Plethysms: permutations, weights and Schur functions

Mark Wildon (joint work with Rowena Paget)
Outline

- §1 Motivation: Examples of plethysms
- §2 Main result: Minimal and maximal constituents of $s_\nu \circ s_\mu$
§1 Polynomial representations of $\text{GL}(V)$

Let $V$ be a finite-dimensional $\mathbb{C}$-vector space.

- The natural representation of $\text{GL}(V)$ is irreducible.

$V \otimes V \cong \text{Sym}^2 V \oplus \wedge^2 V$.

$V \otimes V^3 \cong \text{Sym}^3 V \oplus \wedge^3 V \oplus \ldots$.

$u = (v_1 \wedge v_2) \otimes v_1 \in \wedge^2 V \otimes V$ is highest weight, weight $(2,1)$.

Why highest weight? Check $u$ killed by Lie algebra action of $e \in \mathfrak{gl}(V)$, defined by $e(v_2) = v_1$, $e(v_i) = 0$ if $i \neq 2$:

$eu = (ev_1 \wedge v_2) \otimes v_1 + (v_1 \wedge ev_2) \otimes v_1 + (v_1 \wedge v_2) \otimes ev_1 = 0 + 0 + 0$.

Two isomorphic complementary submodules are generated by $(v_1 \otimes v_2 - v_2 \otimes v_1) \otimes v_1$ and $v_1 \otimes (v_1 \otimes v_2 - v_2 \otimes v_1)$.

Generally $V \otimes V^r \cong \bigoplus_{\lambda \in \text{Par}(r)} (\Delta_\lambda V) \oplus d_\lambda V$ where $\Delta_\lambda V$ is the unique irreducible representation of $\text{GL}(V)$ of highest weight $\lambda$.

For instance $\text{Sym}^n V = \Delta_{(n)} V$, $\wedge^n V = \Delta_{(1^n)} V$.
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  - Two isomorphic complementary submodules are generated by $(v_1 \otimes v_2 - v_2 \otimes v_1) \otimes v_1$ and $v_1 \otimes (v_1 \otimes v_2 - v_2 \otimes v_1)$. 

Generally $V \otimes r \cong \bigoplus_{\lambda \in \text{Par}(r)} (\Delta_{\lambda} V) \oplus d_{\lambda}$ where $\Delta_{\lambda} V$ is the unique irreducible representation of $\text{GL}(V)$ of highest weight $\lambda$. For instance $\text{Sym}^n V = \Delta_{\binom{n}{2}} V$, $\wedge^n V = \Delta_{(1^n)} V$. 

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$$ V \otimes^r \cong \bigoplus_{\lambda \in \text{Par}(r)} (\Delta^\lambda V)^{d_\lambda} $$

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Plethysm: Composing polynomial representations

Consider $\text{Sym}^2(\text{Sym}^2 V) \rightarrow \text{Sym}^4 V: (uv)(u'v') \mapsto uvu'v'$.

- Kernel is $\Delta^{(2,2)} V$. Why? $(v_1 v_1)(v_2 v_2) - (v_1 v_2)(v_1 v_2)$ is highest weight, of weight $(2, 2)$. 
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- $\text{Sym}^2(\text{Sym}^2 V) \cong \Delta^{(2,2)} V \oplus \Delta^{(4)} V$. 

Next step up:

$\phi \in \text{Sym}^4(\text{Sym}^2 V) = \bigotimes_4 \text{Sym}^2 V$ may

- Vanish doubly on $C$: $(Y_{11} Y_{22} - Y_{21} Y_{12})^2$
- Vanish singly on $C$: $Y_{21} Y_{11} (Y_{11} Y_{22} - Y_{21} Y_{12})$

