\lambda(h_\alpha) + 2m).
\]
as required. Note that this equation holds even when \( V_\lambda = 0 \), since the argument just given shows that both sides are zero.

If \( \lambda(h_\alpha) \leq 0 \) then a similar calculation (see Exercise 3.4) shows that \( f_\alpha e_\alpha \) acts as the scalar \( -\sum_{b=0}^{\infty} n_{\lambda-ba}(\lambda - ba, \alpha) \) on \( V_\lambda \). Now \( \sum_{c=-\infty}^{\infty} n_{\lambda+ca}(\lambda + ca, \alpha) = 0 \) since each irreducible summand \( U^{(i)} \) contributes the sum of the \( h_\alpha \) eigenvalues on \( U^{(i)} \), which is 0 by (a) in Section 1. Adding these two equations we get the required formula. \( \square \)

Lemma 3.4. Let \( \lambda \in \Lambda \). If \( v \in V_\lambda \) then
\[
\frac{1}{2} \sum_{j=1}^{\ell} t_{\alpha_j} h_{\alpha_j} \cdot v = (\lambda, \lambda)v.
\]
Proof. We saw earlier that \( \frac{1}{2} t_{\alpha_1}, \ldots, \frac{1}{2} t_{\alpha_\ell} \) and \( h_{\alpha_1}, \ldots, h_{\alpha_\ell} \) are dual bases of \( H^\ast \) with respect to the Killing form \( \kappa \) on \( H \times H \). By Exercise 3.2(ii)

\[
\frac{1}{2} \sum_{i=1}^\ell \lambda(t_{\alpha_i})\lambda(h_{\alpha_i}) = (\ell, \ell)
\]
as required. \( \Box \)

Since \( \Gamma \) is central in \( U(L) \) it acts as a scalar on \( V \), say \( \gamma \). Let \( \lambda \in \Lambda \). By Lemmas 3.2, 3.3 and 3.4, we have

\[
n_\lambda \gamma = \text{Tr}_{V_\lambda}(f_\alpha e_\alpha) = (\lambda, 2\delta)n_\lambda + 2 \sum_{\alpha \in \Phi^+} \sum_{m=1}^\infty n_{\lambda + m\alpha}(\lambda + m\alpha, \alpha) + (\lambda, \lambda)n_\lambda.
\]

Recall that \( V \) has highest weight \( \mu \). Since \( e_\alpha \cdot V_\mu = 0 \) for all \( \alpha \in \Phi^+ \), \( n_\mu = 1 \), and \( (\lambda, 2\delta) + (\lambda, \lambda) = ||\lambda + \delta||^2 - ||\delta||^2 \), the previous equation implies

\[
\gamma = ||\mu + \delta||^2 - ||\delta||^2.
\]

Comparing these two equations we obtain

\[
(||\mu + \delta||^2 - ||\lambda + \delta||^2)n_\lambda = 2 \sum_{\alpha \in \Phi^+} \sum_{m=1}^\infty n_{\lambda + m\alpha}(\lambda + m\alpha, \alpha)
\]
as stated in Freudenthal’s Formula. For an immediate application of Freudenthal’s Formula see Exercise 3.5 below.

4. Statement of Weyl Character Formula

Formal exponentials and characters. For each \( \lambda \in \Lambda \) we introduce a formal symbol \( e(\lambda) \) which we call the formal exponential of \( \lambda \). Let \( Q[\Lambda] \) denote the abelian group with \( \mathbb{Z} \)-basis \( \{ e(\lambda) : \lambda \in \Lambda \} \). We make \( Q[\Lambda] \) into an algebra by defining the multiplication on basis elements by

\[
e(\lambda)e(\lambda') = e(\lambda + \lambda') \quad \text{for} \; \lambda, \lambda' \in \Lambda.
\]

Note that \( e(0) = 1 \) and that each \( e(\lambda) \) is invertible, with inverse \( e(-\lambda) \). This definition is motivated by 1-parameter subgroups: see Exercise 4.1. Given an \( L \)-module \( V \), we define the formal character of \( L \) by

\[
\chi_V = \sum_{\lambda \in \Lambda} (\dim V_\lambda)e(\lambda) \in Q[\Lambda].
\]

