Generalized Foulkes modules and decomposition numbers of the symmetric group

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Schur functions

Let λ be a partition. Recall that a *semistandard tableau* of shape λ is a filling of the boxes of the Young diagram of λ so that the rows are weakly increasing from left to right, and the columns are strictly increasing from top to bottom. For example,

$$T = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 3 \\ \hline 2 & 3 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array}$$

is a semistandard tableaux of shape $(4, 2, 1)$ with

$$x^T = x_1^3 x_2 x_3^2 x_4.$$

The *Schur function* for λ is the symmetric function

$$s_\lambda = \sum_T x^T$$

where the sum is over all semistandard Young tableaux of shape λ. If we only allow numbers between 1 and N to appear then s_λ becomes the character of the representation $\Delta^\lambda(E)$ where E is an N-dimensional complex vector space, and Δ^λ is a Schur functor.

Decomposition matrix of S_6 in characteristic 3

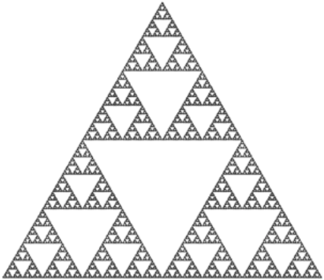
	(6)	(5,1)	(4,2)	(3,3)	(4,1,1)	(3,2,1)	(2,2,1,1)
(6)	1						
$(5,1)$	1	1					
$(4,2)$	·	·	1				
$(3,3)$	·	1	·	1			
$(4,1,1)$	·	1	·	·	1		
$(3,2,1)$	1	1	·	1	1	1	
$(2,2,1,1)$	·	·	·	·	·	·	1
$(2,2,2)$	1	·	·	·	·	1	·
$(3,1,1,1)$	·	·	·	·	1	1	·
$(2,1,1,1,1)$	·	·	·	1	·	1	·
$(1,1,1,1,1,1)$	·	·	·	1	·	·	·

S_6 in characteristic 3: two-row partitions

	(6)	**(5,1)**	**(4,2)**	**(3,3)**	(4,1,1)	(3,2,1)	(2,2,1,1)
(6)	1						
(5,1)	1	1					
(4,2)	·	·	1				
(3,3)	·	1	·	1			
(4,1,1)	·	1	·	·	1		
(3,2,1)	1	1	·	1	1	1	
(2,2,1,1)	·	·	·	·	·	·	1
(2,2,2)	1	·	·	·	·	1	·
(3,1,1,1)	·	·	·	·	1	1	·
(2,1,1,1,1)	·	·	·	1	·	1	·
(1,1,1,1,1,1)	·	·	·	1	·	·	·

General form of the two-row decomposition matrix

1										
1	1									
	1	1								
	1	1	1							
1	1		1	1						
1			1	1	1					
			1		1	1				
			1		1	1	1			
			1	1	1		1	1		
			1	1			1	1	1	
			1				1		1	(1)



S_6 in characteristic 3: separated into blocks

	(6)	(5,1)	(3,3)	(4,1,1)	(3,2,1)
(6)	1				
$(5,1)$	1	1			
$(3,3)$	·	1	1		
$(4,1,1)$	·	1	·	1	
$(3,2,1)$	1	1	1	1	1
$(2,2,2)$	1	·	·	·	1
$(3,1,1,1)$	·	·	·	1	1
$(2,1,1,1,1)$	·	·	1	·	1
$(1,1,1,1,1,1)$	·	·	1	·	·

	(4,2)
$(4,2)$	1

	(2,2,1,1)
$(2,2,1,1)$	1

S_6 in characteristic 3: $-\otimes \mathrm{sgn}$ involution

	(6)	(5,1)	(4,1,1)	(3,2,1)	(3,3)
(6)	1				
$(5,1)$	1	1			
$(4,1,1)$		1	1		
$(3,3)$		1			1
$(3,2,1)$	1	1	1	1	1
$(2,2,2)$	1			1	
$(3,1,1,1)$			1	1	
$(2,1,1,1,1)$				1	1
$(1,1,1,1,1,1)$					1