Such functions are in kernel of $\text{Sym}^4(\text{Sym}^2 V) \to \text{Sym}^8 V$, so $\text{Sym}^4(\text{Sym}^2 V) \cong \Delta^{(4,4)} V \oplus \Delta^{(6,2)} V \oplus \Delta^{(8)} V$. 
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  - $\text{Sym}^2 V = \langle v_1 v_1, 2v_1 v_2, v_2 v_2 \rangle_\mathbb{C}$
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- \( \text{Sym}^2(\text{Sym}^2 V) \cong \Delta^{(2,2)} V \oplus \Delta^{(4)} V \).
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  - \( \text{Sym}^2 V = \langle v_1 v_1, 2v_1 v_2, v_2 v_2 \rangle_\mathbb{C} \)
  - \( \mathcal{O}(\text{Sym}^2 V) = \mathbb{C}[Y_{11}, Y_{12}, Y_{22}] \)
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- Take $\dim V = 2$. Geometrically:
  - $\text{Sym}^2 V = \langle \nu_1 \nu_1, 2 \nu_1 \nu_2, \nu_2 \nu_2 \rangle_{\mathbb{C}}$
  - $\mathcal{O}(\text{Sym}^2 V) = \mathbb{C}[Y_{11}, Y_{12}, Y_{22}]$
  - Let $C$ be the image of the squaring map $V \hookrightarrow \text{Sym}^2 V$,
    \[
    \alpha \nu_1 + \beta \nu_2 \mapsto \alpha^2 \nu_1 \nu_1 + 2\alpha\beta \nu_1 \nu_2 + \beta^2 \nu_2 \nu_2
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    $$\alpha v_1 + \beta v_2 \mapsto \alpha^2 v_1v_1 + 2\alpha\beta v_1v_2 + \beta^2 v_2v_2$$
  - $C = \text{Zeros}(Y_{11}Y_{22} - Y_{12}^2)$; the $\text{GL}(V)$-submodule of $
    \mathcal{O}(\text{Sym}^2 V)$ generated by $Y_{11}Y_{22} - Y_{12}^2$ is $\Delta^{(2,2)} V$. 

Next step up: $f \in \text{Sym}^4(\text{Sym}^2 V) = \mathcal{O}(\text{Sym}^2 V)$ may

- Vanish doubly on $C$: $(Y_{11}Y_{22} - Y_{12}^2)^2$
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Next step up: $f \in \text{Sym}^4(\text{Sym}^2 V) = \mathcal{O}(\text{Sym}^2 V)_4$ may
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- $\text{Sym}^2(\text{Sym}^2 V) \cong \Delta^{(2,2)} V \oplus \Delta^{(4)} V$.
- Take $\dim V = 2$. Geometrically:
  - $\text{Sym}^2 V = \langle v_1 v_1, 2v_1 v_2, v_2 v_2 \rangle_C$
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  - let $\mathcal{C}$ be the image of the squaring map $V \hookrightarrow \text{Sym}^2 V$,
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Plethysm: Symmetric groups and wreath products

Take $\dim V \geq 4$. So $S_4 \leq \text{GL}(V)$: $\begin{pmatrix} \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix}$.

Weight space $(1, 1, 1, 1)$ inside $\text{Sym}_2(\text{Sym}_2(V))$ is

$\langle (v_1 v_2)(v_3 v_4), (v_1 v_3)(v_2 v_4), (v_1 v_4)(v_2 v_3) \rangle$.

Identify $(v_1 v_2)(v_3 v_4)$ with the set partition $\{\{1, 2\}, \{3, 4\}\}$.

Stabiliser $S_2 \wr S_2 = (S_2 \times S_2) \rtimes S_2 = \langle (1 2), (3 4) \rangle \rtimes \langle (1 3)(2 4) \rangle$.

Weight space is permutation module $C^{\uparrow S_4}_{S_2 \wr S_2}$.

Character $\chi(2, 2) + \chi(4)$, corresponding to $\Delta(2, 2) \oplus \Delta(4)$. 
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Take \( \dim V \geq 4 \). So \( S_4 \leq GL(V) \): \((1234) \mapsto \begin{pmatrix} \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \end{pmatrix} \).

▶ Weight space \((1, 1, 1, 1)\) inside \( \text{Sym}^2(\text{Sym}^2 V) \) is
\[
\langle (v_1 v_2)(v_3 v_4), (v_1 v_3)(v_2 v_4), (v_1 v_4)(v_2 v_3) \rangle.
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- Identify \((v_1 v_2)(v_3 v_4)\) with the set partition \(\{\{1, 2\}, \{3, 4\}\}\).
- Stabiliser \( S_2 \wr S_2 = (S_2 \times S_2) \rtimes S_2 = \langle (12), (34) \rangle \rtimes \langle (13)(24) \rangle. \)
Take $\dim V \geq 4$. So $S_4 \leq \text{GL}(V)$: $(1234) \mapsto \begin{pmatrix} 
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 1 & \cdot & \cdot & \cdot \\
 \cdot & 1 & \cdot & \cdot \\
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\end{pmatrix}$.

- Weight space $(1, 1, 1, 1)$ inside $\text{Sym}^2(\text{Sym}^2 V)$ is
  $\langle (v_1 v_2)(v_3 v_4), (v_1 v_3)(v_2 v_4), (v_1 v_4)(v_2 v_3) \rangle$.
- Identify $(v_1 v_2)(v_3 v_4)$ with the set partition $\{\{1, 2\}, \{3, 4\}\}$.
- Stabiliser $S_2 \wr S_2 = (S_2 \times S_2) \rtimes S_2 = \langle (12), (34) \rangle \rtimes \langle (13)(24) \rangle$.

