Plethysm of symmetric functions: modular isomorphisms and stability

Mark Wildon

Heilbronn Institute for Mathematical Research, Bristol University

Representation Theory in Birmingham, May 2025

LMS Celebrating New Appointments: Stacey Law

Joint work with: Álvaro Gutiérrez, Alvaro L. Martinez
Eoghan McDowell, Rowena Paget, Michał Szwej

§1 Motivation: the Wronskian isomorphism and categorification

§2 Polynomial representations and plethysms of Schur functions

§3 Maximal summands of plethysms

§4 Foulkes' Conjecture and plethysm stability

§1 Motivation: the Wronskian isomorphism

Let V be a vector space.

- $$\begin{aligned}\mathrm{Sym}^2 V &= V^{\otimes 2}/\langle v\otimes w - w\otimes v : v, w \in V\rangle \\ &= \langle vw : v \in V, w \in V\rangle\end{aligned}$$
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Observation. $\mathrm{Sym}^2\,\mathbb{C}^n$ and $\bigwedge^2 \mathbb{C}^{n+1}$ both have dimension $\binom{n+1}{2}$.

- Proof. If $e_1, \ldots, e_n$ is a basis for $\mathbb{C}^n$ then $\mathrm{Sym}^2\mathbb{C}^n$ has basis $e_1^2, \ldots, e_n^2, e_1 e_2, \ldots, e_{n-1} e_n$, of size $n + \binom{n}{2}$.

Question. Asked by მამუკა ჯიბლაძე on MathOverflow: Is there a natural isomorphism between these vector spaces?

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Answer. Yes!

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Answer. Yes! Let E be the 2-dimensional natural representation of $\mathrm{SL}_2(\mathbb{C})$. Then $\mathrm{Sym}^{n-1}E$ is n-dimensional and

$$\mathrm{Sym}^2\mathrm{Sym}^{n-1}E \cong_{\mathrm{SL}_2(\mathbb{C})} \bigwedge^2 \mathrm{Sym}^n E.$$

Are there nice isomorphisms $S^2(k^n) \cong \Lambda^2(k^{n+1})$?

Asked 1 year, 1 month ago Active 1 year, 1 month ago Viewed 349 times

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This might be forced to migrate to math.SE but let me still risk it.

The spaces $S^2(k^n)$ and $\Lambda^2(k^{n+1})$ from the title have equal dimensions. Is there a *natural* isomorphism between them?

⋮

share cite edit close flag edited Jan 15 '19 at 10:52

asked Jan 15 '19 at 9:45
მამუკა ჯიბლაძე
13.9k ● 3 ● 50 ● 125

19

Let E be a 2-dimensional k-vector space. The Wronksian isomorphism is an isomorphism of $\mathrm{SL}(E)$-modules $\bigwedge^m S^{m+r-1}(E) \cong S^m S^r(E)$. It is easiest to deduce it from the corresponding identity in symmetric functions (specialized to 1 and q), but it can also be defined explicitly: see for example Section 2.5 of this paper of Abdesselam and Chipalkatti.

In particular, identifying $S^n(E)$ with the homogeneous polynomial functions on E of degree n, their definition becomes the map $\wedge^2 S^n(E) \to S^2 S^{n-1}(E)$ defined by

$$f \wedge g \mapsto \frac{\partial f}{\partial X}\frac{\partial g}{\partial Y} - \frac{\partial f}{\partial Y}\frac{\partial g}{\partial X}.$$

Now $S^n(E) \cong k^{n+1}$ and $S^{n-1}(E) \cong k^n$, so we have the required isomorphism $S^2 k^n \cong \wedge^2 k^{n+1}$.

share cite edit delete flag edited Jan 15 '19 at 11:49

answered Jan 15 '19 at 11:09
Mark Wildon
8,018 ● 1 ● 32 ● 51

add a comment

Action of $\mathrm{SL}_2(F)$ on $\bigwedge^2 \mathrm{Sym}^2 E$ where $E = \langle v, w \rangle$

$$\begin{array}{c} \begin{array}{cc} v & w \end{array} \\ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \end{array} \longmapsto \begin{array}{c} \begin{array}{ccc} v^2 \wedge vw & w^2 \wedge vw & v^2 \wedge w^2 \end{array} \\ \begin{pmatrix} \alpha^3\delta - \alpha^2\beta\gamma & \alpha\beta^2\delta - \alpha\beta^2\gamma & 2\alpha^2\beta\delta - 2\alpha\beta^2\gamma \\ -\alpha\gamma^2\delta + \beta\gamma^3 & \alpha\delta^3 - \beta\gamma\delta^2 & 2\beta\gamma^2\delta - 2\alpha\gamma\delta^2 \\ \alpha^2\gamma\delta - \alpha\gamma^2\beta & \beta^2\gamma\delta - \alpha\beta\delta^2 & \alpha^2\delta^2 - \beta^2\gamma^2 \end{pmatrix} \end{array}$$

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- Even after the sign flip, this is not the matrix for $\mathrm{Sym}^2 E$.

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- Even after the sign flip, this is not the matrix for $\mathrm{Sym}^2 E$. The matrices are not even conjugate if $\mathrm{char}\, F = 2$!

