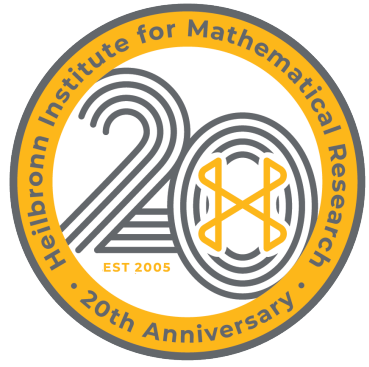
An introduction to plethysms of symmetric functions

Mark Wildon

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Cardiff GAPT Seminar, October 2026

Joint work with: Álvaro Gutiérrez, Álvaro L. Martínez
Eoghan McDowell, Rowena Paget, Michał Szwej
also starring: David Hemmer, Stacey Law, Michael Smith, Pavel Turek

§1 Motivation: the Wronskian isomorphism
§2 Combinatorial definition of plethysm
§3 Modular plethysms for the symmetric group
§4 Foulkes' Conjecture and plethysm stability

§1 Motivation: the Wronskian isomorphism

Let V be a complex vector space.

- $\operatorname{Sym}^2 V = V^{\otimes 2}/\langle v \otimes w - w \otimes v : v, w \in V \rangle$
  $= \langle vw : v \in V, w \in V \rangle$
- $\bigwedge^2 V = V^{\otimes 2}/\langle v \otimes v : v \in V \rangle$
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Observation. $\operatorname{Sym}^2 \mathbb{C}^n$ and $\bigwedge^2 \mathbb{C}^{n+1}$ both have dimension $\binom{n+1}{2}$.

- Part of a proof. If $e_1, \ldots, e_n$ is a basis for $\mathbb{C}^n$ then $\operatorname{Sym}^2 \mathbb{C}^n$ has basis $e_1e_2, e_1e_3, \ldots, e_{n-1}e_n, e_1^2, \ldots, e_n^2$ of size $\binom{n}{2} + n$.

Question. Asked by **მამუკა ჯიბლაძე** on MathOverflow: Is there a natural isomorphism between these vector spaces?

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Answer. Yes!

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Question. Asked by **მამუკა ჯიბლაძე** on MathOverflow: Is there a natural isomorphism between these vector spaces?

Answer. Yes! Let E be the 2-dimensional natural representation of $\operatorname{SL}_2(\mathbb{C})$. Then $\operatorname{Sym}^{n-1} E$ is n-dimensional and

$$\operatorname{Sym}^2 \operatorname{Sym}^{n-1} E \cong \bigwedge^2 \operatorname{Sym}^n E$$

where the isomorphism $\cong$ **is of representations of** $\operatorname{SL}_2(\mathbb{C})$**.**

Are there nice isomorphisms $S^2(k^n) \cong \Lambda^2(k^{n+1})$?

Asked 1 year, 1 month ago Active 1 year, 1 month ago Viewed 349 times

12

This might be forced to migrate to math.SE but let me still risk it.

The spaces $S^2(k^n)$ and $\Lambda^2(k^{n+1})$ from the title have equal dimensions. Is there a *natural* isomorphism between them?

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edited Jan 15 '19 at 10:52

asked Jan 15 '19 at 9:45



მამუკა ჯიბლაძე

13.9k • 3 • 50 • 125

19

Let E be a 2-dimensional k-vector space. The Wronksian isomorphism is an isomorphism of $\mathrm{SL}(E)$-modules $\bigwedge^m S^{m+r-1}(E) \cong S^m S^r(E)$. It is easiest to deduce it from the corresponding identity in symmetric functions (specialized to 1 and q), but it can also be defined explicitly: see for example Section 2.5 of this paper of Abdesselam and Chipalkatti.

In particular, identifying $S^n(E)$ with the homogeneous polynomial functions on E of degree n, their definition becomes the map $\wedge^2 S^n(E) \to S^2 S^{n-1}(E)$ defined by

$$f \wedge g \mapsto \frac{\partial f}{\partial X}\frac{\partial g}{\partial Y} - \frac{\partial f}{\partial Y}\frac{\partial g}{\partial X}.$$

Now $S^n(E) \cong k^{n+1}$ and $S^{n-1}(E) \cong k^n$, so we have the required isomorphism $S^2 k^n \cong \wedge^2 k^{n+1}$.

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edited Jan 15 '19 at 11:49

answered Jan 15 '19 at 11:09



Mark Wildon

8,018 • 1 • 32 • 51

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Action of $\mathrm{SL}_2(\mathbb{F})$ on $\bigwedge^2 \mathrm{Sym}^2 E$ where $E = \langle X, Y \rangle$

$$\begin{array}{c} \begin{matrix} X & Y \end{matrix} \\ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \end{array} \longmapsto \begin{array}{c} \begin{matrix} X^2 \wedge XY & Y^2 \wedge XY & X^2 \wedge Y^2 \end{matrix} \\ \begin{pmatrix} \alpha^3\delta - \alpha^2\beta\gamma & \beta^3\gamma - \alpha\beta^2\delta & 2\alpha^2\beta\delta - 2\alpha\beta^2\gamma \\ -\alpha\gamma^2\delta + \beta\gamma^3 & \alpha\delta^3 - \beta\gamma\delta^2 & 2\beta\gamma^2\delta - 2\alpha\gamma\delta^2 \\ \alpha^2\gamma\delta - \alpha\gamma^2\beta & \beta^2\gamma\delta - \alpha\beta\delta^2 & \alpha^2\delta^2 - \beta^2\gamma^2 \end{pmatrix} \end{array}$$

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- Even after the sign flip, this is not the matrix for $\mathrm{Sym}^2 E$. The matrices need not even be conjugate if $\mathrm{char}\ \mathbb{F} = 2$! Instead it's $\mathrm{SL}_2(\mathbb{F})$ acting on $\mathrm{Sym}_2 E = \langle X \otimes X, Y \otimes Y, X \otimes Y + Y \otimes X \rangle$.
- Thus $(\mathrm{Sym}^2 E)^\star \cong_{\mathrm{SL}_2(\mathbb{F})} \bigwedge^2 \mathrm{Sym}^2 E$ and the duality is critical.