- Weight space is permutation module $\mathbb{C}^{\uparrow S_4}_{S_2 \wr S_2}$
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The weight space is the permutation module \( \mathbb{C}^{\uparrow_{S_4} S_2 \wr S_2} \).

- Character \( \chi^{(2,2)} + \chi^{(4)} \), corresponding to \( \Delta^{(2,2)} \oplus \Delta^{(4)} \).
Imprimitivity is surprisingly primitive!

Let \( f(X) \in \mathbb{Q}[X] \) be irreducible with roots \( \alpha_1, \ldots, \alpha_d \in \mathbb{C} \).

Then \( \text{Gal}(f) \) acts on \( K = \mathbb{Q}[\alpha_1, \ldots, \alpha_d] \), permuting the roots \( \alpha_1, \ldots, \alpha_d \) transitively.
Imprimitivity is surprisingly primitive!

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- Let \( L = \mathbb{Q}[\sqrt{\alpha_1}, \ldots, \sqrt{\alpha_d}] \). Then \( \text{Gal}(L/K) \leq C_2 \times \cdots \times C_2 \) and
Imprimitivity is surprisingly primitive!

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For example, \( X^3 - 12X - 4 = (X - \alpha)(X - \beta)(X - \gamma) \) has Galois group \( S\{\alpha, \beta, \gamma\} \). Since \( \alpha \beta \gamma = 4 \in \mathbb{Q}^\times \), \( \text{Gal}(X^6 - 12X^2 - 4) \) is a proper subgroup of \( C_2 \wr S_3 \):

\[
\text{Gal}(L/\mathbb{Q}) = \left\langle \left( \sqrt{\alpha}, -\sqrt{\alpha} \right), \left( \sqrt{\beta}, -\sqrt{\beta} \right), \left( \sqrt{\gamma}, -\sqrt{\gamma} \right) \right\rangle 
\times \left\langle \left( \sqrt{\alpha}, \sqrt{\beta} \right), \left( -\sqrt{\alpha}, -\sqrt{\beta} \right), \left( \sqrt{\alpha}, \sqrt{\beta}, \sqrt{\gamma} \right), \left( -\sqrt{\alpha}, -\sqrt{\beta}, -\sqrt{\gamma} \right) \right\rangle 
\leq C_2 \wr S_3
\]

\[
\text{Gal}(L/K) = \left\langle \left( \sqrt{\alpha}, -\sqrt{\alpha} \right), \left( \sqrt{\beta}, -\sqrt{\beta} \right), \left( \sqrt{\gamma}, -\sqrt{\gamma} \right) \right\rangle 
\leq C_2 \wr S_3
\]
Foulkes’ Conjecture

Let $\Omega^{(mn)}$ be the set of all set partitions of $\{1, 2, \ldots, mn\}$ into $n$ sets each of size $m$.

Conjecture (Foulkes)

If $m \leq n$ then there is an injective map of $S_{mn}$-representations

$$\langle \Omega^{(rn)} \rangle_C \to \langle \Omega^{(mn)} \rangle_C.$$

$\phi^{(rn)} = \chi^{(2n)} + \chi^{(2n-2, 2)} + \chi^{(2n-4, 4)} + \cdots$

$\phi^{(2n)} = \chi^{2 \lambda} \to H$ Hence FC holds when $m = 2$.

These are the only multiplicity-free Foulkes characters for $mn \geq 18$ (Saxl, 1980).
Foulkes’ Conjecture

Let $\Omega^{(mn)}$ be the set of all set partitions of $\{1, 2, \ldots, mn\}$ into $n$ sets each of size $m$.

Conjecture (Foulkes)

If $m \leq n$ then there is an injective map of $S_{mn}$-representations

$$\langle \Omega^{(nm)} \rangle_C \to \langle \Omega^{(mn)} \rangle_C.$$

Equivalently, there is an injective map of $\text{GL}(V)$-representations

$$\text{Sym}^m(\text{Sym}^n V) \to \text{Sym}^n(\text{Sym}^m V).$$
Foulkes’ Conjecture

Let $\Omega^{(mn)}$ be the set of all set partitions of $\{1, 2, \ldots, mn\}$ into $n$ sets each of size $m$.

Conjecture (Foulkes)

If $m \leq n$ then there is an injective map of $S_{mn}$-representations $\langle \Omega^{(nm)} \rangle_{\mathbb{C}} \rightarrow \langle \Omega^{(mn)} \rangle_{\mathbb{C}}$.