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- Even after the sign flip, this is not the matrix for $\mathrm{Sym}^2 E$. The matrices are not even conjugate if $\mathrm{char}\, F = 2$! Instead it is $\mathrm{SL}_2(F)$ acting on $\mathrm{Sym}_2 E = \langle v \otimes v, w \otimes w, v \otimes w + w \otimes v \rangle$.
- Thus $(\mathrm{Sym}^2 E)^\star \cong_{\mathrm{SL}_2(F)} \bigwedge^2 \mathrm{Sym}^2 E$ and the duality is critical.

Duality and the modular Wronskian isomorphism

Theorem (McDowell–W 2020)

Let F be any field. Let M, $d \in \mathbb{N}$ and let E be the 2-dimensional natural representation of $\mathrm{SL}_2(F)$. There is an explicit isomorphism

$$\mathrm{Sym}_M \mathrm{Sym}^d E \cong_{\mathrm{SL}_2(F)} \bigwedge^M \mathrm{Sym}^{d+M-1} E.$$

Here $\mathrm{Sym}_M V$ is the invariant subspace of $V^{\otimes M}$ under the position permutation action of S_M and $\mathrm{Sym}^M V$ is the usual quotient of $V^{\otimes M}$.

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As a corollary we obtain a modular version of Hermite Reciprocity.

Corollary (Hermite 1854 over $\mathbb{C}$, McDowell–W 2020)

Let F be any field. Let M, $N \in \mathbb{N}$ and let E be the natural 2-dimensional representation of $\mathrm{GL}_2(F)$. Then

$$\mathrm{Sym}_M \mathrm{Sym}^N E \cong \mathrm{Sym}^N \mathrm{Sym}_M E$$

by an explicit map.

Decategorifying the Wronskian isomorphism

We start with

- $\mathrm{Sym}_M \mathrm{Sym}^d E \cong_{\mathrm{SL}_2(F)} \bigwedge^M \mathrm{Sym}^{d+M-1} E$

Taking dimensions, noting that $\mathrm{Sym}^d E$ has dimension $d+1$, gives

- $\left(\!\!\binom{d+1}{M}\!\!\right) = \binom{d+M}{M}$

where $\left(\!\binom{a}{b}\!\right)$ is the number of b-multisets of a set of size a.

But we went down too far! Instead take traces of $\begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix}$:

- $h_M(1, q, \dots, q^d) = q^{-\binom{M}{2}} e_M(1, q, \dots, q^{d+M-1})$
- $\displaystyle\sum_\lambda q^{|\lambda|} = \begin{bmatrix} M+d \\ M \end{bmatrix}$

where the sum is over all partitions in a $M \times d$ box.

Are there natural isomorphisms $S^{(2,1)}(k^{m+1}) \cong k^2 \otimes W$?

Asked 1 year, 6 months ago Modified 1 year, 6 months ago Viewed 340 times

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In this popular 2019 MO question, user მამუკა ჯიბლაძე asked:

> The spaces $S^2(k^n)$ and $\Lambda^2(k^{n+1})$ from the title have equal dimensions. Is there a *natural* isomorphism between them?

The answer was affirmative: let $E = k^2$, then the *Wronskian isomorphism* gives an identification $\bigwedge^m(S^{m+r-1}(E)) \to S_m(S^r(E))$ as $\mathrm{SL}(E)$-modules.

(Here, $S_m(V) = (V^{\otimes m})^{S_m}$ and $S^m(V)$ is the symmetric power, as a quotient of $V^{\otimes m}$.)

The recent paper Modular plethystic isomorphisms for two-dimensional linear groups of McDowell–Wildon contains the proof of this over any field.

My question is in a similar vein. Let ∇^λ be the Schur functor of the partition λ, and consider $\lambda_0 = (2,1)$. Then, the $\mathrm{SL}(E)$-character of $\nabla^{\lambda_0}(S^m(E))$ is simply

$$[2]\begin{bmatrix} m+2 \\ 3 \end{bmatrix},$$

where the brackets indicate that these are quantum numbers and binomials, so for instance $[2] = q + q^{-1}$.

Notice that this holds *for any* m. This suggests the possibility that there exists a natural isomorphism $\nabla^{\lambda_0}(S^m(E)) \cong E \otimes V_m$, for some "uniformly defined" $\mathrm{SL}(E)$-module V_m of dimension $\left[\begin{smallmatrix} m+2 \\ 3 \end{smallmatrix}\right]$.

> Is there such a natural isomorphism?

rt.representation-theory lie-groups lie-algebras binary-quadratic-forms Edit tags

Share Cite Edit Follow Close Flag edited Nov 18, 2023 at 19:28 asked Nov 15, 2023 at 21:45 Alvaro Martinez

$\mathrm{SL}_2(F)$-isomorphisms beyond symmetric powers

Theorem (Alvaro L. Martinez–W 2024)

Let F be any field and let E be the natural 2-dimensional representation of $\mathrm{SL}_2(F)$. Then

$$\Delta^{(2,1^{N-1})}\mathrm{Sym}^d E \cong_{\mathrm{SL}_2(F)} \mathrm{Sym}^{N-1}E \otimes \bigwedge^{N+1} \mathrm{Sym}^{d+1}E$$

This decategorifies to $s_{(2,1^{N-1})}(1, q, \ldots, q^d) = \begin{bmatrix} N \\ 1 \end{bmatrix} \begin{bmatrix} d+2 \\ N+1 \end{bmatrix}$.