Modular Wronskian isomorphism

$\mathrm{Sym}^M \mathbb{C}^{d+1}$ and $\bigwedge^M \mathbb{C}^{d+M}$ both have dimension $\binom{d+M}{M}$, because

$$\left(\!\!\binom{d+1}{M}\!\!\right) = \binom{d+M}{M}$$

where $\left(\!\!\binom{a}{b}\!\!\right)$ is the number of b-multisets of a set of size a.

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Theorem (McDowell–W 2020)

Let $\mathbb{F}$ be any field. Let M, $d \in \mathbb{N}$ and let E be the 2-dimensional natural representation of $\mathrm{SL}_2(\mathbb{F})$. There is an explicit isomorphism

$$\mathrm{Sym}_M \mathrm{Sym}^d E \cong_{\mathrm{SL}_2(\mathbb{F})} \bigwedge^M \mathrm{Sym}^{d+M-1} E.$$

Here $\mathrm{Sym}_M V$ is the invariant subspace of $V^{\otimes M}$ under the position permutation action of S_M and $\mathrm{Sym}^M V$ is degree M polynomials in V.

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Corollary (Hermite 1854, Aprodu *et al* 2019, McDowell–W 2020)

Let $\mathbb{F}$ be any field. Let M, $N \in \mathbb{N}$ and let E be the natural 2-dimensional representation of $\mathrm{GL}_2(\mathbb{F})$. Then

$$\mathrm{Sym}_M \mathrm{Sym}^N E \cong \mathrm{Sym}^N \mathrm{Sym}_M E.$$

Are there natural isomorphisms $S^{(2,1)}(k^{m+1}) \cong k^2 \otimes W$?

Asked 1 year, 6 months ago Modified 1 year, 6 months ago Viewed 340 times

7

In this popular 2019 MO question, user მამუკა ჯიბლაძე asked:

> The spaces $\mathrm{S}^2(k^n)$ and $\Lambda^2(k^{n+1})$ from the title have equal dimensions. Is there a *natural* isomorphism between them?

The answer was affirmative: let $E = k^2$, then the *Wronskian isomorphism* gives an identification $\bigwedge^m(S^{m+r-1}(E)) \to S_m(S^r(E))$ as $\mathrm{SL}(E)$-modules.

(Here, $S_m(V) = (V^{\otimes m})^{S_m}$ and $S^m(V)$ is the symmetric power, as a quotient of $V^{\otimes m}$.)

The recent paper Modular plethystic isomorphisms for two-dimensional linear groups of McDowell–Wildon contains the proof of this over any field.

My question is in a similar vein. Let ∇^λ be the Schur functor of the partition λ, and consider $\lambda_0 = (2,1)$. Then, the $\mathrm{SL}(E)$-character of $\nabla^{\lambda_0}(S^m(E))$ is simply

$$[2]\begin{bmatrix} m+2 \\ 3 \end{bmatrix},$$

where the brackets indicate that these are quantum numbers and binomials, so for instance $[2] = q + q^{-1}$.

Notice that this holds *for any* m. This suggests the possibility that there exists a natural isomorphism $\nabla^{\lambda_0}(S^m(E)) \cong E \otimes V_m$, for some "uniformly defined" $\mathrm{SL}(E)$-module V_m of dimension $\begin{bmatrix} m+2 \\ 3 \end{bmatrix}$.

> Is there such a natural isomorphism?

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edited Nov 18, 2023 at 19:28

asked Nov 15, 2023 at 21:45
Alvaro Martinez

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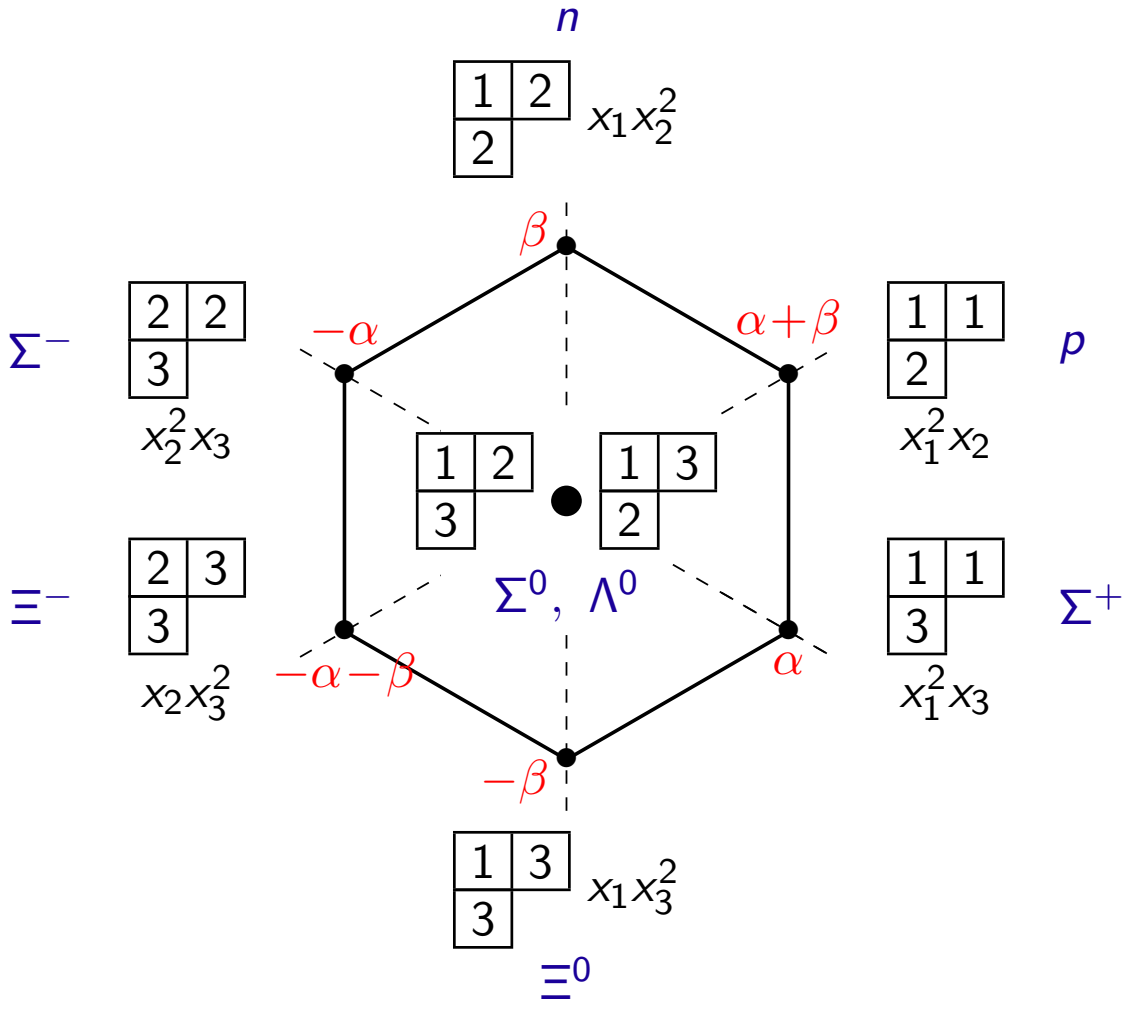
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$$V \otimes V \otimes V \cong \mathrm{Sym}^3 V \oplus \bigwedge^3 V \oplus \Delta^{(2,1)}(V) \oplus \Delta^{(2,1)}(V).$$

