Character deflations, wreath products and Foulkes’ Conjecture

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Outline

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§2 Deflations
§3 Combinatorial rule for deflated character values
§4 Applications
§1 Foulkes’ Conjecture

Let $S_r$ be the group of all permutations of $\Omega = \{1, 2, \ldots, r\}$. It is often useful to consider actions of $S_r$ on other sets.
§1 Foulkes’ Conjecture

Let $S_r$ be the group of all permutations of $\Omega = \{1, 2, \ldots, r\}$. It is often useful to consider actions of $S_r$ on other sets.

Here we can find $S_4$ acting on (amongst other things):

- the set $\{\alpha, \beta, \gamma, \delta\}$
- the set $\{\alpha \beta + \gamma \delta, \alpha \gamma + \beta \delta, \alpha \delta + \beta \gamma\}$ of size 3,
- the field extension $\mathbb{Q}(\alpha, \beta, \gamma, \delta)$,
- the 4-dimensional $\mathbb{Q}$-vector space $\langle \alpha, \beta, \gamma, \delta \rangle_{\mathbb{Q}}$. 
Linear representations

Let $\mathbf{C} \Omega = \langle e_1, e_2, \ldots, e_r \rangle$. This is the natural permutation representation of $S_r$ where the elements of $S_r$ act by permutation matrices.

Vector space decomposition:

$$\mathbf{C} \Omega = \langle e_1 + e_2 + \cdots + e_r \rangle \bigoplus \langle e_i - e_j : 1 \leq i < j \leq r \rangle.$$  

Each summand is preserved (i.e. mapped into itself) by the action of $S_r$. No proper subspace of either summand is preserved, so each summand is an irreducible representation of $S_r$.

Different permutation representations of $S_r$ can be compared by looking at the multiplicities of their irreducible constituents.
Example: comparing different linear representations

- $S_4$ acting on $\{1, 2, 3, 4\}$, point stabiliser $S_3$
- $S_4$ acting on $\{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}$, point stabiliser $S_2 \times S_2$

\[
\begin{align*}
(12) & \mapsto \begin{pmatrix}
1 & \cdot & \cdot \\
\cdot & \cdot & 1 \\
. & . & .
\end{pmatrix} & \quad (12) & \mapsto \begin{pmatrix}
1 & \cdot & \cdot & \cdot & . \\
\cdot & \cdot & \cdot & 1 & . \\
\cdot & 1 & \cdot & . & . \\
\cdot & . & 1 & \cdot & . \\
. & . & . & . & 1
\end{pmatrix}
\end{align*}
\]

Remarkable fact: any representation $\rho: S_r \to \text{GL}(V)$ is determined (up to a suitable notion of isomorphism) by its character $\phi(g) = \text{tr}(\rho(g))$ for $g \in S_r$.

Moreover, the multiplicity of an irreducible representation with character $\chi$ in $\rho$ is $\langle \phi, \chi \rangle = \frac{1}{r!} \sum_{g \in S_r} \phi(g) \chi(g)$.
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\[
(234) \mapsto \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix}
\]

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$$(12) \mapsto \begin{pmatrix} 1 \\ -1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \quad (12) \mapsto \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & -1 \\ 0 & -1 \end{pmatrix}$$

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\]

Remarkable fact: any representation $\rho : S_r \to \text{GL}(V)$ is determined (up to a suitable notion of isomorphism) by its character

\[\phi(g) = \text{tr}(\rho(g)) \quad \text{for } g \in S_r.\]

Moreover, the multiplicity of an irreducible representation with character $\chi$ in $\rho$ is

\[
\langle \phi, \chi \rangle = \frac{1}{r!} \sum_{g \in S_r} \phi(g)\chi(g).
\]
Foulkes’ Conjecture

Let $a, b \in \mathbb{N}$.

Let $\Omega^{(a,b)}$ be the collection of set partitions of $\{1, 2, \ldots, ab\}$ into $b$ sets each of size $a$, acted on by $S_{ab}$.

Let $C\Omega^{(a,b)}$ the corresponding permutation representation of $S_{ab}$.

Let $\phi^{(a,b)}$ be the character of $C\Omega^{(a,b)}$. So if $g \in S_{ab}$ then $\phi^{(a,b)}$ is the number of set partitions in $\Omega^{(a,b)}$ that are fixed by $g$.

Conjecture (Foulkes’ Conjecture)

If $a < b$ and $\chi$ is an irreducible character of $S_{ab}$ then

$$\langle \phi^{(a,b)}, \chi \rangle \geq \langle \phi^{(b,a)}, \chi \rangle.$$
Murnaghan–Nakayama Rule

Let $\lambda$ be a partition of $r$ and let $\gamma = (\gamma_1, \ldots, \gamma_k)$ be such that $\gamma_1 + \cdots + \gamma_k = r$. A border-strip tableau of shape $\lambda$ and type $\gamma$ is an assignment of the numbers from the set $\{1, 2, \ldots, k\}$ to the boxes of the diagram of $\lambda$ such that

(i) The boxes labelled $i$ form a border-strip of length $\gamma_i$;
(ii) The boxes labelled by numbers $\leq i$ form the diagram of a partition.