Equivalently, if $\phi^{(mn)}$ is the character of $\langle \Omega^{(mn)} \rangle_{\mathbb{C}}$, then $\langle \phi^{(nm)}, \chi^{\lambda} \rangle \leq \langle \phi^{(mn)}, \chi^{\lambda} \rangle$ for all $\lambda \in \text{Par}(mn)$. 
Foulkes’ Conjecture

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$$\langle \phi^{(nm)}, \chi^\lambda \rangle \leq \langle \phi^{(mn)}, \chi^\lambda \rangle$$

for all $\lambda \in \text{Par}(mn)$.

$$\phi^{(n^2)} = \chi^{(2n)} + \chi^{(2n-2,2)} + \chi^{(2n-4,4)} + \cdots$$

$$\phi^{(2^{n})} = \sum_{\lambda \in \text{Par}(n)} \chi^{2\lambda}$$

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If $m \leq n$ then there is an injective map of $S_{mn}$-representations
\[\langle \Omega^{(nm)} \rangle_{\mathbb{C}} \rightarrow \langle \Omega^{(mn)} \rangle_{\mathbb{C}}.\]

Equivalently, if $\phi^{(mn)}$ is the character of $\langle \Omega^{(mn)} \rangle_{\mathbb{C}}$, then
\[\langle \phi^{(nm)}, \chi^\lambda \rangle \leq \langle \phi^{(mn)}, \chi^\lambda \rangle \text{ for all } \lambda \in \text{Par}(mn).\]

\[\phi^{(n^2)} = \chi^{(2n)} + \chi^{(2n-2,2)} + \chi^{(2n-4,4)} + \ldots\]

\[\phi^{(2^n)} = \sum_{\lambda \in \text{Par}(n)} \chi^{2\lambda}\]

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Foulkes’ Conjecture

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Conjecture (Foulkes)

If $m \leq n$ then there is an injective map of $S_{mn}$-representations $\langle \Omega^{(n^m)} \rangle_C \rightarrow \langle \Omega^{(m^n)} \rangle_C$.

Equivalently, if $\phi^{(m^n)}$ is the character of $\langle \Omega^{(m^n)} \rangle_C$, then $\langle \phi^{(n^m)}, \chi^\lambda \rangle \leq \langle \phi^{(m^n)}, \chi^\lambda \rangle$ for all $\lambda \in \text{Par}(mn)$.

\[
\phi^{(n^2)} = \chi^{(2n)} + \chi^{(2n-2,2)} + \chi^{(2n-4,4)} + \ldots \\
\phi^{(2^n)} = \sum_{\lambda \in \text{Par}(n)} \chi^{2\lambda}
\]

- Hence FC holds when $m = 2$.
- These are the only multiplicity-free Foulkes characters for $mn \geq 18$ (Saxl, 1980).
Decomposition Numbers

- Giannelli, MW 2014: results on decomposition numbers of symmetric groups obtained from local structure of $\langle \Omega^{(2^n)} \rangle$ over fields of prime characteristic.

\[(1, 2, 3, 4) \mapsto \begin{pmatrix} \nu_1 & \nu_2 & \nu_3 & \nu_4 \\ . & . & . & 1 \\ 1 & . & . & . \\ . & 1 & . & . \\ . & . & 1 & . \end{pmatrix}\]
Decomposition Numbers

- Giannelli, MW 2014: results on decomposition numbers of symmetric groups obtained from local structure of $\langle \Omega^{(2^n)} \rangle$ over fields of prime characteristic.

$$(1, 4, 3) \mapsto \begin{pmatrix}
v_1 & v_2 & v_3 & v_4 \\
\cdot & \cdot & 1 & \cdot \\
\cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 \\
1 & \cdot & \cdot & \cdot \\
\end{pmatrix}$$
Decomposition Numbers

- Giannelli, MW 2014: results on decomposition numbers of symmetric groups obtained from local structure of $\langle \Omega^{(2^n)} \rangle$ over fields of prime characteristic.

$$(1, 2)(3, 4) \mapsto \begin{pmatrix} v_1 & 1 & . & . \\ 1 & . & . & . \\ . & . & . & 1 \\ . & . & 1 & . \end{pmatrix}$$
Decomposition Numbers

- Giannelli, MW 2014: results on decomposition numbers of symmetric groups obtained from local structure of $\langle \Omega^{(2^n)} \rangle$ over fields of prime characteristic.

$$(1, 2)(3, 4) \mapsto \begin{pmatrix} v_1 & v_2 & v_3 & v_4 \\ \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \end{pmatrix}$$

In the new basis

$$w_1 = v_1 + v_2 + v_3 + v_4$$
$$w_2 = v_2 - v_1$$
$$w_3 = v_3 - v_1$$
$$w_4 = v_4 - v_1$$
Decomposition Numbers