And when $\dim V = 3$ it is the adjoint representation of $\mathrm{SL}_3(\mathbb{C})$, or $\mathfrak{sl}_3(\mathbb{C})$, or $\mathfrak{su}_3(\mathbb{C})$, or $\mathrm{SU}_3(\mathbb{C})$. This is the Gell-Mann eightfold way.



$\mathrm{SL}_2(\mathbb{F})$-isomorphisms beyond symmetric powers

Theorem (Álvaro L. Martínez–W 2024)

Let $\mathbb{F}$ be any field and let E be the natural 2-dimensional representation of $\mathrm{SL}_2(\mathbb{F})$. Then

$$\Delta^{(2,1^{N-1})}\mathrm{Sym}^d E \cong_{\mathrm{SL}_2(\mathbb{F})} \mathrm{Sym}^{N-1} E \otimes \bigwedge^{N+1} \mathrm{Sym}^{d+1} E$$

This decategorifies to $s_{(2,1^{N-1})}(1, q, \ldots, q^d) = q^{\binom{N}{2}} \begin{bmatrix} N \\ 1 \end{bmatrix}_q \begin{bmatrix} d+2 \\ N+1 \end{bmatrix}_q$.

Thm (Álvaro Gutiérrez–Álvaro L. Martínez–Michał Szwej–W 2025)

Let $\mathbb{F}$ be any field and let E be the natural 2-dimensional representation of $\mathrm{SL}_2(\mathbb{F})$. Then

$$\Delta^{(M,1^{N-1})}\mathrm{Sym}^d E \cong_{\mathrm{SL}_2(\mathbb{F})} \bigwedge^{M-1} \mathrm{Sym}^{M+N-3} E \otimes \bigwedge^{M+N-1} \mathrm{Sym}^{M+d-1} E.$$

§2 Combinatorial definition of plethysm

Given a tableau t let $x^t = x_1^{a_1} x_2^{a_2} \dots$ where a_i is the number of entries of t equal to i. We say t has *weight* $(a_1, a_2, \dots)$.

Definition (Schur function)

Let μ be a partition. The *Schur function* s_μ is the generating function enumerating semistandard tableaux of shape μ by weight:

$$s_\mu = \sum_{t \in \mathrm{SSYT}(\mu)} x^t.$$

For instance

$$\begin{aligned} s_{(2)}(x_1, x_2, \dots) &= x^{\ytableaushort{11}} + x^{\ytableaushort{12}} + x^{\ytableaushort{22}} + x^{\ytableaushort{13}} + \cdots \\ &= x_1^2 + x_1 x_2 + x_2^2 + x_1 x_3 + \cdots \end{aligned}$$

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Setting $V = \mathbb{C}^d$, $s_\mu(x_1, \dots, x_d)$ is the trace of $\mathrm{diag}(x_1, \dots, x_d)$ acting on $\Delta^\mu(V)$. So $s_{(n)}(x_1, \dots, x_d)$ is the character of $\mathrm{Sym}^n V$.

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Definition (Plethysm of Schur functions)

Let μ and ν be partitions. Let $\mathrm{SSYT}(\mu) = \{t(1), t(2), \dots\}$. The *plethysm product* of s_ν and s_μ is $s_\nu \circ s_\mu = s_\nu(x^{t(1)}, x^{t(2)}, \dots)$.

$\mathrm{SL}_2(\mathbb{C})$-plethysms and the plethystic model

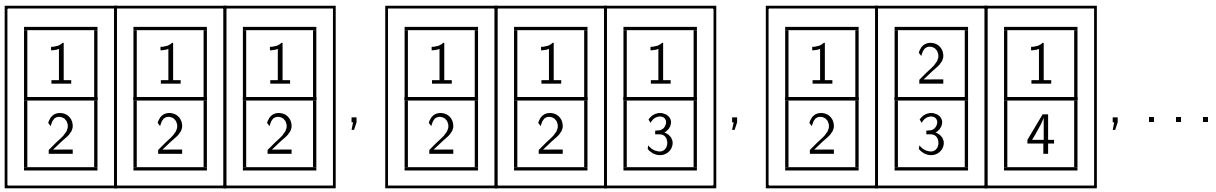
- $(s_{(M)} \circ s_{(d)})(1, q) = s_{(M)}(1, q, \ldots, q^d)$
  enumerates M-multisubsets of $\{0, 1, \ldots, d\}$.
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More generally, the plethysm product $s_\nu \circ s_\mu$ enumerates plethystic semistandard tableaux of outer shape ν and inner shape μ.