Weyl group. Let \( S_\beta : H^*_R \rightarrow H^*_R \) denote the reflection corresponding to \( \beta \in \Phi \) as defined by

\[
S_\beta(\theta) = \theta - \frac{2(\theta, \beta)}{(\beta, \beta)} \beta \quad \text{for} \; \theta \in H^*_R.
\]

The alternative forms \( S_\beta(\theta) = \theta - (\theta, \beta)\beta = w - \theta(h_\beta)\alpha \) are often useful. By definition the Weyl group of \( L \) is the group generated by the \( S_\beta \) for \( \beta \in \Phi \). We define \( \varepsilon(w) = 1 \) if \( w \) is a product of an even number of reflections, and
\(\varepsilon(w) = -1\) otherwise. The Weyl group \(W\) acts on \(Q[\Lambda]\) by \(w \cdot e(\lambda) = e(w(\lambda))\) for \(w \in W\) and \(\lambda \in \Lambda\).

**Symmetric and antisymmetric elements.** We say that an element \(f \in Q[\Lambda]\) is symmetric if \(w \cdot f = f\) for all \(w \in W\) and antisymmetric if \(w \cdot f = \varepsilon(w)f\) for all \(w \in W\). By Exercise 4.3(iv), \(f \in Q[\Lambda]\) is antisymmetric if and only if

\[
    f = g \sum_{w \in W} \varepsilon(w) w \cdot e(\delta)
\]

for some symmetric \(g\).

**Weyl Character Formula.** We may now state the main result. By the result on antisymmetric elements of \(Q[\Lambda]\) just mentioned, the right-hand side in the formula below is a well-defined symmetric element of \(Q[\Lambda]\).

**Theorem 4.1** (Weyl Character Formula). Let \(V\) be the irreducible \(L\)-module of highest weight \(\mu \in \Lambda\). Then

\[
    \chi_V = \sum_{w \in W} \varepsilon(w) w \cdot e(\mu + \delta) / \sum_{w \in W} \varepsilon(w) w \cdot e(\delta)
\]

Some applications of the Weyl Character Formula are given in Exercises 4.4, 4.5 and 4.6. Kostant’s Multiplicity Formula (see for instance [2, §8.2]) is also a quick corollary.

## 5. Proof of the Weyl Character Formula

The following proof is adapted from Igusa’s notes [4]. For calculations it will be convenient to extend \(Q[\Lambda]\) to a larger ring \(Q[\frac{1}{2}\Lambda]\) by adjoining a square root \(e(\frac{1}{2}\alpha)\) for each \(\alpha \in \Phi\). We then complete \(Q[\frac{1}{2}\Lambda]\) to the algebra \(Q[\frac{1}{2}\Lambda]\) of formal power series generated by the \(e(\frac{1}{2}\lambda)\) for \(\lambda \in \Lambda\). For example, in \(Q[\frac{1}{2}\Lambda]\) we have \(\sum_{s=0}^{\infty} e(\lambda)^s = \frac{1}{1 - e(\lambda)}\).

We shall also need the *Laplacian operator* \(\Delta : Q[\frac{1}{2}\Lambda] \to Q[\frac{1}{2}\Lambda]\), defined by \(\Delta(e(\lambda)) = ||\lambda||^2 e(\lambda)\) for \(\lambda \in \frac{1}{2}\Lambda\), and the bilinear form \(\{ , \}\) on \(Q[\frac{1}{2}\Lambda]\) defined by

\[
    \{e(\lambda), e(\mu)\} = (\lambda, \mu)e(\lambda + \mu) \text{ for } \lambda, \mu \in \frac{1}{2}\Lambda.
\]

See Exercise 4.3(i) and (iv) for some motivation for \(\Delta\). These gadgets are related by the following lemma.