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$$(1, 2)(3, 4) \mapsto \begin{pmatrix} v_1 & v_2 & v_3 & v_4 \\ \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \end{pmatrix} \mapsto \begin{pmatrix} w_1 & w_2 & w_3 & w_4 \\ 1 & \cdot & \cdot & \cdot \\ \cdot & -1 & -1 & -1 \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \end{pmatrix}$$

In the new basis

$$w_1 = v_1 + v_2 + v_3 + v_4$$
$$w_2 = v_2 - v_1$$
$$w_3 = v_3 - v_1$$
$$w_4 = v_4 - v_1$$
Decomposition Numbers

- Giannelli, MW 2014: results on decomposition numbers of symmetric groups obtained from local structure of $\langle \Omega^{(2^n)} \rangle$ over fields of prime characteristic.

\[
\begin{pmatrix}
v_1 & v_2 & v_3 & v_4 \\
\cdot & 1 & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & 1 & \cdot \\
\end{pmatrix}
\mapsto
\begin{pmatrix}
1 & \cdot & \cdot & \cdot \\
\cdot & -1 & -1 & -1 \\
\cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot \\
\end{pmatrix}
\]

In the new basis
\[
\begin{align*}
w_1 &= v_1 + v_2 + v_3 + v_4 \\
w_2 &= v_2 - v_1 \\
w_3 &= v_3 - v_1 \\
w_4 &= v_4 - v_1
\end{align*}
\]
Decomposition Numbers

- Giannelli, MW 2014: results on decomposition numbers of symmetric groups obtained from local structure of $\langle \Omega^{(2^n)} \rangle$ over fields of prime characteristic.

\[(1, 2)(3, 4) \mapsto \begin{pmatrix} w_2 & w_3 & w_4 \\ -1 & -1 & -1 \\ . & . & 1 \\ . & 1 & . \end{pmatrix} \mapsto \begin{pmatrix} z & w_3 & w_4 \\ 1 & 1 & 1 \\ . & 1 & . \\ . & . & 1 \end{pmatrix}\]

In the rational basis
\[w_2 = v_2 - v_1\]
\[w_3 = v_3 - v_1\]
\[w_4 = v_4 - v_1\]

In the $\mathbb{F}_2$-basis
\[z = v_1 + v_2 + v_3 + v_4\]
\[w_3 = v_3 - v_1\]
\[w_4 = v_4 - v_1\]
Decomposition Numbers

- Giannelli, MW 2014: results on decomposition numbers of symmetric groups obtained from local structure of \( \langle \Omega^{(2^n)} \rangle \) over fields of prime characteristic.

\[(1, 2, 3, 4) \mapsto \begin{pmatrix} w_2 & w_3 & w_4 \\ -1 & -1 & -1 \\ 1 & . & . \\ . & 1 & . \end{pmatrix} \mapsto \begin{pmatrix} z & w_3 & w_4 \\ 1 & 1 & 1 \\ . & 1 & 1 \\ . & . & 1 \end{pmatrix}\]

In the rational basis
\[w_2 = v_2 - v_1\]
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\[w_4 = v_4 - v_1\]

In the \( \mathbb{F}_2 \)-basis
\[z = v_1 + v_2 + v_3 + v_4\]
\[w_3 = v_3 - v_1\]
\[w_4 = v_4 - v_1\]
Giannelli, MW 2014: results on decomposition numbers of symmetric groups obtained from local structure of \( \langle \Omega^{(2^n)} \rangle \) over fields of prime characteristic.

\[
(1, 2, 3, 4) \mapsto \begin{pmatrix} w_2 & w_3 & w_4 \\ -1 & -1 & -1 \\ 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \end{pmatrix} \mapsto \begin{pmatrix} z & w_3 & w_4 \\ 1 & 1 & 1 \\ \cdot & 1 & 1 \\ \cdot & \cdot & 1 \end{pmatrix}
\]

In the rational basis

\[
w_2 = v_2 - v_1 \\
w_3 = v_3 - v_1 \\
w_4 = v_4 - v_1
\]

In the \( F_2 \)-basis

\[
z = v_1 + v_2 + v_3 + v_4 \\
w_3 = v_3 - v_1 \\
w_4 = v_4 - v_1
\]

Hence \( S^{(3,1)}_{F_2} \) has a trivial submodule.
The quotient is a 2-dimensional simple \( F_2 S_4 \)-module
Decomposition matrix of $\mathbb{F}_3 S_6$

<table>
<thead>
<tr>
<th></th>
<th>(6)</th>
<th>(5,1)</th>
<th>(4,2)</th>
<th>(3,3)</th>
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<th>(3,2,1)</th>
<th>(2,2,1,1)</th>
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Decomposition matrix of $\mathbb{F}_3 S_6$: two-row partitions