- $s_{(3)} \circ s_{(1,1)} = s_{(3,3)} + s_{(2,2,1,1)} + s_{(1,1,1,1,1,1)}$ enumerates 3-multisets of 2-sets:

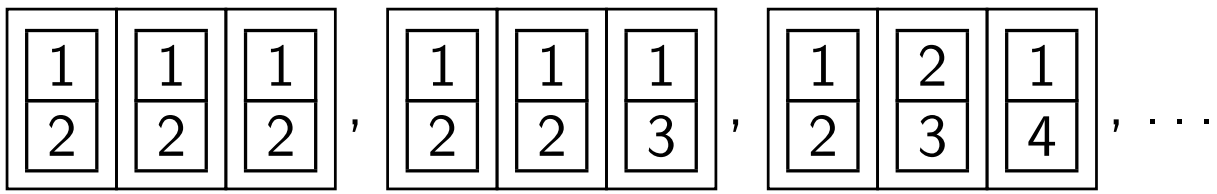


$\mathrm{SL}_2(\mathbb{C})$-plethysms and the plethystic model

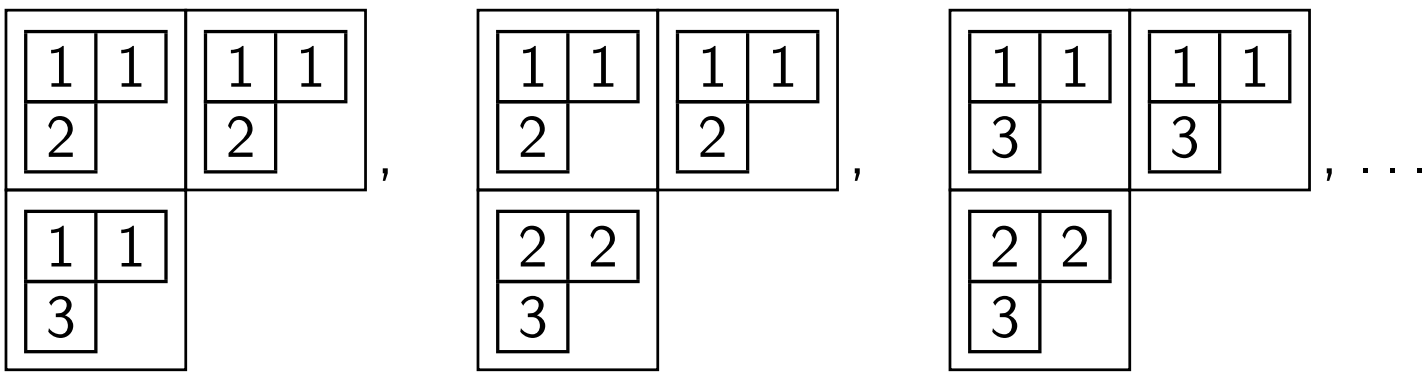
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- $s_{(3)} \circ s_{(1,1)} = s_{(3,3)} + s_{(2,2,1,1)} + s_{(1,1,1,1,1,1)}$ enumerates 3-multisets of 2-sets:



- $s_{(2,1)} \circ s_{(2,1)} = s_{(6,2,1)} + s_{(5,4)} + \cdots + s_{(2,2,2,2,1)}$ enumerates:



Stanley's Problem 9

By definition of the Hall inner product, $\langle f, s_\lambda \rangle$ is the multiplicity of s_λ as a summand of the symmetric function f.

Problem (Stanley's Problem 9, 2000)

Find a combinatorial interpretation of the plethysm coefficients $\langle s_{(n)} \circ s_{(m)}, s_\lambda \rangle$ *that makes it clear they are non-negative.*

Equivalently, find a combinatorial interpretation for the multiplicity of the irreducible $\mathrm{GL}_d(\mathbb{C})$-module $\Delta^\lambda(V)$ in $\mathrm{Sym}^n \mathrm{Sym}^m V$.

§3 Modular plethysms for the symmetric group

Problem (Decomposition numbers)

Determine the composition factors of Specht modules over fields of prime characteristic.

	(6)	(5,1)	(4,2)	(3,3)	(4,1,1)	(3,2,1)	(2,2,1,1)
(6)	1						
$(5,1)$	1	1					
$(4,2)$	·	·	1				
$(3,3)$	·	1	·	1			
$(4,1,1)$	·	1	·	·	1		
$(3,2,1)$	1	1	·	1	1	1	
$(2,2,1,1)$	·	·	·	·	·	·	1
$(2,2,2)$	1	·	·	·	·	1	·
$(3,1,1,1)$	·	·	·	·	1	1	·
$(2,1,1,1,1)$	·	·	·	1	·	1	·
$(1,1,1,1,1,1)$	·	·	·	1	·	·	·

For S_6 in characteristic 3, the Specht module $S^{(3,3)}$ has composition factors labelled by $(5,1)$ and $(3,3)$.

Even partitions and plethysms

For $n \in \mathbb{N}$,

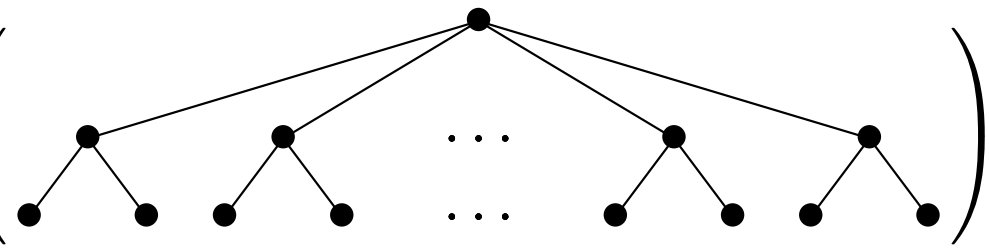
$$s_{(n)} \circ s_{(2)} = \sum_{\lambda \in \mathrm{Par}(n)} s_{2\lambda}$$

where 2λ is the *even* partition obtained by doubling each part of λ. Equivalently, for the symmetric group,

$$\mathbb{C}\uparrow_{S_2 \wr S_n}^{S_{2n}} = \bigoplus_{\lambda \in \mathrm{Par}(n)} S^{2\lambda}$$

where

$$S_2 \wr S_n = \mathrm{Aut}\left(\right.$$



$$\left. \right)$$

Given a partition γ, let $\mathcal{E}(\gamma)$ be the set of even partitions obtained from γ by adding the least possible number of disjoint p-hooks.

- For example if $p = 3$ then $\mathcal{E}\left(\ydiagram{2,1}\right) = \left\{ \right.$ $\left. \right\}$

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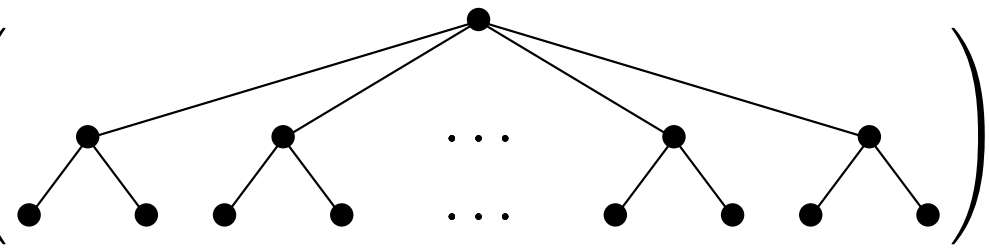
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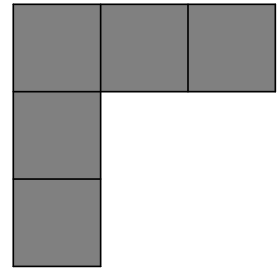


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Given a partition γ, let $\mathcal{E}(\gamma)$ be the set of even partitions obtained from γ by adding the least possible number of disjoint p-hooks.