**Lemma 5.1.** Let \(f, g \in Q[\frac{1}{2}\Lambda]\). Then

\[
    \Delta(fg) = f\Delta(g) + \Delta(f)g + 2\{f, g\}.
\]

**Proof.** By linearity it is sufficient to prove the lemma when \(f = e(\lambda)\) and \(g = e(\mu)\) for some \(\lambda, \mu \in \frac{1}{2}\Lambda\). In this case it states that

\[
    ||\lambda + \mu||^2 e(\lambda + \mu) = e(\lambda)||\mu||^2 e(\mu) + ||\lambda^2|| e(\lambda) e(\mu) + 2(\lambda, \mu)e(\lambda + \mu)
\]

which is obvious. \(\square\)
Proof of Weyl Character Formula. Let $Q$ denote the denominator in the Weyl Character Formula. We begin the proof with Freudenthal’s formula in the form

$$
(||\mu + \delta||^2 - ||\delta||^2)n_\lambda = (||\lambda||^2 + (\lambda, 2\delta))n_\lambda + 2 \sum_{\alpha \in \Phi^+} \sum_{m=1}^{\infty} (\lambda + m\alpha, \alpha)n_{\lambda + m\alpha}.
$$

Multiply both sides by $e(\lambda)$ and sum over all $\lambda \in \Lambda$ to get

$$
(||\mu + \delta||^2 - ||\delta||^2)\chi_V = \Delta(\chi_V) + \sum_{\lambda \in \Lambda} (\lambda, 2\delta)n_\lambda e(\lambda) + X
$$

where

$$
X = 2 \sum_{\lambda \in \Lambda} \sum_{\alpha \in \Phi^+} \sum_{m=1}^{\infty} (\lambda + m\alpha, \alpha)n_{\lambda + m\alpha} e(\lambda)
$$

Now multiply through by $Q$ and replace $2\delta$ with $\sum_{\alpha \in \Phi^+} \alpha$ to combine the second two summands on the right-hand side of (5). This gives

$$
(||\mu + \delta||^2 - ||\delta||^2)Q\chi_V = Q\Delta(\chi_V) + Q \sum_{\lambda \in \Lambda} \sum_{\alpha \in \Phi^+} (\lambda, \alpha)n_\lambda e(\lambda) \frac{e(\alpha) + 1}{e(\alpha) - 1}.
$$

Since $Q\chi_V$ is antisymmetric, it follows from Exercise 4.3(i) that $Q\chi_V = \sum_{w \in W} \varepsilon(w) w \cdot e(\mu + \delta)$ if and only if $\Delta(Q\chi_V) = ||\delta||^2 Q\chi_V$. Again by this exercise, $\Delta(Q) = ||\delta||^2 Q$. Hence it is sufficient to prove

$$
\Delta(Q\chi_V) - \Delta(Q)\chi_V - Q\Delta(\chi_V) = Q \sum_{\lambda \in \Lambda} \sum_{\alpha \in \Phi^+} (\lambda, \alpha)n_\lambda e(\lambda) \frac{e(\alpha) + 1}{e(\alpha) - 1}.
$$

By Lemma 5.1, the left-hand side in (6) is $2\{Q, \chi_V\}$. So finally, it is sufficient to prove that

$$
2\{Q, \sum_{\lambda \in \Lambda} n_\lambda e(\lambda)\} = Q \sum_{\alpha \in \Phi^+} \frac{e(\alpha) + 1}{e(\alpha) - 1} \sum_{\lambda \in \Lambda} (\lambda, \alpha)n_\lambda e(\lambda)
$$

which, by linearity, follows from the relation

$$
2\{Q, e(\nu)\} = Q \sum_{\alpha \in \Phi^+} \frac{e(\alpha) + 1}{e(\alpha) - 1} (\nu, \alpha)e(\nu) \quad \text{for } \nu \in \Lambda,
$$

proved in Exercise 5.2 below. \qed
Exercises

Exercise 1.1. Let $E = \langle u, v \rangle$ be the natural 2-dimensional $\mathfrak{sl}_2(\mathbb{C})$-module. Show that $\text{Sym}^d E$ is irreducible for each $d \in \mathbb{N}$.

Exercise 1.2. Let $V$ be a finite-dimensional $\mathfrak{sl}_2(\mathbb{C})$-module.

(i) Show that $V$ contains an $h$-eigenvector $v$ such that $e \cdot v = 0$.

(ii) Show that the submodule of $V$ generated by $v$ is $d$-dimensional if and only if $h \cdot v = dv$.

(iii) Deduce that any irreducible $\mathfrak{sl}_2(\mathbb{C})$-module is isomorphic to $\text{Sym}^d E$ for some $d \in \mathbb{N}_0$.

Exercise 2.1. Show that a Cartan subalgebra (as defined in Section 2) is abelian.

