The probability that two elements of a finite group are conjugate

Mark Wildon (with Simon R. Blackburn and John R. Britnell)

Royal Holloway
University of London

Outline

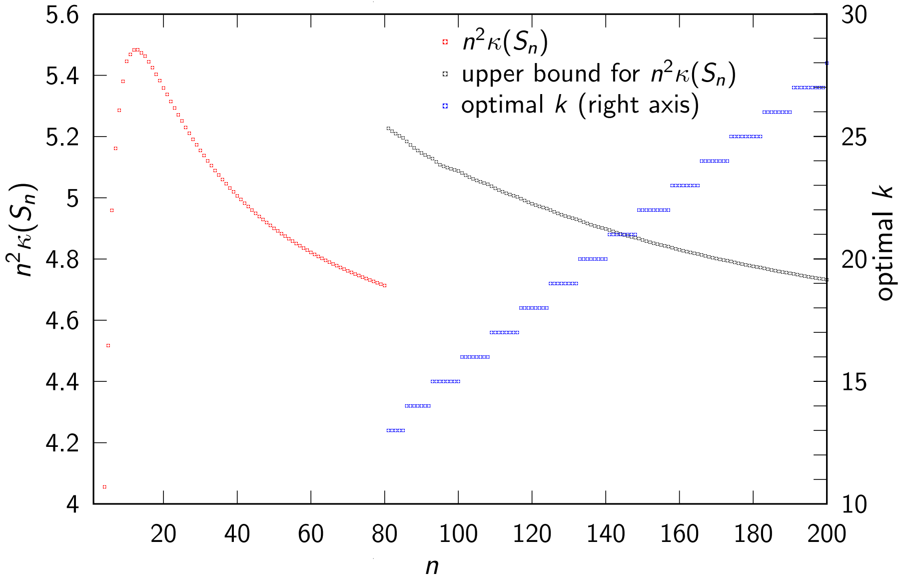
§1 Small $\kappa(G)$

§2 Limit points of $\kappa(G)$

§3 Large $\kappa(G)$

§4 κ for symmetric groups

Inductive proof of Theorem 3



Conjugate commuting probabilities

Let $\rho(G)$ be the probability that if $g, h \in G$ are chosen uniformly and independently at random, then g and h have conjugates that commute.

The method used to prove Theorem 3 also give the analogous result for $\rho(S_n)$.

Theorem 4

For all $n \in \mathbf{N}$ we have

$$\rho(S_n) \leq \frac{C_\rho}{n^2}$$

where $C_\rho = 10^2 \rho(S_{10}) \approx 11.42$. Moreover if $A_\rho = \sum_{n=1}^{\infty} \rho(S_n)$ then $\rho(S_n) \sim \frac{A_\rho}{n^2}$ as $n \to \infty$.

Conjugate commuting probabilities

Let $\rho(G)$ be the probability that if $g, h \in G$ are chosen uniformly and independently at random, then g and h have conjugates that commute.

The method used to prove Theorem 3 also give the analogous result for $\rho(S_n)$.

Theorem 4

For all $n \in \mathbf{N}$ we have

$$\rho(S_n) \leq \frac{C_\rho}{n^2}$$

where $C_\rho = 10^2 \rho(S_{10}) \approx 11.42$. Moreover if $A_\rho = \sum_{n=1}^{\infty} \rho(S_n)$ then $\rho(S_n) \sim \frac{A_\rho}{n^2}$ as $n \to \infty$.

Corollary 5

Let $g, h \in S_n$ be chosen independently and uniformly at random. If g and h have conjugates that commute then, provided n is sufficiently large, the probability that g and h are conjugate is at least $42/65 \approx 0.646$, and at most $43/61 \approx 0.710$.

Thank you. Any questions?

Other results and questions

- If G and H are isoclinic groups then $\kappa(G) = \kappa(H)$. The groups G such that $|G : Z(G)| = 4$ form a single isoclinism class.
- If $N \lhd G$ then $\kappa(G) < \kappa(G/N)$. Using this we can define κ for a profinite group G as $\lim_N \kappa(G/N)$. Could it be that every limit point of κ is 'explained' by a profinite group?
- What can be said about $\rho(G)$ for a general group G?