- For example if $p = 3$ then $\mathcal{E}\left(\ydiagram{2,1}\right) = \Big\{(6,2),(4,4),(4,2,2)\Big\}$

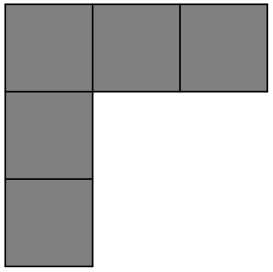
Decomposition Numbers: 3-block of S_{14} with core $(3,1,1)$

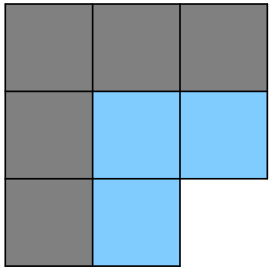


	$(12,1^2)$	$(9,4,1)$	$(9,3,2)$	$(8,4,2)$	$(6^2,2)$	$(6,4^4)$	$(6,4,2^2)$	$(6,3,2^2,1)$	$(5,4,2^2,1)$	$(4^2,2^2,1^2)$
$(12,1^2)=\langle 2\rangle$	1									
$(9,4,1)=\langle 2,2\rangle$	1	1								
$(9,3,2)=\langle 2,1\rangle$	2	1	1							
$(8,4,2)=\langle 1\rangle$	1	1	1	1						
$(6^2,2)=\langle 1,2\rangle$				1	1					
$(6,4^4)=\langle 1,2,2\rangle$			1	1	1	1				
$(6,4,2^2)=\langle 2,2,2\rangle$	1	1	1	1	1	1	1			
$(6,3,2^2,1)=\langle 1,1,2\rangle$	2	1	1				1	1		
$(5,4,2^2,1)=\langle 1,1\rangle$	1	1	1		1	1	1	1	1	
$(4^2,2^2,1^2)=\langle 3\rangle$	1		1		1	1		1	1	1
$(9,1^5)=\langle 2,3\rangle$			1							
$(6,4,1^4)=\langle 2,2,3\rangle$							1			
$(6,3,2,1^3)=\langle 1,2,3\rangle$			1			1	1	1		
$(6,2^3,1^2)=\langle 3,2\rangle$								1		
$(6,1^8)=\langle 2,3,3\rangle$						1				
$(5,4,2,1^3)=\langle 1,3\rangle$					2	1	1	1	1	
$(3^4,1^2)=\langle 3,1\rangle$	1		1			1				1
$(3^2,2^4)=\langle 1,1,3\rangle$	1									1
$(3^2,2^2,1^4)=\langle 1,1,1\rangle$					1	1			1	1
$(3^2,2,1^6)=\langle 1,3,3\rangle$					2	1			1	
$(3,2^3,1^5)=\langle 3,3\rangle$					1				1	
$(3,1^{11})=\langle 3,3,3\rangle$					1					

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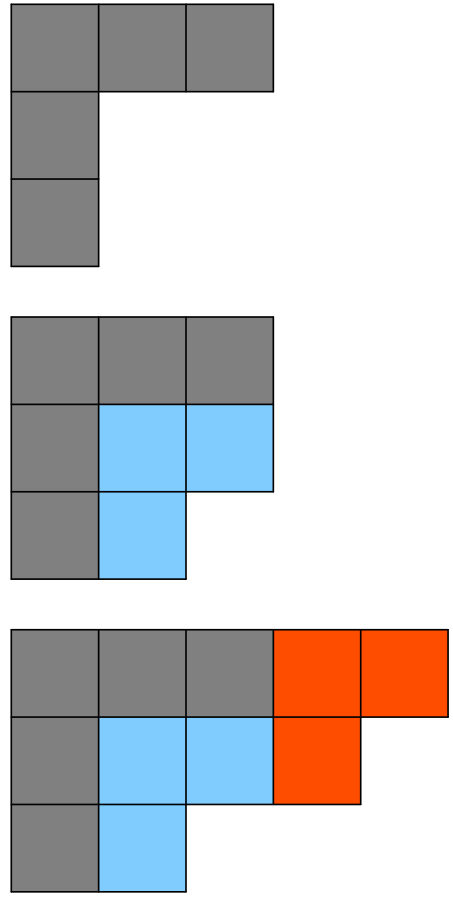
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$(6^2,2)=\langle 1,2\rangle$				1	1					
$(6,4^4)=\langle 1,2,2\rangle$			1	1	1	1				
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$(6,4,1^4)=\langle 2,2,3\rangle$							1			
$(6,3,2,1^3)=\langle 1,2,3\rangle$			1			1	1	1		
$(6,2^3,1^2)=\langle 3,2\rangle$								1		
$(6,1^8)=\langle 2,3,3\rangle$						1				
$(5,4,2,1^3)=\langle 1,3\rangle$					2	1	1	1	1	
$(3^4,1^2)=\langle 3,1\rangle$	1		1			1				1
$(3^2,2^4)=\langle 1,1,3\rangle$	1									1
$(3^2,2^2,1^4)=\langle 1,1,1\rangle$					1	1			1	1
$(3^2,2,1^6)=\langle 1,3,3\rangle$					2	1			1	
$(3,2^3,1^5)=\langle 3,3\rangle$					1				1	
$(3,1^{11})=\langle 3,3,3\rangle$					1					