Theorem (Álvaro Gutiérrez–Alvaro L. Martinez–Michał Szwej–W)

Let F be any field and let E be the natural 2-dimensional representation of $\mathrm{SL}_2(F)$. Then

$$\Delta^{(M,1^{N-1})}\mathrm{Sym}^d E \cong_{\mathrm{SL}_2(F)} \bigwedge^{M-1} \mathrm{Sym}^{M+N-3}E \otimes \bigwedge^{M+N-1} \mathrm{Sym}^{M+d-1}E.$$

§2 Polynomial representations and plethysms of Schur functions

- Polynomial representations of $\mathrm{GL}(E)$ with $E = \langle e_1, e_2, e_3 \rangle \cong \mathbb{C}^3$.

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  - $E \otimes E \otimes E \cong \mathrm{Sym}^3 E \oplus \bigwedge^3 E \ \oplus \ ?$

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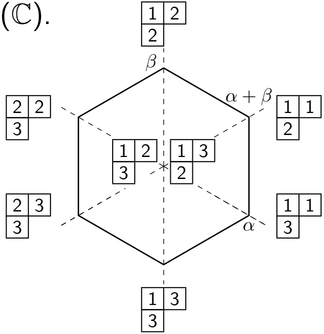
$\nabla^{(2,1)}(E) \subseteq \mathrm{Sym}^2 E \otimes E$ has basis $F(t)$ a for t semistandard tableaux of shape $(2,1)$ with entries from $\{1,2,3\}$:

$$F\left(\begin{array}{|c|c|}\hline a & b \\ \hline c & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array}\right) = e_a e_b \otimes e_c - e_c e_b \otimes e_a \in \mathrm{Sym}^2 E \otimes E.$$

You might also know it as the adjoint representation of the Lie algebra $\mathrm{sl}_3(\mathbb{C})$.



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  - Tensor product: $\mathrm{Sym}^2 E \otimes \mathrm{Sym}^2 E$
  - Symmetric power of symmetric power: $\mathrm{Sym}^2 \mathrm{Sym}^2 E$ with basis $(e_1^2)(e_1^2)$, $(e_1^2)(e_2^2)$, $(e_1^2)(e_1 e_2)$, $(e_2^2)(e_2^2)$, $(e_2^2)(e_1 e_2)$, $(e_1 e_2)(e_1 e_2)$

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- Symmetric functions
  - $s_{(2)}(y_1, y_2, y_3) = y_1^2 + y_1 y_2 + y_1 y_3 + y_2^2 + y_2 y_3 + y_3^3$
  - $$\begin{aligned} s_{(2,1)}(y_1, y_2, y_3) &= y^{\ytableaushort{11,2}} + y^{\ytableaushort{11,3}} + y^{\ytableaushort{12,2}} + y^{\ytableaushort{12,3}} + y^{\ytableaushort{13,2}} + \cdots + y^{\ytableaushort{22,3}} + y^{\ytableaushort{23,3}} \\ &= y_1^2 y_2 + y_1^2 y_3 + y_1 y_2^2 + 2 y_1 y_2 y_3 + \cdots + y_2^2 y_3 + y_2 y_3^2 \end{aligned}$$

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  - Multiplication: $s_{(2)}(x_1, x_2)^2 = (x_1^2 + x_2^2 + x_1 x_2)^2$
  - Evaluate $s_{(2)}(y_1, y_2, y_3)$ at monomials in $s_{(2)}(x_1, x_2)$ to get

$$s_{(2)}(x_1^2, x_2^2, x_1 x_2) = (x_1^2)(x_1^2) + (x_1^2)(x_2^2) + (x_1^2)(x_1 x_2) + \cdots + (x_1 x_2)(x_1 x_2).$$

§2 Polynomial representations and plethysms of Schur functions

- Polynomial representations of $\mathrm{GL}(E)$ with $E = \langle e_1, e_2, e_3 \rangle \cong \mathbb{C}^3$.
  - $E \otimes E \cong \mathrm{Sym}^2 E \oplus \bigwedge^2 E$
  - $E \otimes E \otimes E \cong \mathrm{Sym}^3 E \oplus \bigwedge^3 E \oplus \nabla^{(2,1)} E \oplus \nabla^{(2,1)} E$

  Now take $E = \langle e_1, e_2 \rangle \cong \mathbb{C}^2$
  - Tensor product: $\mathrm{Sym}^2 E \otimes \mathrm{Sym}^2 E$
  - Symmetric power of symmetric power: $\mathrm{Sym}^2 \mathrm{Sym}^2 E$ with basis $(e_1^2)(e_1^2)$, $(e_1^2)(e_2^2)$, $(e_1^2)(e_1 e_2)$, $(e_2^2)(e_2^2)$, $(e_2^2)(e_1 e_2)$, $(e_1 e_2)(e_1 e_2)$