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<th>(4,2)</th>
<th>(3,3)</th>
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<th>(3,2,1)</th>
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General form of the two-row decomposition matrix

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<td>Type III without the extra 1.</td>
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</table>

<table>
<thead>
<tr>
<th>Type IV.</th>
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![Triangle Diagram]
Decomposition matrix of $\mathbb{F}_3 S_6$: separated into blocks

<table>
<thead>
<tr>
<th></th>
<th>(6)</th>
<th>(5,1)</th>
<th>(3,3)</th>
<th>(4,1,1)</th>
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Decomposition matrix of $\mathbb{F}_2S_{10}$: separated into blocks

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<th>(8,2)</th>
<th>(7,3)</th>
<th>(6,4)</th>
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<th>(5,3,2)</th>
<th>(7,2,1)</th>
<th>(5,4,1)</th>
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<tr>
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### Decomposition Numbers: 3-block of $S_{12}$ with core $(3, 1, 1)$

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## Decomposition Numbers: 3-block of $S_{12}$ with core (3, 1, 1)

|                | (12, $1^2$) | (9, 4, 1) | (9, 3, 2) | (8, $2^2$) | (6, $4^2$) | (6, 3, $2^2$, 1) | (5, 4, $2^2$, 1) | (4, $2^2$, 1$^2$) | (9, $1^5$) | (6, 4, $1^4$) | (6, 3, 2, 1$^3$) | (6, $2^3$, 1$^2$) | (6, 1$^8$) | (5, 4, 2, 1$^3$) | (3$^4$, 1$^2$) | (3$^2$, 2$^4$) | (3$^2$, $2^2$, 1$^4$) | (3$^2$, 2, 1$^6$) | (3, $2^3$, 1$^5$) | (3, 1$^{11}$) |
|----------------|-------------|-----------|-----------|------------|------------|------------------|------------------|------------------|-------------|-------------|------------------|-----------------|----------|------------------|-----------------|----------|------------------|-------------|
| (12, $1^2$)    | 1           |           |           |            |            |                  |                  |                  |             |             |                  |                  |          |                  |                  |          |                  |             |
| (9, 4, 1)      | 1           | 1         |           |            |            |                  |                  |                  |             |             |                  |                  |          |                  |                  |          |                  |             |
| (9, 3, 2)      | 2           | 1         | 1         |            |            |                  |                  |                  |             |             |                  |                  |          |                  |                  |          |                  |             |
| (8, $2^2$)     | 1           | 1         | 1         | 1          |            |                  |                  |                  |             |             |                  |                  |          |                  |                  |          |                  |             |
| (6$^2$, 2)     |             |           |           |            | 1          |                  |                  |                  |             |             |                  |                  |          |                  |                  |          |                  |             |
| (6, $4^2$)     |             |           |           |            |            |                  |                  |                  | 1           |             |                  |                  |          |                  |                  |          |                  |             |
| (6, 4, $2^2$)  |             |           |           |            |            |                  |                  |                  |             |             |                  |                  |          |                  |                  |          |                  |             |
| (6, 3, $2^2$, 1)|             |           |           |            |            |                  |                  |                  |             |             |                  |                  |          |                  |                  |          |                  |             |
| (5, 4, $2^2$, 1)|             |           |           |            |            |                  |                  |                  |             |             |                  |                  |          |                  |                  |          |                  |             |
| (4$^2$, 2$^2$, 1$^2$) |             |           |           |            |            |                  |                  |                  |             |             |                  |                  |          |                  |                  |          |                  |             |
| (9, $1^5$)     |             |           |           |            |            |                  |                  |                  | 1           |             |                  |                  |          |                  |                  |          |                  |             |
| (6, 4, $1^4$)  |             |           |           |            |            |                  |                  |                  |             | 1           |                  |                  |          |                  |                  |          |                  |             |
| (6, 3, 2, 1$^3$) |             |           |           |            |            |                  |                  |                  |             |             |                  |                  |          |                  |                  |          |                  |             |
| (6, $2^3$, 1$^2$) |             |           |           |            |            |                  |                  |                  |             |             |                  |                  |          |                  |                  |          |                  |             |
| (6, 1$^8$)     |             |           |           |            |            |                  |                  |                  |             |             |                  |                  |          |                  |                  |          |                  |             |
| (5, 4, 2, 1$^3$) |             |           |           |            |            |                  |                  |                  |             |             |                  |                  |          |                  |                  |          |                  |             |
| (3$^4$, 1$^2$)  |             |           |           |            |            |                  |                  |                  |             |             |                  |                  |          |                  |                  |          |                  |             |
| (3$^2$, 2$^4$)  |             |           |           |            |            |                  |                  |                  |             |             |                  |                  |          |                  |                  |          |                  |             |
| (3$^2$, $2^2$, 1$^4$) |             |           |           |            |            |                  |                  |                  |             |             |                  |                  |          |                  |                  |          |                  |             |
| (3$^2$, 2, 1$^6$) |             |           |           |            |            |                  |                  |                  |             |             |                  |                  |          |                  |                  |          |                  |             |
| (3, $2^3$, 1$^5$) |             |           |           |            |            |                  |                  |                  |             |             |                  |                  |          |                  |                  |          |                  |             |
| (3, 1$^{11}$)  |             |           |           |            |            |                  |                  |                  |             |             |                  |                  |          |                  |                  |          |                  |             |
## Decomposition Numbers: 3-block of $S_{12}$ with core $(3, 1, 1)$