Solution. Given $h, k \in H$, we can write $k$ as a sum of $\text{ad} h$ eigenvectors, say $k = k_0 + \sum_{i=1}^n k_i$ where $(\text{ad} h)k_0 = 0$ and $(\text{ad} h)k_i = \lambda_i k_i$. Hence $(\text{ad} h)^2 k = \sum_{i=1}^n \lambda_i^2 k_i$. A useful linear algebra lemma shows that all the $k_i$ are in the Lie subalgebra of $H$ generated by $x$ and $y$. Now $[h, k_i] = \lambda_i k_i$ and so $(\text{ad} k_i)^2 x = [k_i, [k_i, x]] = [k_i, -\lambda_i k_i] = 0$; since $k_i \in H$, $\text{ad} k_i$ is diagonalizable, and so we must have $(\text{ad} k_i)x = 0$. Hence $[h, k] = 0$.

Exercise 2.2. The aim of this exercise is to show that if $H$ is a Cartan subalgebra of $L$ then $H$ is self-centralizing.

(i) Show that $L_0$ is nilpotent. [Hint: use Engel’s theorem and the abstract Jordan decomposition.]

(ii) Show that there is a basis of $L_0$ in which all $\text{ad} x : L \to L$ for $x \in L_0$ are represented by upper-triangular matrices.

(iii) Show that if $x \in L_0$ and $\text{ad} x : L \to L$ is nilpotent then $\text{Tr}(\text{ad} x \circ \text{ad} y) = 0$ for all $y \in L_0$. Deduce that $x = 0$.

(iv) Deduce that every element of $L_0$ is semisimple and hence show that $L_0 = H$.

Exercise 2.3. Let $V$ be a complex vector space. Show that if $x$ and $y \in \mathfrak{gl}(V)$ are such that $[x, y]$ commutes with $x$ then $[x, y]$ is nilpotent. [Hint: there is a quick solution using Lie’s Theorem. For an ad-hoc proof (which then allows this exercise to be used as part of a proof of Lie’s Theorem) first show that $\text{Tr}[x, y]^n = 0$ for all $n \in \mathbb{N}$.]

Exercise 2.4. Let $\alpha$ and $\beta$ be non-perpendicular roots in a root system. Use the fundamental relation (4) to find the possible angles between $\alpha$ and $\beta$ and the possible values of $||\alpha||/||\beta||$.

Exercise 2.5. Find the Killing form of $\mathfrak{sl}_2(\mathbb{C})$ with respect to the basis $e, f, h$ and hence calculate $||\alpha||^2$ where $\alpha$ is the unique root of $\mathfrak{sl}_2(\mathbb{C})$. (In practice
the previous exercise always gives enough information, so this calculation is unnecessary. For example, this is true in Freudenthal’s formula, since \( n_\lambda \) is expressed as a quotient of norms, and in Exercise 4.6, for the same reason.)

**Exercise 2.6.** Let \( V \) be a finite-dimensional \( L \)-module and let \( v \in V \) be a highest-weight vector. Show that the submodule of \( L \) generated by \( v \) is irreducible.

**Exercise 2.7.** Let \( H \) be the Cartan subalgebra of diagonal matrices in \( \mathfrak{sl}_3(\mathbb{C}) \). For \( i \in \{1, 2, 3\} \), let \( \varepsilon_i : H \to \mathbb{C} \) be the function sending \( \text{diag}(a_1, a_2, a_3) \) to \( a_i \). Let \( \alpha = \varepsilon_1 - \varepsilon_2 \) and let \( \beta = \varepsilon_2 - \varepsilon_3 \).

(i) Show that \( \{\alpha, \beta\} \) is a base for the root system \( \Phi \).

(ii) Show that \( ||\alpha|| = ||\beta|| \) and that the angle between \( \alpha \) and \( \beta \) is \( 2\pi/3 \).

(iii) Find the fundamental dominant integral weights \( \omega_1, \omega_2 \) corresponding to this base in terms of \( \alpha \) and \( \beta \).

(iv) Show that \( \omega_1 = \varepsilon_1 \) and \( \omega_2 = \varepsilon_1 + \varepsilon_2 \). (Since \( \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0 \) other, equivalent, expressions for \( \omega_1 \) and \( \omega_2 \) are also possible.)