- Symmetric functions
  - $s_{(2)}(y_1, y_2, y_3) \;= y_1^2 + y_1 y_2 + y_1 y_3 + y_2^2 + y_2 y_3 + y_3^3$
  - $s_{(2,1)}(y_1, y_2, y_3) = y^{\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}} + y^{\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}} + y^{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}} + y^{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}} + y^{\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}} + \cdots + y^{\begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}} + y^{\begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}}$
    $= y_1^2 y_2 + y_1^2 y_3 + y_1 y_2^2 + 2 y_1 y_2 y_3 + \cdots + y_2^2 y_3 + y_2 y_3^2$
  - Multiplication: $s_{(2)}(x_1, x_2)^2 = (x_1^2 + x_2^2 + x_1 x_2)^2$
  - Evaluate $s_{(2)}(y_1, y_2, y_3)$ at monomials in $s_{(2)}(x_1, x_2)$ to get

$$s_{(2)}(x_1^2, x_2^2, x_1 x_2) = (x_1^2)(x_1^2) + (x_1^2)(x_2^2) + (x_1^2)(x_1 x_2) + \cdots + (x_1 x_2)(x_1 x_2).$$

This is the plethysm $(s_{(2)} \circ s_{(2)})(x_1, x_2)$, obtained by evaluating $s_{(2)}$ at the monomials x_1^2, x_2^2, $x_1 x_2$ in $s_{(2)}(x_1, x_2)$.

Combinatorial definition of plethysm

Given a tableau t let $x^t = x_1^{a_1} x_2^{a_2} \dots$ where a_i is the number of entries of t equal to i. We say t has *weight* $(a_1, a_2, \dots)$.

Definition (Schur function)

Let μ be a partition. The *Schur function* s_μ is the generating function enumerating semistandard tableaux of shape μ by weight:

$$s_\mu = \sum_{t \in \mathrm{SSYT}(\mu)} x^t.$$

For instance

$$\begin{aligned} s_{(2)}(x_1, x_2, \dots) &= x^{\begin{array}{|c|c|}\hline 1 & 1 \\ \hline \end{array}} + x^{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array}} + x^{\begin{array}{|c|c|}\hline 2 & 2 \\ \hline \end{array}} + x^{\begin{array}{|c|c|}\hline 1 & 3 \\ \hline \end{array}} + \cdots \\ &= x_1^2 + x_1 x_2 + x_2^2 + x_1 x_3 + \cdots \end{aligned}$$

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Equivalently, $s_\mu(x_1, \ldots, x_d)$ is the trace of $\mathrm{diag}(x_1, \ldots, x_n)$ acting on $\nabla^\mu(E)$. For instance $s_{(n)}(x_1, \ldots, x_d)$ is the character of $\mathrm{Sym}^n E$.

Definition (Plethysm of Schur functions)

Let μ and ν be partitions. Let $\mathrm{SSYT}(\mu) = \{t(1), t(2), \ldots\}$. The *plethystic product* of s_ν and s_μ is $s_\nu \circ s_\mu = s_\nu(x^{t(1)}, x^{t(2)}, \ldots)$.

Stanley's Problem 9

By definition of the Hall inner product, $\langle f, s_\lambda \rangle$ is the multiplicity of s_λ as a summand of the symmetric function f.

Problem (Stanley's Problem 9, 2000)

Find a combinatorial interpretation of the plethysm coefficients $\langle s_{(n)} \circ s_{(m)}, s_\lambda \rangle$ *that makes it clear they are non-negative.*

Equivalently, find a combinatorial interpretation for the multiplicity of the irreducible $\mathrm{GL}_d(\mathbb{C})$-module $\nabla^\lambda(E)$ in $\mathrm{Sym}^n \mathrm{Sym}^m E$.

$\mathrm{SL}_2(\mathbb{C})$-plethysms revisited

Proposition

Let μ and ν be partitions and let d, $e \in \mathbb{N}$. Let E be the natural 2-dimensional representation of $\mathrm{SL}_2(\mathbb{C})$. The following are equivalent

- $\Delta^\mu \mathrm{Sym}^d E \cong_{\mathrm{SL}_2(\mathbb{C})} \Delta^\nu \mathrm{Sym}^e E$;
- $(s_\mu \circ s_{(d)})(q, q^{-1}) = (s_\nu \circ s_{(e)})(q, q^{-1})$;
- $(s_\mu \circ s_{(d)})(1, q) = (s_\nu \circ s_{(e)})(1, q)$ *up to a power of* q;
- $s_\mu(1, q, \ldots, q^d) = s_\nu(1, q, \ldots, q^e)$ *up to a power of* q.