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### Additional Information

- The table represents the decomposition numbers for the 3-block of $S_{12}$ with core $(3, 1, 1)$.
- Each entry corresponds to a particular partition of 12, with the numbers indicating the multiplicity of that partition in the decomposition.
- The entries are organized in a grid format to facilitate visual analysis.
Decomposition Numbers: 3-block of $S_{12}$ with core $(3, 1, 1)$

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<th>$(9, 4, 1)^2$</th>
<th>$(9, 3, 2)^2$</th>
<th>$(8, 4, 1^2)$</th>
<th>$(6, 2^2, 2)$</th>
<th>$(6, 4, 1^4)$</th>
<th>$(6, 3, 2^2, 1)$</th>
<th>$(4, 2^2, 1^2)$</th>
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<td>$(3^2, 2, 1^6) = \langle 1, 3, 3 \rangle$</td>
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<td>$(3, 1^{11}) = \langle 3, 3, 3 \rangle$</td>
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Decomposition Numbers: 3-block of $S_{12}$ with core $(3, 1, 1)$
Foulkes’ Conjecture and Howe’s Conjecture

Let $\Omega^{(m^n)}$ be the set of all set partitions of $\{1, 2, \ldots, mn\}$ into $n$ sets each of size $m$.

Conjecture (Howe 1987)

The $\mathbb{C}S_{mn}$-homomorphism $\theta^{(m^n)} : \langle \Omega^{(n^m)} \rangle_{\mathbb{C}} \to \langle \Omega^{(m^n)} \rangle_{\mathbb{C}}$ defined by

$$\{A_1, \ldots, A_m\} \mapsto \sum \{B_1, \ldots, B_n\},$$

where the sum is over all $\{B_1, \ldots, B_n\} \in \Omega^{(mn)}$ such that $|A_i \cap B_j| = 1$ for all $i$ and $j$, is injective.

▶ Dent, Siemons 2000: FC is true for $m = 3$.

▶ McKay 2007: if $\theta^{(mn)}$ is injective then so is $\theta^{(mn')}$ for all $n' \geq n$. Hence HC and FC hold for $m = 4$.

▶ Müller, Neunhöffer 2005: $\theta^{(5,5)}$ is not injective.

▶ Cheung, Ikenmeyer, Mkrtchyan 2015: $\theta^{(5,6)}$ is injective, hence FC is true for $m = 5$. 
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\]

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Problem

Decompose $\phi^{(3^n)}$ into irreducible characters of $S_{3n}$.

Equivalently, decompose $\text{Sym}^n(\text{Sym}^3 V)$ into irreducible representations of $\text{GL}(V)$.
Open problem

Problem

*Decompose \( \phi(3^n) \) into irreducible characters of \( S_{3n} \).*

Equivalently, decompose \( \text{Sym}^n(\text{Sym}^3 V) \) into irreducible representations of \( \text{GL}(V) \).

It is not hard to show that

\[
\phi(3^n) \downarrow_{S_{3n-1}} = (\phi(3^{n-1}) \times 1_{S_2}) \uparrow^{S_{3n-1}}.
\]

Computational evidence suggests that this property, together with \( \langle \phi(3^n), 1_{S_{3n}} \rangle = 1 \), determines \( \phi(3^n) \) uniquely.
Foulkes’ Conjecture: computational results

- Müller, Neunhöffer 2005: FC is true if $m + n \leq 17$.
- Evseev, Paget, MW 2014: FC is true if $m + n \leq 19$. 
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Plethysm: Symmetric polynomials.

Suppose \( \dim V = d \).