	(4,2)
$(4,2)$	1

	(2,2,1,1)
$(2,2,1,1)$	1

S_6 in characteristic 3: hook partitions

	(6)	(5,1)	(4,1,1)	(3,2,1)	(3,3)
(6)	**1**				
(5, 1)	**1**	**1**			
(4, 1, 1)		**1**	**1**		
(3, 3)		1			1
(3, 2, 1)	1	1	1	1	1
(2, 2, 2)	1			1	
(3, 1, 1, 1)			**1**	**1**	
(2, 1, 1, 1, 1)				**1**	**1**
(1, 1, 1, 1, 1, 1)					**1**

	(4,2)
(4, 2)	1

	(2,2,1,1)
(2, 2, 1, 1)	1

S_6 in characteristic 3: hook partitions

	(6)	(5,1)	(4,1,1)	(3,2,1)	(3,3)
(6)	**1**				
(5, 1)	**1**	**1**			
(4, 1, 1)		**1**	**1**		
(3, 1, 1, 1)			**1**	**1**	
(2, 1, 1, 1, 1)				**1**	**1**
(1, 1, 1, 1, 1, 1)					**1**
(3, 3)		1			1
(3, 2, 1)	1	1	1	1	1
(2, 2, 2)	1			1	

	(4,2)
(4, 2)	1

	(2,2,1,1)
(2, 2, 1, 1)	1

S_6 in characteristic 3: hook partitions

$$\begin{aligned} D &= D^{(5,1)} \\ \textstyle\bigwedge^2 D &= D^{(4,1,1)} \\ \textstyle\bigwedge^3 D &= D^{(3,2,1)} \\ \textstyle\bigwedge^4 D &= D^{(3,3)} \end{aligned}$$

$$\begin{aligned} U &= S^{(5,1)} \\ \textstyle\bigwedge^2 U &= S^{(4,1,1)} \\ \textstyle\bigwedge^3 U &= S^{(3,1,1,1)} \\ \textstyle\bigwedge^4 U &= S^{(2,1,1,1,1)} \\ \textstyle\bigwedge^5 U &= S^{(1,1,1,1,1,1)} \end{aligned}$$

	(6)	(5,1)	(4,1,1)	(3,2,1)	(3,3)
(6)	**1**				
(5, 1)	**1**	**1**			
(4, 1, 1)		**1**	**1**		
(3, 1, 1, 1)			**1**	**1**	
(2, 1, 1, 1, 1)				**1**	**1**
(1, 1, 1, 1, 1, 1)					**1**
(3, 3)		1			1
(3, 2, 1)	1	1	1	1	1
(2, 2, 2)	1			1	

	(4,2)
(4, 2)	1

	(2,2,1,1)
(2, 2, 1, 1)	1

S_6 in characteristic 3: outer automorphism

	(6)	**(5,1)**	(4,1,1)	**(3,2,1)**	(3,3)
(6)	1				
$(\mathbf{5},\mathbf{1})$	1	1			
$(\mathbf{4},\mathbf{1},\mathbf{1})$		1	1		
$(\mathbf{3},\mathbf{3})$		1			1
$(3,2,1)$	1	1	1	1	1
$(\mathbf{2},\mathbf{2},\mathbf{2})$	1			1	
$(\mathbf{3},\mathbf{1},\mathbf{1},\mathbf{1})$			1	1	
$(\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$				1	1
$(1,1,1,1,1,1)$					1

	(4,2)
$(4,2)$	1

	(2,2,1,1)
$(2,2,1,1)$	1

3-Block of S_{14} with core $(3,1,1)$ [M. Fayers, 2002]