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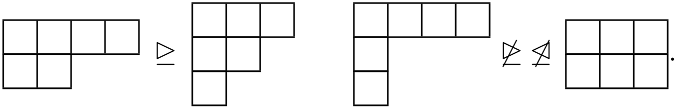
For example, Hermite Reciprocity

- $\mathrm{Sym}^M \mathrm{Sym}^N E \cong_{\mathrm{GL}_2(\mathbb{C})} \mathrm{Sym}^N \mathrm{Sym}^M E$;
- $s_{(M)} \circ s_{(N)}(1, q) = s_{(N)} \circ s_{(M)}(1, q)$;
- $s_{(M)}(1, q, \dots, q^N) = s_{(N)}(1, q, \dots, q^M)$;
- $|\mathrm{Par}_{M \times N}(n)| = |\mathrm{Par}_{N \times M}(n)|$ for all $n \in \mathbb{N}_0$.

where $\mathrm{Par}_{a \times b}(n)$ is the partitions of n fitting into an $a \times b$ box.

§3: Maximal summands in plethysms

A partition λ *dominates* a partition κ if the Young diagram of κ can be obtained from the Young diagram of λ by repeatedly moving boxes downwards. For instance



Quiz. Choose partitions κ and λ of n (a very large number) uniformly at random. What, roughly, is the chance that κ and λ are comparable in the dominance order?

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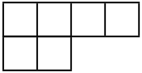
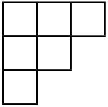
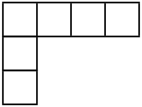
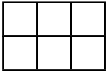
 $\trianglerighteq$   $\ntrianglerighteq\ \ntrianglelefteq$ .

Quiz. Choose partitions κ and λ of n (a very large number) uniformly at random. What, roughly, is the chance that κ and λ are comparable in the dominance order?

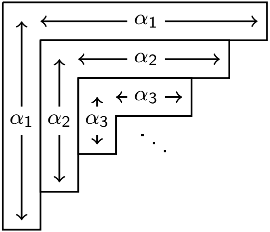
Answer. Asymptotically 0, by a theorem of Pittel (1997).

n	5	6	10	20	30	35
$p_{\mathrm{comparable}}$	1	0.967	0.904	0.782	0.716	0.694

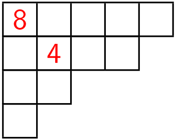
But no problem if you guessed something else: the convergence is very slow, and the small cases are misleading.

Most plethysms have many different maximal summands.

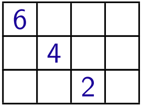
Extreme example: $s_{(1^n)} \circ s_{(2)}$. Let $n \in \mathbb{N}$. Given a partition α of n with distinct parts, let $2[\alpha]$ be the partition of $2n$ with leading diagonal hook lengths $2\alpha_1, 2\alpha_2, \ldots$



$2[4,2]$



$2[3,2,1]$



The plethysm $s_{(1^n)} \circ s_{(2)}$ corresponding to $\bigwedge^n \mathrm{Sym}^2 E$ is

$$s_{(1^n)} \circ s_2 = \sum_{\alpha \in \mathrm{Par}_{\mathrm{distinct}}(n)} s_{2[\alpha]}$$

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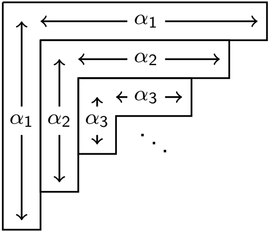

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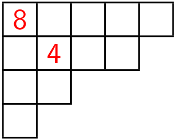

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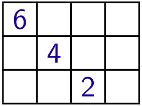

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For instance, if $n = 6$ then

$$s_{(1^6)} \circ s_2 = s_{(7,1^5)} + s_{(6,3,1,1,1)} + s_{(5,4,2,1)} + s_{(4,4,4)}$$

and $(7, 1^5), (6, 3, 1, 1, 1), (5, 4, 2, 1), (4, 4, 4)$ are all incomparable.

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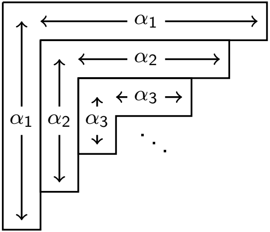
Extreme example: $s_{(1^n)} \circ s_{(2)}$. Let $n \in \mathbb{N}$. Given a partition α of n with distinct parts, let $2[\alpha]$ be the partition of $2n$ with leading diagonal hook lengths $2\alpha_1, 2\alpha_2, \ldots$



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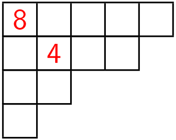

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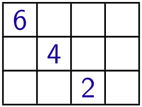

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in which $2[\alpha]$ and $2[\beta]$ are incomparable for all distinct α and β.

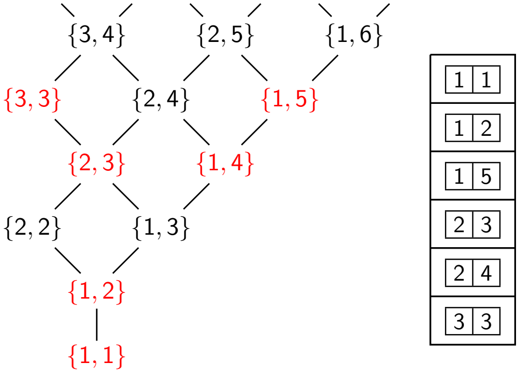
Thus every constituent of $s_{(1^n)} \circ s_{(2)}$ is both maximal *and* minimal. All of them are determined by our theorem.

