Commuting conjugacy classes in groups: an overview

Mark Wildon (joint work with John Britnell)
(1) Introduction
(2) Finite symmetric groups
(3) General linear groups
Let $G$ be a group. For $x, g \in G$ define the conjugate of $x$ by $g$ to be $x^g = g^{-1}xg$. The conjugacy class of $x$ is $x^G = \{x^g : g \in G\}$.
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**Definition**
Say that classes $C$ and $D$ commute, and write $C \sim D$, if there exist $x \in C$, $y \in D$ such that $xy = yx$. 
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Say that classes $C$ and $D$ \textit{commute}, and write $C \sim D$, if there exist $x \in C$, $y \in D$ such that $xy = yx$. 

\begin{tikzpicture}
\node at (0,0) {$\{ (12) , \ldots \}$};
\node at (1,-1) {$\{ (12)(34), \ldots \}$};
\node at (2.5,0) {$\{ \text{id} \}$};
\node at (5,0) {$\{ (123), \ldots \}$};
\node at (0,-2.5) {$\{ (1234), \ldots \}$};
\node at (5.5,-2.5) {$\text{Sym}(4)$};
\draw[->] (0,0) -- (1,-1);
\draw[->] (0,0) -- (0,-2.5);
\draw[->] (1,-1) -- (2.5,0);
\draw[->] (2.5,0) -- (5,0);
\draw[->] (0,-2.5) -- (5.5,-2.5);
\end{tikzpicture}
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\[ x^G \sim y^G \iff x \text{ commutes with } y^g \text{ for some } g \in G \]
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\iff \text{Cent}_G(x) = \{ h \in G : hx = xh \} \text{ meets } y^G.
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$$\iff \text{Cent}_G(x) = \{h \in G : hx = xh\} \text{ meets } y^G.$$ 

The commuting relation therefore determines which conjugacy classes meet Cent$_G(x)$.

(2) If $G$ is finite then $\sim$ determines

$$Z(G) = \{x \in G : xy = yx \text{ for all } y \in G\}.$$
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Proof: Suppose \( x^G \) commutes with every class. Then \( \text{Cent}_G(x) \) meets every class so
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\bigcup_{g \in G} \text{Cent}_G(x)^g = G.
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But the conjugates of a proper subgroup of \( G \) cannot cover \( G \). Hence \( \Cent_G(x) = G \). \( \square \)
Note that $\text{Cent}_G(x)^g$ is the stabiliser of $x^g$ in the conjugacy action of $G$ on $x^G$. So

$$\bigcup_{g \in G} \text{Cent}_G(x)^g$$

is the set of elements of $G$ fixing at least one element of $x^G$. In 1870, Jordan showed that any non-trivial finite transitive permutation group contains an element without fixed points. So unless $\text{Cent}_G(x) = G$, when the action is trivial, the conjugates of $\text{Cent}_G(x)$ do not cover $G$. 

Traité des substitutions
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(3) If $G$ is infinite then $Z(G)$ cannot be determined by $\sim$. Let $X$ be an infinite set and let

$$G = \text{FSym}(X) = \left\{ g : X \to X : \begin{array}{c} g \text{ bijective} \\ X \setminus \text{Fix } g \text{ finite} \end{array} \right\}.$$
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Then any two classes $x^G, y^G \in G$ commute. But $G$ is not abelian.
Conjugacy classes in $\text{Sym}(n)$ are labelled by partitions of $n$.

For example, if $g = (2345)(67) \in \text{Sym}(7)$ then $g^{\text{Sym}(7)}$ consists of all permutations whose cycle decomposition has a 4-cycle, a 2-cycle and a fixed point. The labelling partition is $(4, 2, 1)$.
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Theorem
The classes in $\text{Sym}(n)$ corresponding to partitions $\lambda$ and $\mu$ commute if and only if there is a partition $\nu$ which is a coarsening of both $\lambda$ and $\mu$. 
Probabilistic questions

This part is joint work with Simon Blackburn (RHUL).

**Theorem**

Let $P(n)$ be the probability that if two elements $g, h \in \text{Sym}(n)$ are chosen uniformly at random then $g^{\text{Sym}(n)} \sim h^{\text{Sym}(n)}$. Then there is a constant $C \approx 6.2$ such that $P(n) \sim \frac{C}{n^2}$ as $n \to \infty$. 
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**Sketch proof:** Most permutations in $\text{Sym}(n)$ have a long cycle, of length $> n/\log n$. If $g$ has a long cycle of length $\ell$ and $g^{\text{Sym}(n)} \sim h^{\text{Sym}(n)}$ then, almost always, $h$ also has a long cycle of length $\ell$. We use this to get a recurrence for $P(n)$. Some analysis then shows that $P(n) \sim C/n^2$ where

$$C = \sum_{n=0}^{\infty} P(n)$$
Marrying in symmetric groups

Say that an even permutation is marriable if it commutes with an odd permutation.