	$(12,1^2)$	$(9,4,1)$	$(9,3,2)$	$(8,4,2)$	$(6^2,2)$	$(6,4^4)$	$(6,4,2^2)$	$(6,3,2^2,1)$	$(5,4,2^2,1)$	$(4^2,2^2,1^2)$
$(12,1^2)=\langle 2\rangle$	1									
$(9,4,1)=\langle 2,2\rangle$	1	1								
$(9,3,2)=\langle 2,1\rangle$	2	1	1							
$(8,4,2)=\langle 1\rangle$	1	1	1	1						
$(6^2,2)=\langle 1,2\rangle$				1	1					
$(6,4^4)=\langle 1,2,2\rangle$			1	1	1	1				
$(6,4,2^2)=\langle 2,2,2\rangle$	1	1	1	1	1	1	1			
$(6,3,2^2,1)=\langle 1,1,2\rangle$	2	1	1				1	1		
$(5,4,2^2,1)=\langle 1,1\rangle$	1	1	1		1	1	1	1	1	
$(4^2,2^2,1^2)=\langle 3\rangle$	1		1		1	1		1	1	1
$(9,1^5)=\langle 2,3\rangle$			1							
$(6,4,1^4)=\langle 2,2,3\rangle$							1			
$(6,3,2,1^3)=\langle 1,2,3\rangle$			1			1	1	1		
$(6,2^3,1^2)=\langle 3,2\rangle$								1		
$(6,1^8)=\langle 2,3,3\rangle$						1				
$(5,4,2,1^3)=\langle 1,3\rangle$					2	1	1	1	1	
$(3^4,1^2)=\langle 3,1\rangle$	1		1			1				1
$(3^2,2^4)=\langle 1,1,3\rangle$	1									1
$(3^2,2^2,1^4)=\langle 1,1,1\rangle$					1	1			1	1
$(3^2,2,1^6)=\langle 1,3,3\rangle$					2	1			1	
$(3,2^3,1^5)=\langle 3,3\rangle$					1				1	
$(3,1^{11})=\langle 3,3,3\rangle$					1					

A more general result

Applying similar arguments to the twisted Foulkes modules $H^{(2^n)} \otimes \mathrm{sgn}_{S_k}\uparrow^{S_{2n+k}}$ gives analogous results for partitions with exactly k odd parts.

Theorem (Giannelli–MW)

Let p be an odd prime and let $k \in \mathbf{N}$. Let γ be a p-core and let $v_k(\gamma)$ be the minimum number of p-hooks that, when added to γ, give a partition with exactly k odd parts. Suppose that

$$v_k(\gamma) < v_{k-mp}(\gamma)$$

for all $m \in \mathbf{N}$. Let $\mathcal{O}$ be the set of partitions with exactly k odd parts that can be obtained from γ by adding $v_k(\gamma)$ p-hooks. Then the only non-zero rows in the column of the decomposition matrix labelled by λ are 1s in rows labelled by partitions in $\mathcal{O}$.

Example of more general theorem

Take $p = 3$ and $k = 2$. Start with the empty 3-core $\varnothing$ and try to reach a partition with 2 odd parts. This can't be done by adding one 3-hook. But it can be done by adding two 3-hooks, giving

$$\mathcal{O} = \{(5,1),(4,1,1),(3,3),(3,2,1)\}.$$

The column of the decomposition matrix labelled by $(5,1)$ is as predicted by the theorem.

	(6)	**(5,1)**	(4,2)	(3,3)	(4,1,1)	(3,2,1)	(2,2,1,1)
(6)	1						
(5,1)	1	**1**					
(4,2)	·	·	1				
(3,3)	·	**1**	·	1			
(4,1,1)	·	**1**	·	·	1		
(3,2,1)	1	**1**	·	1	1	1	
(2,2,1,1)	·	·	·	·	·	·	1
(2,2,2)	1	·	·	·	·	1	·
(3,1,1,1)	·	·	·	·	1	1	·
(2,1,1,1,1)	·	·	·	1	·	1	·
(1,1,1,1,1,1)	·	·	·	1	·	·	·