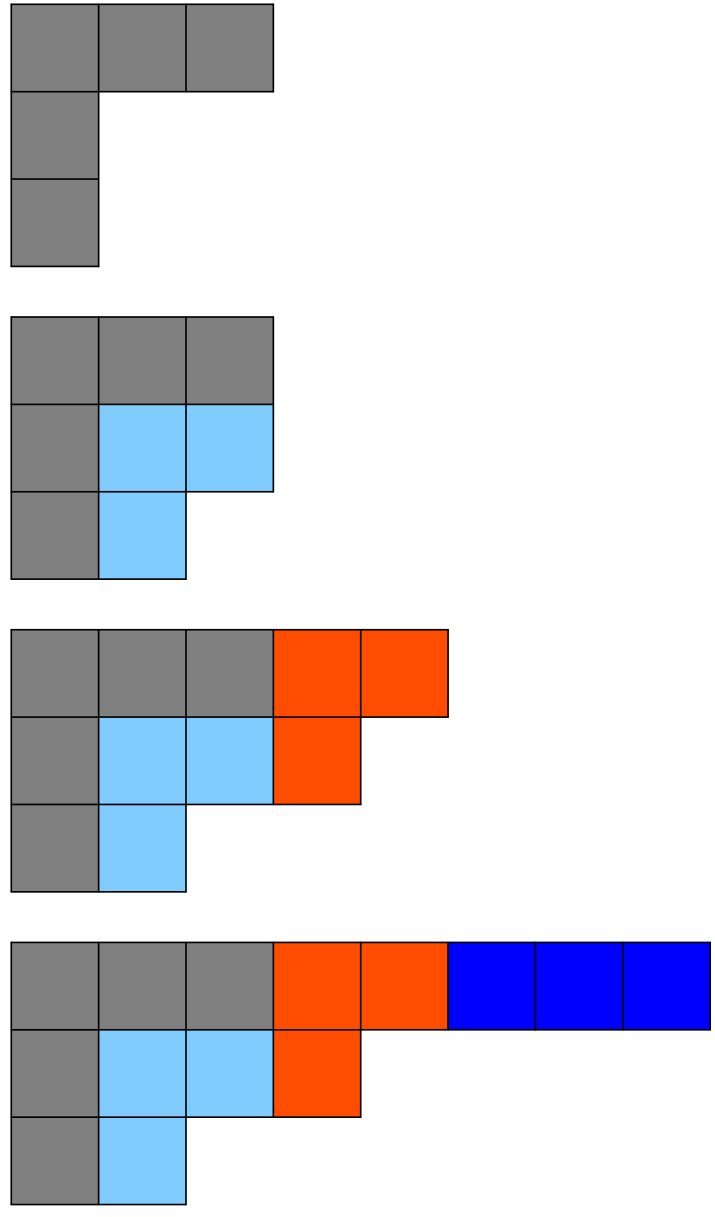
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$(6,4,2^2)=\langle 2,2,2\rangle$	1	1	1	1	1	1	1			
$(6,3,2^2,1)=\langle 1,1,2\rangle$	2	1	1				1	1		
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$(6,3,2,1^3)=\langle 1,2,3\rangle$			1			1	1	1		
$(6,2^3,1^2)=\langle 3,2\rangle$								1		
$(6,1^8)=\langle 2,3,3\rangle$						1				
$(5,4,2,1^3)=\langle 1,3\rangle$					2	1	1	1	1	
$(3^4,1^2)=\langle 3,1\rangle$	1		1			1				1
$(3^2,2^4)=\langle 1,1,3\rangle$	1									1
$(3^2,2^2,1^4)=\langle 1,1,1\rangle$					1	1			1	1
$(3^2,2,1^6)=\langle 1,3,3\rangle$					2	1			1	
$(3,2^3,1^5)=\langle 3,3\rangle$					1				1	
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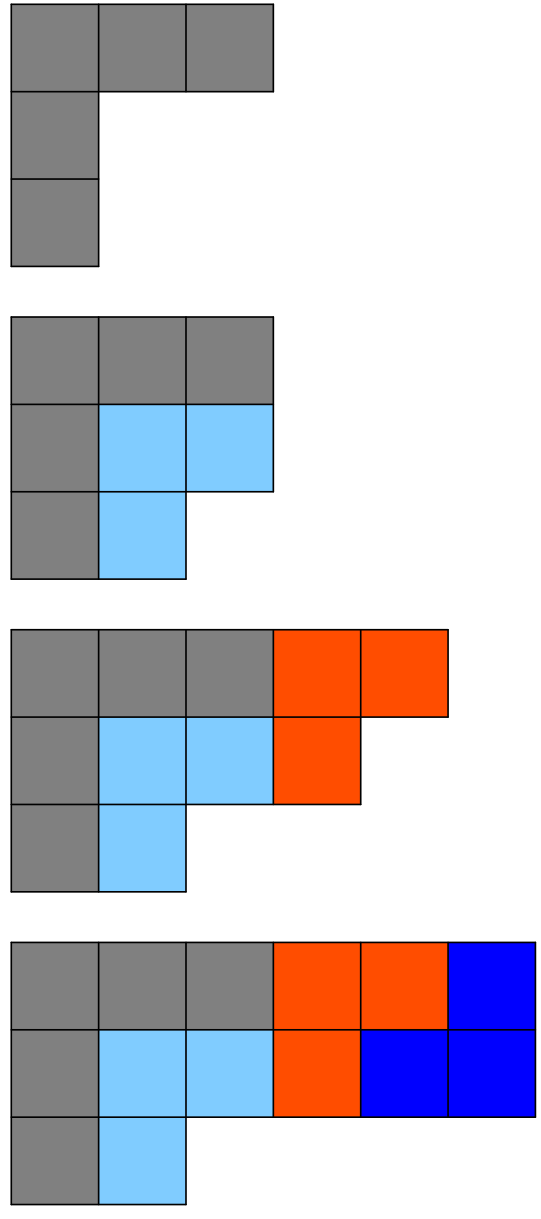
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$(6^2,2)=\langle 1,2\rangle$				1	1					
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$(6,3,2,1^3)=\langle 1,2,3\rangle$			1			1	1	1		
$(6,2^3,1^2)=\langle 3,2\rangle$								1		
$(6,1^8)=\langle 2,3,3\rangle$						1				
$(5,4,2,1^3)=\langle 1,3\rangle$					2	1	1	1	1	
$(3^4,1^2)=\langle 3,1\rangle$	1		1			1				1
$(3^2,2^4)=\langle 1,1,3\rangle$	1									1
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$(6,4,1^4)=\langle 2,2,3\rangle$							1			
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$(3^4,1^2)=\langle 3,1\rangle$	1		1			1				1
$(3^2,2^4)=\langle 1,1,3\rangle$	1									1
$(3^2,2^2,1^4)=\langle 1,1,1\rangle$					1	1			1	1
$(3^2,2,1^6)=\langle 1,3,3\rangle$					2	1			1	
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Plethysms predict columns of decomposition matrices

Theorem (Giannelli–W 2014)

Let p be an odd prime and let γ be a p-core. Let $\lambda \in \mathcal{E}(\gamma)$ be maximal. The column of the decomposition matrix labelled by λ has entries 0 and 1. Moreover its non-zero entries are in rows labelled by $\mathcal{E}(\gamma)$.

