A tour of Foulkes’ Conjecture

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Outline

(1) Foulkes’ Conjecture
(2) Irreducible representations of symmetric groups
(3) Set families and maps
Let $S_r$ be the symmetric group on $\Omega = \{1, 2, \ldots, r\}$.

Let $C\Omega = \langle e_1, e_2, \ldots, e_r \rangle$. This is the natural permutation representation of $S_r$ where element of $S_r$ act by permutation matrices.
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Vector space decomposition:

$$C\Omega = \langle e_1 + e_2 + \cdots + e_r \rangle \bigoplus \langle e_i - e_j : 1 \leq i < j \leq r \rangle.$$
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Each summand is preserved by the action of $S_r$. No proper subspace of either summand is preserved, so each is an irreducible representation of $S_r$. 
Foulkes’ Conjecture

- Let $a, b \in \mathbb{N}$.
- Let $\Omega^b_a$ be the collection of set partitions of $\{1, 2, \ldots, ab\}$ into $b$ sets each of size $a$, acted on by $S_{ab}$.
- Let $\mathcal{C}_a^b$ be the corresponding permutation representation of $S_{ab}$.

If $U$ an irreducible representation of $S_{ab}$, let $[\mathcal{C}_a^b : U]$ denote the number of summands of $\mathcal{C}_a^b$ isomorphic to $U$. 
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Conjecture (Foulkes’ Conjecture)

*If $a < b$ and $U$ is an irreducible representation of $S_{ab}$ then*

$$[C\Omega^b_a : U] \geq [C\Omega^a_b : U].$$
Progress so far

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Proved for:

- \( a = 2 \), Thrall 1942;
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- $a + b \leq 20$, Mueller–Neunhöffer 2005;
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- \( a = 4 \), McKay 2008.
Other settings for Foulkes’ Conjecture

Representations of symmetric groups

Polynomial representations of $\text{GL}(E)$ where $E = \mathbb{C}^n$  
Symmetric functions
Other settings for Foulkes’ Conjecture

Representations of symmetric groups

Schur functor

Characteristic map

Polynomial representations of $\text{GL}(E)$ where $E = \mathbb{C}^n$

Symmetric functions

Formal characters
Other settings for Foulkes’ Conjecture

Representations of symmetric groups

Schur functor

$S^r E$

Polynomial representations of $GL(E)$ where $E = \mathbb{C}^n$

Characteristic map

$h_r$

$h_r(x, y) = x^r + x^{r-1}y + \cdots + y^r$

Symmetric functions

Formal characters
§2 Irreducible representations of $S_r$

Indexed by partitions of $r$, e.g. $(5, 2, 2) \in \text{Par}(9)$.

Specht module $S^\lambda$ is irreducible representation labelled by $\lambda$. Linearly spanned by all $\lambda$-tableaux: e.g. if $\lambda = (4, 2)$ then $S^{(4,2)}$ consists of all linear combinations of

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
5 & 6 \\
\end{array}, \quad \begin{array}{cccc}
4 & 3 & 5 & 6 \\
1 & 2 \\
\end{array}, \quad \begin{array}{cccc}
6 & 5 & 4 & 3 \\
2 & 1 \\
\end{array}, \ldots
\]

Satisfies Garnir relations:

- Column swaps:
  \[
  \begin{array}{cccc}
  4 & 3 & 5 & 6 \\
  1 & 2 \\
  \end{array} = \begin{array}{cccc}
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\end{array}
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- Shuffles. Related to determinantal identities:

\[
\begin{vmatrix}
a & b & \alpha & \beta \\
c & d & \gamma & \delta \\
\end{vmatrix} = \begin{vmatrix}
a & \beta & b & \alpha \\
c & \delta & d & \gamma \\
\end{vmatrix} - \begin{vmatrix}
a & \alpha & \beta & b \\
c & \gamma & \delta & d \\
\end{vmatrix}
\]
Hook formula

The set of tableaux whose rows increase from left to right, and whose columns increase from top to bottom form a basis for $S^\lambda$.

The hook formula states that if $\lambda$ is a partition of $r$

$$\dim S^\lambda = \frac{r!}{\prod_{\alpha \in \lambda} h_\alpha}$$

where $h_\alpha$ is the hook-length of the box $\alpha$. For example, the red box has hook-length 6 and $\dim S^{(5,4,2,1)} = \frac{12!}{8.6.4.3.1.6.4.2.1.3.1.1}$. 

\begin{array}{cccc}
8 & 6 & 4 & 3 \\
6 & 4 & 2 & 1 \\
3 & 1 & & \\
1 & & & \\
\end{array}
A set family of shape \((a^b)\) is a collection \(\mathcal{P}\) of \(b\) different sets each of size \(a\).

Say that \(\mathcal{P}\) has type \(\lambda\) if there are \(\lambda_i\) sets containing \(i\).
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Say that \(\mathcal{P}\) has type \(\lambda\) if there are \(\lambda_i\) sets containing \(i\).

If \(X, Y\) are sets of size \(a\), say that \(X\) is majorized by \(Y\) if one can write \(X = \{x_1, \ldots, x_a\}\) and \(Y = \{y_1, \ldots, y_a\}\) where \(x_1 \leq y_1, \ldots, x_a \leq y_a\).

Say that \(\mathcal{P}\) is closed if \(Y \in \mathcal{P}, X \preceq Y \implies X \in \mathcal{P}\)

Theorem

Let \(a\) be odd. If there is a closed set family of shape \((a^b)\) and type \(\lambda\) then \([\mathcal{C} \Omega^b_a : S^\lambda] \geq 1\).
Minimal constituents

- Say that a set family $\mathcal{P}$ is minimal if $\mathcal{P}$ has minimal type (in the dominance order) for its shape.
- Say that $S^\lambda$ is a minimal constituent of $\mathcal{C}^b_a$ if $[\mathcal{C}^b_a : S^\lambda] \geq 1$ and $\lambda$ is minimal with this property.

**Theorem**

If $a$ is odd then $S^\lambda$ is a minimal constituent of $\mathcal{C}^b_a$ if and only if there is a minimal set family of shape $(a^b)$ and type $\lambda$. 
Minimal constituents

- Say that a set family \( \mathcal{P} \) is \textit{minimal} if \( \mathcal{P} \) has minimal type (in the dominance order) for its shape.
- Say that \( S^\lambda \) is a \textit{minimal constituent} of \( \mathcal{C}\Omega^b_a \) if \([\mathcal{C}\Omega^b_a : S^\lambda] \geq 1 \) and \( \lambda \) is minimal with this property.

\textbf{Theorem}

If \( a \) is odd then \( S^\lambda \) is a minimal constituent of \( \mathcal{C}\Omega^b_a \) if and only if there is a minimal set family of shape \((a^b)\) and type \( \lambda \).

\textbf{Theorem}

Let \( \mathcal{P} \) be a set family. Then

\[ \mathcal{P} \text{ unique of its type} \implies \mathcal{P} \text{ minimal} \implies \mathcal{P} \text{ closed}. \]

None of these implications is reversible.