(iv) Express the highest weight of the natural, dual natural and adjoint representations of \( \mathfrak{sl}_3(\mathbb{C}) \) as \( \mathbb{Z} \)-linear combinations of \( \omega_1 \) and \( \omega_2 \).

**Exercise 3.1.** Recall that \( \delta = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha \) and that \( B = \{\alpha_1, \ldots, \alpha_\ell\} \) is a base for \( \Phi \).

(i) Show that if \( \beta \in \Phi^+ \) and \( \beta \neq \alpha_i \) then \( S_{\alpha_i}(\beta) \in \Phi^+ \)

(ii) Show that \( S_{\alpha_i}(\delta) = \delta - \alpha_i \) for all \( i \).

(iii) Show that \( \delta = \omega_1 + \cdots + \omega_\ell \) and deduce that \( \delta \in \Lambda \).

**Solution.** (i) Since \( \beta \neq \alpha_i \) and \( k\alpha_i \) is a root if and only if \( k \in \{+1, -1\} \) (see, for example, [1, Proposition 10.9]), there exists \( j \) such that \( \alpha_j \) appears with a strictly positive coefficient in the expression for \( \beta \) as a \( \mathbb{Z} \)-linear combination of \( \alpha_1, \ldots, \alpha_n \). Now \( \alpha_j \) has the same coefficient in

\[
S_{\alpha_i}(\beta) = \beta - (\beta, \alpha_i)\alpha_i,
\]

and so it follows that \( S_{\alpha_i}(\beta) \in \Phi^+ \).

(ii) Since \( S_{\alpha_i} \) permutes \( \Phi^+ \setminus \{\alpha_i\} \) and \( S_{\alpha_i}(\alpha_i) = -\alpha_i \), we have

\[
S_{\alpha_i}(\delta) = \frac{1}{2} \sum_{\beta \in \Phi} S_{\alpha_i}(\beta) = \frac{1}{2} \sum_{\beta \in \Phi} S_{\alpha_i}(\beta) - \alpha_i = \delta - \alpha_i
\]

as required.

(iii) By definition \( \langle \alpha_i, \omega_j \rangle = 0 \) if \( i \neq j \) and \( \langle \alpha_i, \omega_j \rangle = 1 \). Hence

\[
S_{\alpha_j} \left( \sum_{i=1}^{\ell} \omega_i \right) = \sum_{i=1}^{\ell} \omega_i - \omega_j + S_{\alpha_j}(\omega_j) = \sum_{i=1}^{\ell} \omega_i - \omega_j + \omega_j - \alpha_j = \sum_{i=1}^{\ell} \omega_i - \alpha_j.
\]
Hence by (ii), \(-\delta + \sum_{i=1}^{\ell} \omega_i\) is invariant under the generators \(S_{\alpha_1}, \ldots, S_{\alpha_\ell}\) of \(W\). Hence \(\delta = \sum_{i=1}^{\ell} \omega_i \in \Lambda\).

**Exercise 3.2.** Let \(B : V \to V\) be a non-degenerate symmetric bilinear form on an \(n\)-dimensional vector space \(V\). Suppose that \(x_1, \ldots, x_n\) and \(y_1, \ldots, y_n\) are dual bases for \(V\), so

\[
B(x_i, y_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}
\]

Let \(\theta \in V^*\) and let \(t_\theta\) be the unique element such that \(B(t_\theta, v) = \theta(v)\) for all \(v \in V\). Let \(v \in V\).

(i) Show that \(v = \sum_{i=1}^{n} B(x_i, v)y_i = \sum_{j=1}^{n} B(v, y_j)x_j\).

(ii) Hence show that \(B(t_\theta, t_\theta) = \sum_{k=1}^{n} \theta(x_k)\theta(y_k)\).

**Solution.** (i) For each \(j\) we have \(B\left(\sum_{i=1}^{n} B(x_i, v)y_i, x_j\right) = B(x_j, v)\), hence \(B(-v + \sum_{i=1}^{n} B(x_i, v)y_i, x_j) = 0\) for all \(j\). Since \(x_1, \ldots, x_n\) is a basis of \(V\) and \(B\) is non-degenerate, it follows that \(v = \sum_{i=1}^{n} B(x_i, v)y_i, x_j\), as required. Similarly one finds that \(v = \sum_{j=1}^{n} B(v, y_j)x_j\).