Theorem (Paget–W 2018)

The maximal constituents of the plethysm $s_\nu \circ s_\mu$ are precisely the maximal weights of the plethystic semistandard tableaux of outer shape ν and inner shape μ.

A plethystic semistandard tableaux of outer shape (1^n) and inner shape (m) is the same as a set of n distinct m-multisets of $\mathbb{N}$, ordered by the majorization order.

Taking $m = 2$ we get the decomposition of $s_{(1^n)} \circ s_{(2)}$. For $n = 6$:



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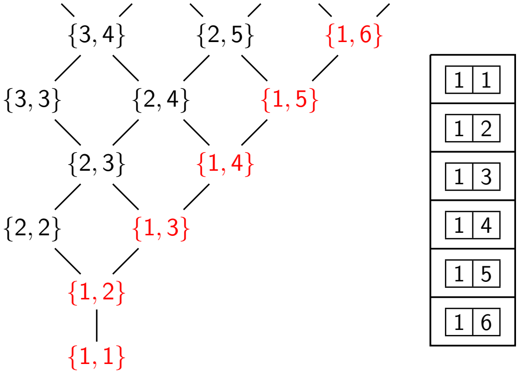

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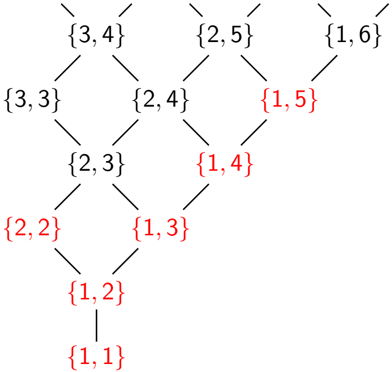

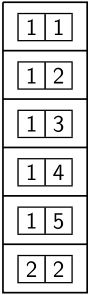

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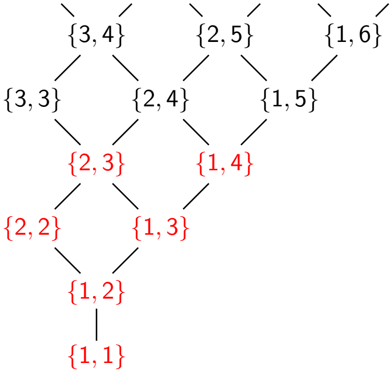

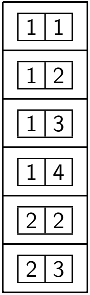

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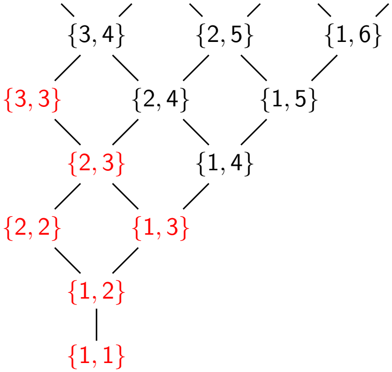

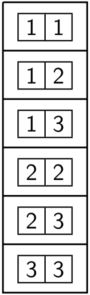

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A plethystic semistandard tableaux of outer shape (1^n) and inner shape (m) is the same as a set of n distinct m-multisets of $\mathbb{N}$, ordered by the majorization order.

- The 2018 proof uses the symmetric group.
- In 2020 with Melanie de Boeck we gave a shorter proof using polynomial representations of $\mathrm{GL}_n(\mathbb{C})$.
- Work in 2022–23 gives a still shorter combinatorial proof, with an explicit 'gap' result on the separation between maximal and minimal summands.

§4: Foulkes' Conjecture and plethysm stability

Conjecture (Foulkes 1950)

If $m \leq n$ then $s_{(n)} \circ s_{(m)} - s_{(m)} \circ s_{(n)}$ is a non-negative integer linear combination of Schur functions.

Equivalently

- There is an injective homomorphism of $\mathrm{GL}(E)$-modules $\mathrm{Sym}^m \mathrm{Sym}^n E \to \mathrm{Sym}^n \mathrm{Sym}^m E$ when $\dim E = mn$.
- There is an injective homomorphism of $\mathbb{C}S_{mn}$-modules $\mathbb{C}\big\uparrow_{S_n \wr S_m}^{S_{mn}} \to \mathbb{C}\big\uparrow_{S_m \wr S_n}^{S_{mn}}$
- Let S_{mn} act on set partitions of $\{1, \ldots, mn\}$. The permutation character for the action on m sets of size n contains the permutation character for the action on n sets of size m.