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Theorem (Hemmer–Turek 2026)

Let p be an odd prime and let γ be a p-core. Let ϑ be a weak composition of length p having a zero. Then there is a partition λ obtained from γ by adding $w_\vartheta(\gamma)$ p-hooks such that the column of the decomposition matrix labelled by λ has entries 0 and 1. Moreover its non-zero entries lie precisely in the rows labelled by $E_\vartheta(\gamma)$.

Plethysms give a column in every decomposition matrix of every block (i.e. fixed chosen p-core) of every symmetric group.

§4: Foulkes' Conjecture and plethysm stability

Conjecture (Foulkes 1950)

If $m \leq n$ then $s_{(n)} \circ s_{(m)} - s_{(m)} \circ s_{(n)}$ is a non-negative integer linear combination of Schur functions.

Equivalently

- There is an injective homomorphism of $\mathrm{GL}(V)$-modules $\mathrm{Sym}^m \mathrm{Sym}^n V \to \mathrm{Sym}^n \mathrm{Sym}^m V$ when $\dim V = m$.
- Let S_{mn} act on set partitions of $\{1, \ldots, mn\}$. The permutation character for the action on n sets of size m contains the permutation character for the action on m sets of size n.

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Proved for $m + n \leq 20$, Anton Evseev, Paget, W (2008), and for $m + n \leq 26$, Michael Smith (2026) and when

- $m = 2$ Thrall (1942)
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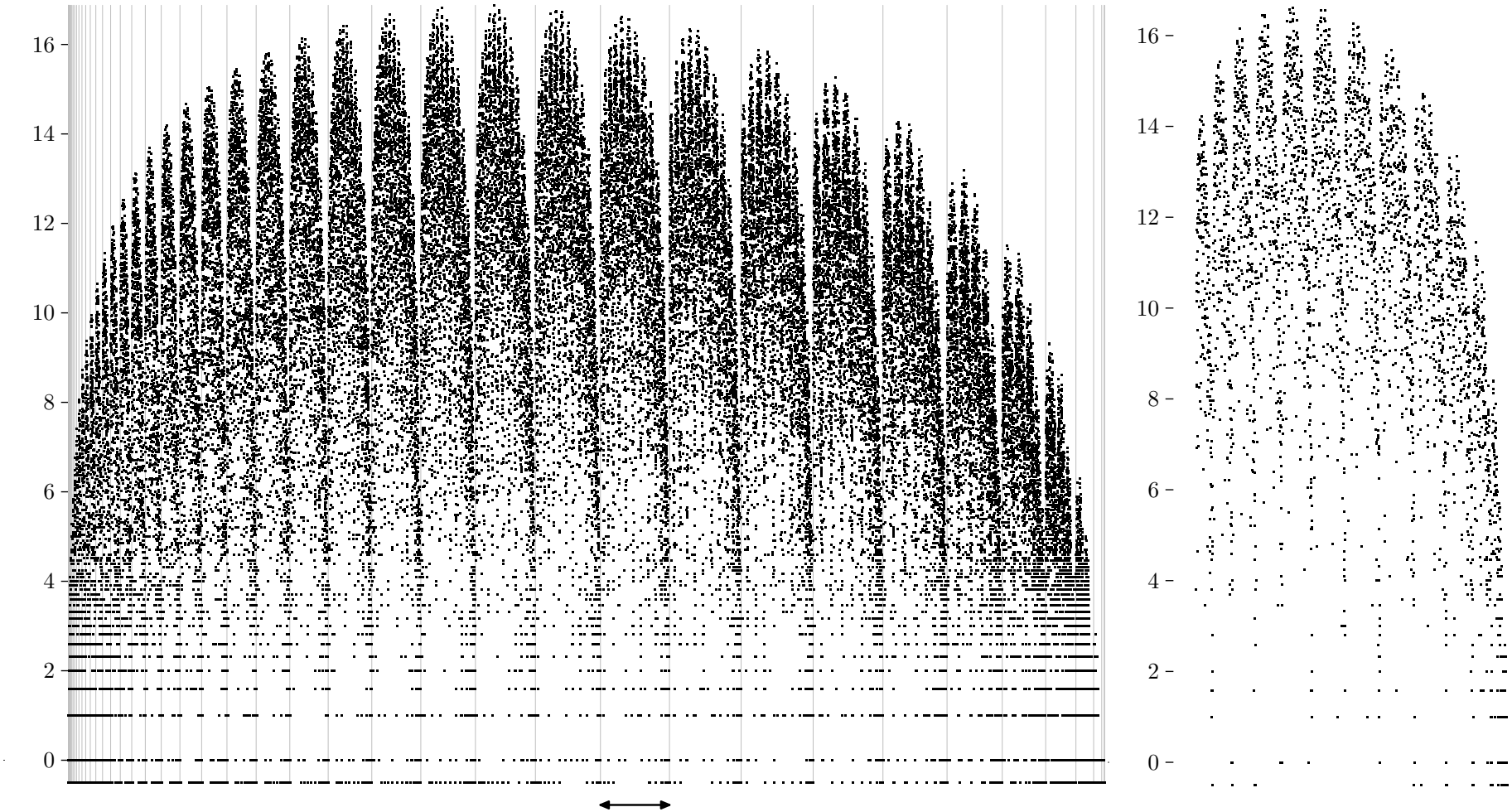
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- $m = 6$ OpenAI (October 6th, 2026), only Lean formalized if $n \geq 30$