(ii) We have \(t_\theta = \sum_{i=1}^{n} B(x_i, t_\theta)y_i\) and \(t_\theta = \sum_{j=1}^{n} B(t_\theta, y_j)x_j\). Hence

\[
(t_\theta, t_\theta) = \sum_{k=1}^{n} B(x_k, t_\theta)B(t_\theta, y_k) = \sum_{k=1}^{n} t_\theta(x_k)t_\theta(y_k)
\]

as required.

**Exercise 3.3.** Prove Lemma 3.1. [Hint: Show that \(\sum_{k=1}^{n} x_k[y_k, w] = \sum_{k=1}^{n} x_k[y_k, w] + \sum_{k=1}^{n} [x_k, w]y_k\) for \(w \in L\), and then use Exercise 3.2(i) to express \([y_k, w]\) as a linear combination of \(y_1, \ldots, y_n\) and \([x_k, w]\) as a linear combination of \(x_1, \ldots, x_n\).]

**Solution.** Since \(U(L)\) is generated, as an algebra, by \(L\), it is sufficient to prove that \(\sum_{k=1}^{n} x_k[y_k, w] = 0\) for each \(w \in L\). A routine calculation gives the result stated in the hint that

\[
\sum_{k=1}^{n} x_k[y_k, w] = \sum_{k=1}^{n} x_k[y_k, w] + \sum_{k=1}^{n} [x_k, w]y_k.
\]

By Exercise 3.2(i) we have \([y_k, w] = \sum_{i=1}^{n} \kappa(x_i, [y_k, w])y_i\) and \([x_k, w] = \sum_{j=1}^{n} \kappa([x_k, w], y_j)x_j\). Substituting we get

\[
\sum_{k=1}^{n} x_k[y_k, w] = \sum_{k=1}^{n} \sum_{i=1}^{n} \kappa(x_i, [y_k, w])x_ky_i + \sum_{k=1}^{n} \sum_{j=1}^{n} \kappa([x_k, w], y_j)x_jy_k.
\]
Now change the summation variables in the second sum and use the associativity of the Killing form to get

\[ \sum_{k=1}^{n} x_k y_k, w = \sum_{k=1}^{n} \sum_{i=1}^{n} \kappa(x_i, [y_k, w]) x_k y_i + \sum_{i=1}^{n} \sum_{k=1}^{n} \kappa([x_i, w], y_k) x_k y_i \]

\[ = \sum_{k=1}^{n} \sum_{i=1}^{n} (-\kappa(x_i, w) + \kappa([x_i, w], y_k)) x_k y_i \]

\[ = 0 \]

as required.

**Exercise 3.4.** Take the notation from Lemma 3.3. Suppose that \( \lambda(h) \leq 0 \).

(i) Deduce from (b) in Section 1 that if \( U^{(i)} \) is a summand with lowest weight \( (\lambda - ba)(h) \) where \( b \in \mathbb{N}_0 \), then \( f_a e_a \) acts on \( U^{(i)}_\lambda \) as the scalar \( (b - \lambda(h)) (b + 1) \).

(ii) Show that the number of summands \( U^{(i)} \) with lowest weight \( (\lambda - ba)(h) \) is \( n_{\lambda - ba} - n_{\lambda - (b+1)\alpha} \).

(iii) Hence show that \( f_a e_a \) acts on \( V_\lambda \) as the scalar \( -\sum_{b=0}^{\infty} n_{\lambda - ba} (\lambda - ba, \alpha) \), as claimed in the proof of Lemma 3.3.

**Solution.** (i) If \( U^{(i)} \) has lowest weight \( (\lambda - ba)(h) \) then \( U^{(i)} \) has highest weight \( -(\lambda - ba)(h) \). If \( v \in U^{(i)}_\lambda \) then

\[ h \cdot v = \lambda(h) = (-\lambda - ba)(h) - 2(b - \lambda(h)) \]

and so taking \( c = b - \lambda(h) \) in (b) in Section 1 gives

\[ f \cdot e \cdot v = (b - \lambda(h))(\lambda - ba)(h) - (b - \lambda(h)) + 1) v \]

\[ = (b - \lambda(h))(b + 1) v \]

as required. Now (ii) follows from (a) in Section 1, in the same way as (c) did, and (iii) is an immediate corollary of (i) and (ii).