Let $\lambda = (5, 4, 2, 1)$ and let $\gamma = (6, 3, 3)$. To find one border-strip tableau of shape $\lambda$ and type $\gamma$: 
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![Diagram of a border-strip tableau]

(Insert diagram here)
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```
1 1 1 1 1 1
1
```

```
1
```

```
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```
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```
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```
   1 1 1 1 1
   1 2 2 2  
   3 3  
   3
```
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Theorem (Murnaghan–Nakayama)

Let $g \in S_r$ have cycle type $\gamma$. Then

$$\chi^\lambda(g) = \sum_T \text{sgn}(T)$$

where the sum is over all border-strip tableaux of shape $\lambda$ and type $\gamma$, and $\text{sgn}(T) = (-1)^{\text{sum of all leg lengths in } T}$. 
Example: \( a = 2, \ b = 6, \ \lambda = (6, 3, 3), \ \gamma = (1, 2, 3) \)
Example: $a = 2, b = 6, \lambda = (6, 3, 3), \gamma = (1, 2, 3)$
Example: $a = 2$, $b = 6$, $\lambda = (6, 3, 3)$, $\gamma = (1, 2, 3)$
4: Applications

- $c_{\lambda, \gamma}$ is independent of the order of the parts of $\gamma$. For example, if $\lambda = (6, 3, 3)$ then $c_{\lambda,(1,2,3)} = +1 + 1 - 1 = 1$ and correspondingly $c_{\lambda,(2,3,1)} = 1$. 

(A special case of) Young's Rule: let $\pi(a \ b)$ be the permutation character of $S_{ab}$ acting on all ordered set partitions of $\{1, 2, \ldots, ab\}$ into $b$ sets each of size $a$. Then $\langle \pi(a \ b), \chi_\lambda \rangle$ is equal to the number of semistandard $\lambda$-tableaux of type $(a \ b)$.

A new recursive formula for Foulkes multiplicities. Fix $a \in \mathbb{N}$. Then $\langle \phi(a \ b), \chi_\lambda \rangle = 1^{b - 1} \sum_{\ell=1}^{\lambda} \sum_{\mu} sgn(\lambda/\mu) \langle \phi(a \ b - \ell), \chi_\mu \rangle$ where the second sum is over all partitions $\mu$ obtainable by removing $a$ border-strips of length $\ell$ from $\lambda$, subject to the constraint in the main theorem.
§4: Applications

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- (A special case of) Young’s Rule: let $\pi^{(a^b)}$ be the permutation character of $S_{ab}$ acting on all ordered set partitions of $\{1, 2, \ldots, ab\}$ into $b$ sets each of size $a$. Then $\langle \pi^{(a^b)}, \chi^\lambda \rangle$ is equal to the number of semistandard $\lambda$-tableaux of type $(a^b)$. 
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- A new recursive formula for Foulkes multiplicities. Fix $a \in \mathbb{N}$. Then

\[
\langle \phi^{(a^b)}, \chi^\lambda \rangle = \frac{1}{b} \sum_{\ell=1}^{b} \sum_{\mu} \text{sgn}(\lambda/\mu) \langle \phi^{(a^{b-\ell})}, \chi^{\mu} \rangle
\]

where the second sum is over all partitions $\mu$ obtainable by removing $a$ border-strips of length $\ell$ from $\lambda$, subject to the constraint in the main theorem.
These graphs show Foulkes multiplicities for all partitions with at most \( b \) parts, arranged in lexicographic order. The \( y \) axis shows\[ \log \frac{\langle \phi^{(a^b)}, \chi^\lambda \rangle}{\langle \phi^{(b^a)}, \chi^\lambda \rangle}. \]

If both numerator and denominator are 0 then the point is artificially placed at \(-1\). If the denominator is 0 but not the numerator then \( \log \langle \phi^{(a^b)}, \chi^\lambda \rangle \) is shown.
$a = 6, \ b = 7$, log comparison of multiplicities
$a = 6, \ b = 8$, log comparison of multiplicities
$a = 6, \ b = 9, \ \text{log comparison of multiplicities}$
$a = 6, \ b = 10, \ \log \text{comparison of multiplicities}$
\( a = 6, \ b = 11, \) log comparison of multiplicities
$a = 6, \ b = 12$, log comparison of multiplicities
$a = 6$, $b = 9$, log comparison of Kostka multiplicities
$a = 6, \ b = 9$, log comparison of multiplicities
Timings

For $a = 6$, varying $b \geq 6$, here are the times to compute all Foulkes multiplicities.

**Symmetrica** is a specialised package for computing with symmetric functions developed by Adelbert Kerber, Axel Kohnert *et al*; it is usually much faster than *Magma* and other more general purpose computer algebra systems. See [www.algorithm.uni-bayreuth.de/en/research/SYMMETRICA/](http://www.algorithm.uni-bayreuth.de/en/research/SYMMETRICA/)

<table>
<thead>
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<th>$b$</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
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<tr>
<td><strong>Symmetrica</strong></td>
<td>0.4s</td>
<td>3.5s</td>
<td>22.6s</td>
<td>272.0s</td>
<td>2710.0s</td>
<td>426m8s</td>
<td>&gt; 2 days</td>
</tr>
<tr>
<td>Recurrence</td>
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<td>3.9s</td>
<td>25.6s</td>
<td>127.7s</td>
<td>454.3s</td>
<td>31m50s</td>
<td>117m3s</td>
</tr>
<tr>
<td>Memory used (Gb)</td>
<td>?</td>
<td>?</td>
<td>0.2</td>
<td>0.2</td>
<td>0.35</td>
<td>0.7</td>
<td>1</td>
</tr>
<tr>
<td>Speed up</td>
<td>0.6</td>
<td>1.1</td>
<td>0.88</td>
<td>2.1</td>
<td>6.0</td>
<td>13.4</td>
<td>&gt; 24.6</td>
</tr>
</tbody>
</table>

To test Foulkes’ Conjecture only the multiplicities for partitions with $\leq a$ parts are needed. This leads to big savings: for example for $a = 6$ and $b = 13$ only 20 minutes are needed (rather than over a day to compute all multiplicities).