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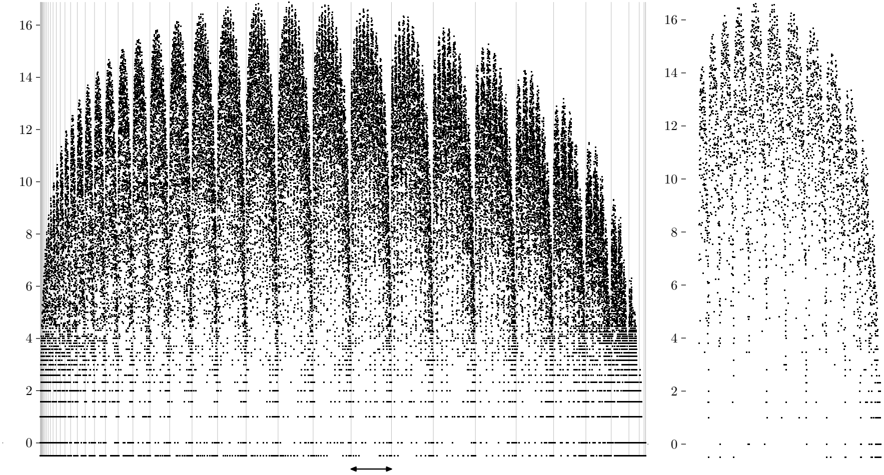
Proved when

- $m = 2$ Thrall (1942)
- $m = 3$ Thrall (1942), Dent and Siemons (2000)
- $m = 4$ McKay (2008),
- $m = 5$ Cheung, Ikenmeyer and Mkrychyan (2015)

and when $m + n \leq 20$, Evseev, Paget and Wildon (2008).

Foulkes Module $\mathrm{Sym}^7\mathrm{Sym}^8 E$

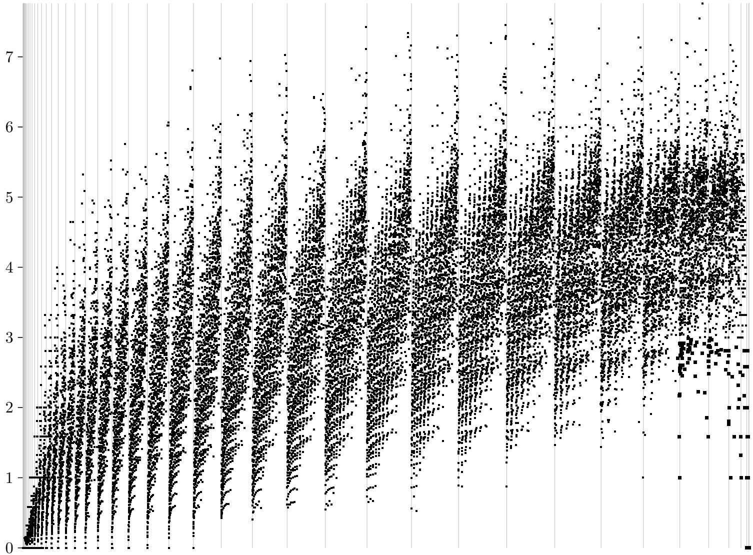
Logarithms of multiplicities of irreducibles $\nabla^\lambda(E)$

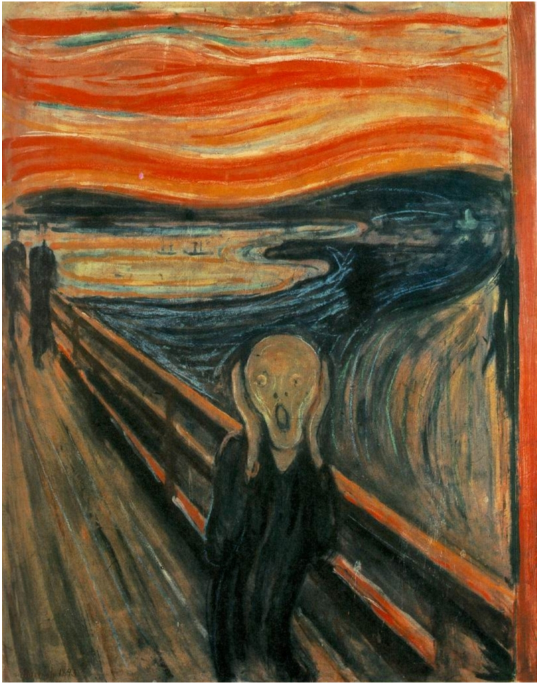


The marked interval, enlarged on right, is all partitions of 56 with first part 19

Foulkes Module $\mathrm{Sym}^7\mathrm{Sym}^8 E$

Logarithmic differences in multiplicities: for big dots, smaller multiplicity is 0.





Theorem (Stability for the Foulkes plethysm)

Let γ be a partition, and let $(mn - |\gamma|; \gamma)$ denote the partition $(mn - |\gamma|, \gamma_1, \ldots, \gamma_\ell)$. The plethysm coefficient

$$\langle s_{(n)} \circ s_{(m)}, s_{(mn-|\gamma|;\gamma)} \rangle$$

is constant for all m and n sufficiently large.