**Exercise 3.5.** Let \( \omega_1, \omega_2 \) be the fundamental dominant weights for \( \mathfrak{sl}_3(\mathbb{C}) \) (see Exercise 2.7). Use Freudenthal’s Formula to determine the dimensions of the weight spaces for the \( \mathfrak{sl}_3(\mathbb{C}) \)-module with highest weight \( 2\omega_1 + \omega_2 \).

**Exercise 4.1.** Let \( \tau : L \to \mathfrak{gl}(V) \) be a representation of \( L \). Let \( G \) be the simply connected Lie group corresponding to \( L \) and let \( \rho : G \to \text{GL}(V) \) be the corresponding representation of \( G \), as defined by

\[ \rho(\exp x) = \exp(\tau(x)) \quad \text{for} \ x \in L. \]

(This defines \( \rho \) on a generating set for \( G \).) Let \( \lambda \in \Lambda \). Show that if \( h \in H \) and \( v \in V_\lambda \) then \( \rho(\exp h)v = \exp(\lambda(h))v \).

**Exercise 4.2.** Show that if \( V \) is an \( L \)-module then \( \chi_V \in \mathbb{Q}[\Lambda] \) is symmetric.
**Exercise 4.3.** Let $\Lambda_{\text{dom}}$ be the set of strictly dominant weights in $\Lambda$.

(i) Given $\lambda \in \Lambda$ define $a(\lambda) = \sum_{w \in W} \varepsilon(w) w \cdot e(\lambda)$. Show that $\Delta(a(\lambda)) = \|\lambda\|^2 a(\lambda)$ and deduce that $\{a(\lambda) : \lambda \in \Lambda_{\text{dom}}\}$ is a $\mathbb{Z}$-basis of $\Delta$-eigenvectors for the set of all antisymmetric elements of $\mathbb{Q}[(\Lambda)]$.

(ii) Show that

$$e(-\delta) \prod_{\alpha \in \Phi^+} (e(\alpha) - 1) = \prod_{\alpha \in \Phi^+} (e(\frac{1}{2}\alpha) - e(-\frac{1}{2}\alpha))$$

and that either side is antisymmetric.

(iii) Show that

$$\sum_{w \in W} \varepsilon(w) w \cdot e(\delta) = \prod_{\alpha \in \Phi^+} (e(\frac{1}{2}\alpha) - e(-\frac{1}{2}\alpha))$$

(iv) Prove that $f \in \mathbb{Q}[(\frac{1}{2}\Lambda)]$ is antisymmetric if and only if

$$f = g \prod_{\alpha \in \Phi^+} (e(\frac{1}{2}\alpha) - e(-\frac{1}{2}\alpha))$$

for some symmetric $g$.

**Solution.** (i) Fix a total order on $\Lambda$ refining the dominance order. Define the degree of an antisymmetric element $f$ to be the greatest weight $\mu$ in this order such that $e(\mu)$ has a non-zero coefficient in $f$. If $\mu$ is the greatest weight of $f$ then $\mu \in \Lambda_{\text{dom}}$ and $\mu$ is acted on regularly by the Weyl group. Hence $f - \sum_{w \in W} \varepsilon(w) w \cdot e(\mu)$ has strictly smaller weight. The result now follows by induction.

(ii) The equality is routine. Recall that $\{\alpha_1, \ldots, \alpha_\ell\}$ is a base for $\Phi$. It follows from Exercise 3.1(i) and (ii) that

$$S_{\alpha_i} \left( \prod_{\alpha \in \Phi^+} (e(\frac{1}{2}\alpha) - e(-\frac{1}{2}\alpha)) \right) = \frac{-e(\frac{1}{2}\alpha_i) + e(-\frac{1}{2}\alpha_i)}{e(\frac{1}{2}\alpha_i) - e(-\frac{1}{2}\alpha_i)} \prod_{\alpha \in \Phi^+} (e(\frac{1}{2}\alpha) - e(-\frac{1}{2}\alpha))$$

$$= - \prod_{\alpha \in \Phi^+} (e(\frac{1}{2}\alpha) - e(-\frac{1}{2}\alpha)).$$

Hence the right-hand side is antisymmetric.