Foulkes representation $\mathrm{Sym}^7\mathrm{Sym}^8 V$

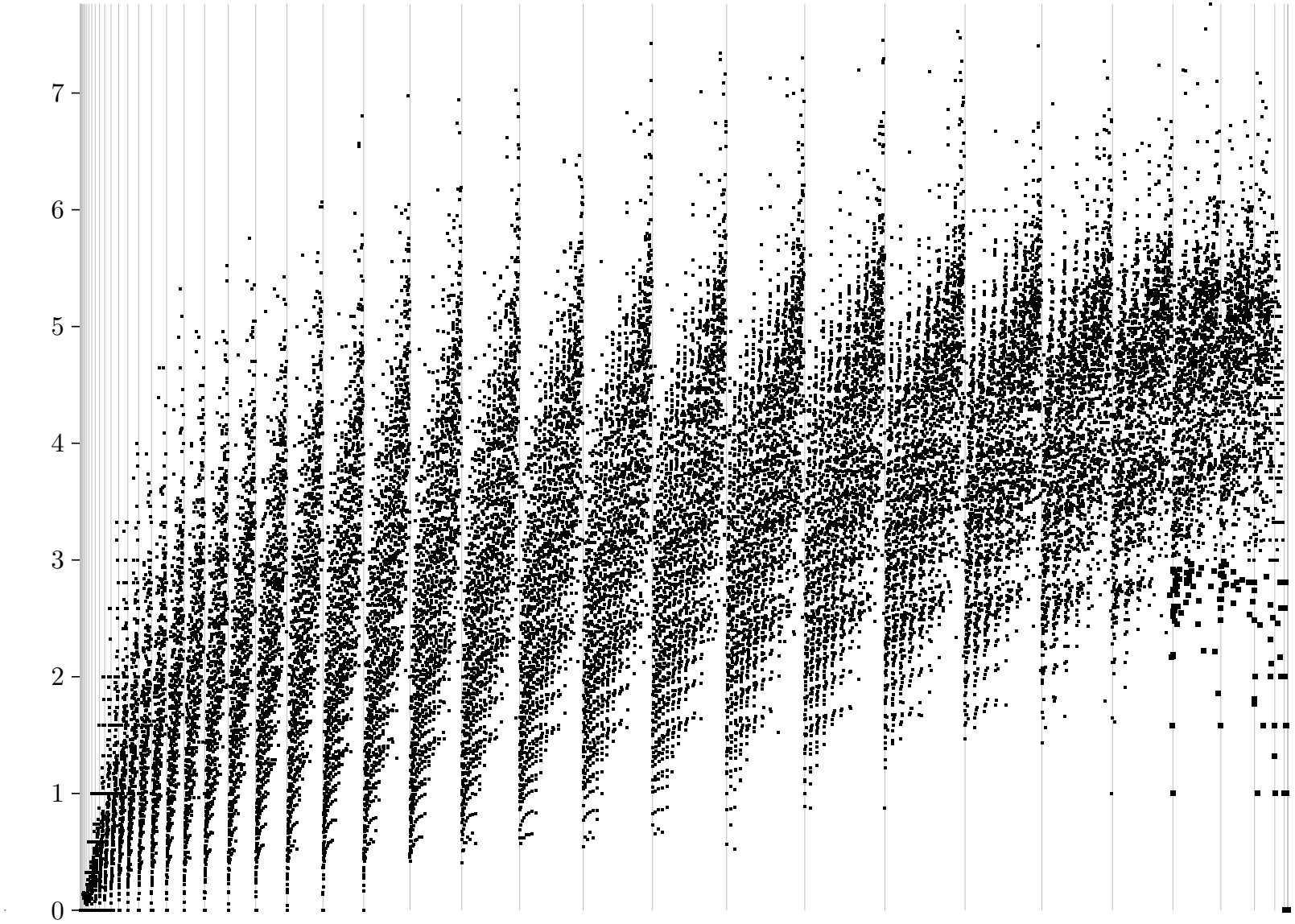
Logarithms of multiplicities of irreducibles $\Delta^\lambda(V)$



The marked interval, enlarged on right, is all partitions of 56 with first part 19

Foulkes representation $\mathrm{Sym}^7\mathrm{Sym}^8 V$

Logarithmic differences in multiplicities compared with $\mathrm{Sym}^8\mathrm{Sym}^7 V$: for big dots, smaller multiplicity is 0.





Stability: order in the chaos

In joint work with Paget we use combinatorial arguments with plethystic semistandard tableaux to give unified proofs of every known stability result on plethysms of two Schur functions. Most are significantly generalized. For instance:

Theorem (Law–Okitani 2021, 2023)

Let $\nu \in \mathrm{Par}(n)$ and $\lambda \in \mathrm{Par}(2n)$. The plethysm coefficient

$$\left\langle s_{\nu \sqcup (1^N)} \circ s_{(2)}, s_{\lambda + (N) \sqcup (1^N)} \right\rangle$$

is constant for N sufficiently large.

Theorem (Paget–W 2025)

Let $\nu \in \mathrm{Par}(n)$, let m, $b \in \mathbb{N}$ and let $\lambda \in \mathrm{Par}(mbn)$. Let $1 \leq d < m$ be such that bd is odd. The plethysm coefficient

$$\left\langle s_{\nu \sqcup (1^N)} \circ s_{(m^b)}, s_{(\lambda \sqcup (d^{Nb})) + N((m-d)^b)} \right\rangle$$

is constant for N at least an explicit bound.

Theorem (Stability for the Foulkes plethysm)

Let γ be a partition, and let $(mn - |\gamma|; \gamma)$ denote the partition $(mn - |\gamma|, \gamma_1, \ldots, \gamma_\ell)$. The plethysm coefficient

$$\left\langle s_{(n)} \circ s_{(m)}, s_{(mn-|\gamma|;\gamma)} \right\rangle$$

is constant for all m and n sufficiently large.

Hence the stable form of Foulkes' Conjecture holds, with equality.
Proved by

- Weintraub (1988): recurrence relation on Schur functions
- Carré–Thibon (1992): vertex operators
- Brion (1993): dominant maps of algebraic varieties
- Manivel (1997): stable embeddings of varieties
- Bowman–Paget (2018): partition algebra
- Bowman–Paget–W (2023): ramified partition algebra, proving stability of $\left\langle s_{(\nu+(n-|\nu|))} \circ s_{(m)}, s_{(mn-|\gamma|;\gamma)} \right\rangle$ for any partition ν.
- Paget–W (2025): plethystic semistandard tableaux, more general

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The BP and BPW proofs are notable as the only ones to give an explicit formula for the multiplicity that is clearly non-negative. This is a significant step towards a solution of Stanley's Problem 9.

Thank you! Any questions?