Theorem
*There is a bijection*

\[
\left\{ \begin{array}{c}
\text{marriable classes} \\
h^{\text{Sym}(n)} \subseteq \text{Alt}(n)
\end{array} \right\} \leftrightarrow \left\{ \begin{array}{c}
\text{all classes} \\
g^{\text{Sym}(n)} \subseteq \text{Sym}(n) \setminus \text{Alt}(n)
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**Proof:** show that given any \( r \) marriable classes, \( C_1, \ldots, C_r \) there are \( r \) classes of odd elements \( D_1, \ldots, D_r \) such that \( C_i \sim D_i \) for each \( i \).
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Let $C = C_1 \cup \cdots \cup C_r$.

Let $X = \{(h, g) : h \in C, \ g \text{ odd}, \ hg = gh\}$. So

$$|X| = \sum_{h \in C} \frac{1}{2} |\text{Cent}(h)| = \frac{n!}{2} \sum_{h \in C} \frac{1}{|h^{\text{Sym}(n)}|} = \frac{n!}{2} r.$$
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Counting the other way we get

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|X| = \sum_{g \in \text{Sym}(n) \setminus \text{Alt}(n)} |\text{Cent}_C(g)|
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where $g^{\text{Sym}(n)} \sim_C$.
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Another application of Hall’s Marriage Theorem

Let $G$ be a group with a finite index subgroup $H$. There exist $g_1, \ldots, g_n \in G$ such that

$$G = g_1H \cup \cdots \cup g_nH = Hg_1 \cup \cdots \cup Hg_n.$$  

This result may also be proved using Hall’s Marriage Theorem.
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**Background to results on $GL_n(F)$**

Fix a field $F$. Given a partition $\lambda$ of $n$, Let $J(\lambda) \in GL_n(F)$ be the unipotent Jordan matrix corresponding to $\lambda$.

A major open problem is to describe the conjugacy classes of $GL_n(F)$ that meet $\text{Cent}_{GL_n(F)} J(\lambda)$. In our language: which classes commute with $J(\lambda)^{GL_n(F)}$?

- Let $D(\lambda)$ be the largest partition such that $J(\lambda) \sim J(D(\lambda))$. In 2009 Iarrobino proved that the map $\lambda \mapsto D(\lambda)$ is idempotent.

- In 2010, Kosir and Oblak found $D(\lambda)$ in the cases where it has at most two parts.

- In 2008, Oblak defined a partition $Q(\lambda)$ and conjectured that $Q(\lambda) = D(\lambda)$. In 2012, Iarrobino and Khattami proved that $D(\lambda) \leq Q(\lambda)$.

Our results reduce the general problem of deciding which classes in $GL_n(F)$ commute to the problem for nilpotent classes over field extensions of $F$. 
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Definition
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The type of \( X \) is the string \( d_1^{\lambda_1} \ldots d_r^{\lambda_r} \) where \( d_i = \deg f_i \).

- Introduced by Steinberg in 1951
- Important in Green’s 1955 construction of the irreducible characters of finite general linear groups.
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Theorem
Let $X, Y \in \text{GL}_n(q)$. Then $X$ and $Y$ have the same type if and only if there exist polynomials $F, G \in \mathbf{F}_q[x]$ such that $F(X) \in Y^{\text{GL}_n(q)}$ and $g(Y) \in X^{\text{GL}_n(q)}$. 
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Corollary
Suppose that $X^{\text{GL}_n(q)} \sim Y^{\text{GL}_n(q)}$. Then any class of the type of $X$ commutes with any class of the type of $Y$. 
Theorem
Let $G = GL_n(F_q)$ and let $X, Y \in G$. Then $Cent_G(X)$ is conjugate to $Cent_G(Y)$ if and only if $X$ and $Y$ have the same type.
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Let $U_q(\lambda) = J(\lambda)^{GL_n(F_q)}$ be the unipotent conjugacy class corresponding to the partition $\lambda$ of $n$. 
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Let $U_q(\lambda) = J(\lambda)^{\text{GL}_n(F_q)}$ be the unipotent conjugacy class corresponding to the partition $\lambda$ of $n$.

**Theorem**

Let $p$ be a prime and let $r \geq 1$. There exists $n \in \mathbb{N}$ such that

$$U_p^a((n, n)) \sim U_p^a((n + 1, n - 1))$$

if and only if $a > r$. 
Future directions

- What is the correct generalization of type for matrices over infinite fields? Probably it involves isomorphism classes of Galois extensions.

- Find all possible determinants of a matrix of a given type. This leads to some interesting problems in arithmetic combinatorics.

- What is the probability that two classes chosen uniformly at random in $\text{Sym}(n)$ commute?