(iii) Both sides are anti-symmetric and the coefficients of $e(\delta)$ agree. The result now follows from (i) since, by Exercise 3.1(iii), $\delta$ is the smallest element of $\Lambda_{\text{dom}}$.

(iv) Sketch: it is sufficient to prove that each $a(\lambda)$ is divisible by $\prod_{\alpha \in \Phi^+} (e(\frac{1}{2}\alpha) - e(-\frac{1}{2}\alpha))$. This follows using that $\mathbb{Q}[(\frac{1}{2}\Lambda)]$ is a UFD.

**Exercise 4.4.** Let $\omega$ be the unique fundamental dominant weight for $\mathfrak{sl}_2(\mathbb{C})$, so $\omega \in \langle h \rangle^*$ is defined by $\omega(h) = 1$. 


(i) Use the results of Section 1 to show that \( V \) is the irreducible \( \mathfrak{sl}_2(\mathbb{C}) \)-module with highest weight \( d\omega \) then
\[
\chi_V = e(d\omega) + e((d-2)\omega) + \cdots + e(-d\omega).
\]
(ii) Check that this is consistent with the Weyl Character Formula.

Exercise 4.5. Let \( \omega_1, \omega_2 \) be the fundamental dominant weights for \( \mathfrak{sl}_3(\mathbb{C}) \) (see Exercise 2.7).

(i) Use the Weyl Character Formula to determine the characters of the finite-dimensional irreducible \( \mathfrak{sl}_3(\mathbb{C}) \)-module \( V \) with highest weight \( a\omega_1 + b\omega_2 \) where \( a, b \in \mathbb{N}_0 \).

(ii) Give a necessary and sufficient condition on \( a \) and \( b \) for \( V \) to have a weight space of dimension at least two.

Exercise 4.6. Deduce from the Weyl Character Formula that if \( V \) is the irreducible \( L \)-module with highest weight \( \lambda \) then
\[
\dim V = \prod_{\alpha \in \Phi^+}(\lambda + \delta, \alpha) / \prod_{\alpha \in \Phi^+}(\lambda, \alpha)
\]

Exercise 5.1. Show that if \( f, g, h \in \mathbb{Q}[\Lambda] \) then
\[
2\{Q, e(\nu)\} = Q \sum_{\alpha \in \Phi^+} \frac{e(\alpha) + 1}{e(\alpha) - 1} (\nu, \alpha) e(\nu)
\]

Solution. By the generalization of Exercise 5.1 to arbitrary products we have
\[
2\{Q, e(\nu)\} = 2 \left\{ \prod_{\alpha \in \Phi^+} \frac{1}{e(\frac{1}{2}\alpha) - e(-\frac{1}{2}\alpha)}, e(\nu) \right\}
= 2 \sum_{\alpha \in \Phi^+} \frac{Q}{e(\frac{1}{2}\alpha) - e(-\frac{1}{2}\alpha)} \left( e(\frac{1}{2}\alpha) - e(-\frac{1}{2}\alpha), e(\nu) \right)
= 2 \sum_{\alpha \in \Phi^+} \frac{Q}{e(\frac{1}{2}\alpha) - e(-\frac{1}{2}\alpha)} \left( (\frac{1}{2}\alpha, \nu) e(\nu + \frac{1}{2}\alpha) + (\frac{1}{2}\alpha, \nu) e(\nu - \frac{1}{2}\alpha) \right)
= \sum_{\alpha \in \Phi^+} \frac{Q}{e(\frac{1}{2}\alpha) - e(-\frac{1}{2}\alpha)} (\alpha, \nu) (e(\frac{1}{2}\alpha) + e(-\frac{1}{2}\alpha)) e(\nu)
= Q \sum_{\alpha \in \Phi^+} \frac{e(\frac{1}{2}\alpha) + e(-\frac{1}{2}\alpha)}{e(\frac{1}{2}\alpha) - e(-\frac{1}{2}\alpha)} (\nu, \alpha) e(\nu)
= Q \sum_{\alpha \in \Phi^+} \frac{e(\alpha) + 1}{e(\alpha) - 1} (\nu, \alpha) e(\nu)
\]
as required.
References