Hence stable Foulkes Conjecture holds, with equality. Proved by

- Weintraub (1988): recurrence relation on Schur functions
- Carré–Thibon (1992): vertex operators
- Brion (1993): dominant maps of algebraic varieties
- Manivel (1997): stable embeddings of varieties
- Bowman–Paget (2018): partition algebra
- Bowman–Paget–W (2023): ramified partition algebra, proving stability of $\langle s_{(\nu+(n-|\nu|))} \circ s_{(m)}, s_{(mn-|\gamma|;\gamma)} \rangle$ for any partition ν.
- Paget, W (2025): plethystic semistandard tableaux, more general

Theorem (Stability for the Foulkes plethysm)

Let γ be a partition, and let $(mn - |\gamma|; \gamma)$ denote the partition $(mn - |\gamma|, \gamma_1, \ldots, \gamma_\ell)$. The plethysm coefficient

$$\langle s_{(n)} \circ s_{(m)}, s_{(mn-|\gamma|;\gamma)} \rangle$$

is constant for all m and n sufficiently large.

Hence stable Foulkes Conjecture holds, with equality. Proved by

- Weintraub (1988): recurrence relation on Schur functions
- Carré–Thibon (1992): vertex operators
- Brion (1993): dominant maps of algebraic varieties
- Manivel (1997): stable embeddings of varieties
- Bowman–Paget (2018): partition algebra
- Bowman–Paget–W (2023): ramified partition algebra, proving stability of $\langle s_{(\nu+(n-|\nu|))} \circ s_{(m)}, s_{(mn-|\gamma|;\gamma)} \rangle$ for any partition ν.
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The BP and BPW proofs are notable as the only ones to give an explicit formula for the multiplicity that is clearly non-negative. This is a significant step towards a solution of Stanley's Problem 9.

Using combinatorial arguments with plethystic semistandard tableaux Paget and I have given unified proofs of every known stability result on plethysms of two Schur functions.

Here are two representative examples.

Theorem (Brion 1993)

Let $\nu \in \mathrm{Par}(n)$, $\mu \in \mathrm{Par}(m)$, $\lambda \in \mathrm{Par}(mn)$. Let $r \in \mathbb{N}$. The plethysm coefficient

$$\left\langle s_\nu \circ s_{\mu + N(1^r)}, s_{\lambda + N(n^r)} \right\rangle$$

is constant for all N at least an explicit bound.

Theorem (Paget–W)

Let $\nu \in \mathrm{Par}(n)$, $\mu/\mu^\star \in \mathrm{SkewPar}(m)$, $\lambda \in \mathrm{Par}(mn)$. Let $r \in \mathbb{N}$. The plethysm coefficient

$$\left\langle s_\nu \circ s_{((\mu + N\kappa)/\mu^\star}, s_{\lambda + Nn\kappa} \right\rangle$$

is constant for all N at least an explicit bound.

Theorem (Law–Okitani 2023)

Let $\nu \in \mathrm{Par}(n)$ and $\lambda \in \mathrm{Par}(mn)$. Let d be odd. The plethysm coefficient

$$\left\langle s_{\nu \sqcup (1^N)} \circ s_{(m)}, s_{\lambda + N(m-d) \sqcup (d^N)} \right\rangle$$

is constant for N sufficiently large.

Our methods generalize the Law–Okitani result further, from (m) to an arbitrary rectangular partition. The proof uses plethystic semistandard tableaux with negative entries and is entirely combinatorial.

Theorem (Paget–W)

Let $\nu \in \mathrm{Par}(n)$, let m, $b \in \mathbb{N}$ and let $\lambda \in \mathrm{Par}(mbn)$. Let d be odd. The plethysm coefficient

$$\left\langle s_{\nu \sqcup (1^N)} \circ s_{(m^b)}, s_{\lambda + N(m-d-1)b + N(1^b) \sqcup (d^N)} \right\rangle$$

is constant for N at least an explicit bound.

The ultimate stability results?

Set $\lambda \oplus (\alpha, \beta) = (\lambda + \beta) \sqcup \alpha'$. Let $\mu/\mu_\star$ be a skew partition and let ν be a partition.

Theorem (Signed inner stability, Paget–W 2025)

Let κ^- and κ^+ be partitions. If $|\kappa^-|$ is even then set $\nu^{(N)} = \nu$ for all N; if $|\kappa^-|$ is odd then set $\nu^{(N)} = \nu$ if N is even and $\nu^{(N)} = \nu'$ if N is odd. Then

$$\left\langle s_{\nu^{(N)}} \circ s_{(\mu \oplus N(\kappa^-, \kappa^+))/\mu_\star}, s_{\lambda \oplus nN(\kappa^-, \kappa^+)} \right\rangle$$

is constant for N at least an explicit bound.

Theorem (Signed outer stability, Paget–W 2025)

Let $R \in \mathbb{N}$. Let (κ^-, κ^+) be a strongly maximal signed weight of shape $\mu/\mu_\star$ and size R. Set $\nu^{(N)} = \nu + (N^R)$ if (κ^-, κ^+) has sign $+1$ and $\nu^{(N)} = \nu \sqcup (R^N)$ if (κ^-, κ^+) has sign -1. Then

$$\left\langle s_{\nu^{(N)}} \circ s_{\mu/\mu_\star}, s_{\lambda \oplus N(\kappa^-, \kappa^+)} \right\rangle$$

is constant for N at least an explicit bound.

Thank you! Any questions?

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Question. What other classical $\mathrm{SL}_2(\mathbb{C})$-isomorphisms have modular analogues?