- A basis of weight vectors for \( \text{Sym}^2 V \) is
  \[
  v_1 v_1, \ v_1 v_2, \ v_2 v_2, \ v_1 v_3, \ldots \ v_d v_d
  \]
  \[
  x_1^2, \ x_1 x_2, \ x_2^2, \ x_1 x_3, \ldots \ x_d^2
  \]

- The *formal character* of \( \text{Sym}^2 V \) is
  \[
  s_2(x_1, \ldots, x_d) = x_1^2 + x_1 x_2 + x_2^2 + x_1 x_3 + \cdots + x_d^2.
  \]

  Formal characters are symmetric polynomials.
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- A basis of weight vectors for \( \text{Sym}^2 V \) is

\[
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&x_1^2 \ x_1 x_2, \ x_2^2, \ x_1 x_3, \ldots \ x_d^2
\end{align*}
\]

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\[
s(2)(x_1, \ldots, x_d) = x_1^2 + x_1 x_2 + x_2^2 + x_1 x_3 + \cdots + x_d^2.
\]

Formal characters are symmetric polynomials.

- The formal character \( h \) of \( \text{Sym}^2(\text{Sym}^2 V) \) is obtained by evaluating \( s(2) \) at the monomials 

\[
\begin{align*}
&(v_1 v_1)(v_1 v_1), \ (v_1 v_1)(v_1 v_2), \ (v_1 v_2)(v_1 v_2), \ (v_1 v_1)(v_2 v_2), \ldots \\
&(x_1^2 x_1^2) \ x_1^2 x_1 x_2, \ x_1 x_2 x_1 x_2 \ x_1^2 x_2^2, \ldots
\end{align*}
\]
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- The formal character of \( \text{Sym}^2 V \) is

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  \[ x_1^2 x_1^2, \ x_1^2 x_1 x_2, \ x_1 x_2 x_1 x_2 \ x_d^2, \ldots \]

  \[ h(x_1, \ldots, x_d) = x_1^4 + x_1^3 x_2 + 2x_1^2 x_2^2 + 2x_1^2 x_2 x_3 + 3x_1 x_2 x_3 x_4 \]
  
  \[ = (x_1^4 + x_1^3 x_2 + x_1^2 x_2^2 + x_1^2 x_2 x_3 + x_1 x_2 x_3 x_4 + \cdots) \]
  
  \[ + (x_1^2 x_2^2 + x_1^2 x_2 x_3 + 2x_1 x_2 x_3 x_4 + \cdots) \]
  
  \[ = s_{(4)}(x_1, \ldots, x_d) + s_{(2,2)}(x_1, \ldots, x_d) \]
Let \( f \) and \( g \) be symmetric polynomials. Assume \( g \) has coefficients in \( \mathbb{N}_0 \) when expressed in the monomial basis. The plethysm \( f \circ g \) is defined by evaluating \( f \) at the monomials of \( g \).

- The formal character of \( \Delta^\nu (\Delta^\mu V) \) is \( s_\nu \circ s_\mu \).
- The corresponding character of \( S_{mn} \) is

\[
(\widehat{(\chi^\mu)^n} \text{Inf}_{S_n} S_{m \wr S_n} \chi^\nu) \uparrow_{S_{m \wr S_n}}^{S_{mn}}
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**Problem (Weak Foulkes’ Conjecture)**

*Show that if $m \leq n$ then $s_{(n)} \circ s_{(m)} - s_{(m)} \circ s_{(n)}$ has non-negative coefficients.*

Equivalently, $S_m \wr S_n$ has at least as many orbits as $S_n \wr S_m$ on the coset space $S_{mn}/S_{\lambda_1} \times S_{\lambda_2} \times \cdots$, for each $\lambda \in \text{Par}(mn)$. 

**πληθυσμόσ: Stanley’s Problem 9**

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- The corresponding character of $S_{mn}$ is

$$
(\left((\chi^\mu)^n\right)^{\text{Inf}}S_m \wr S_n \chi^\nu) \uparrow_{S_m \wr S_n}^{S_{mn}}
$$

**Problem (Weak Foulkes’ Conjecture)**

*Show that if $m \leq n$ then $s_{(n)} \circ s_{(m)} - s_{(m)} \circ s_{(n)}$ has non-negative coefficients.*

Equivalently, $S_m \wr S_n$ has at least as many orbits as $S_n \wr S_m$ on the coset space $S_{mn}/S_{\lambda_1} \times S_{\lambda_2} \times \cdots$, for each $\lambda \in \text{Par}(mn)$.

**Problem (Stanley, 2000)**

*Let $\mu \in \text{Par}(m)$, $\nu \in \text{Par}(n)$, $\lambda \in \text{Par}(mn)$. Find a combinatorial interpretation of the coefficient of $s_\lambda$ in $s_\nu \circ s_\mu$.***
Plethysms and enumeration

Theorem (Read 1959)

\[ \langle s_{2m} \circ s_3, s_{3m} \circ s_2 \rangle \text{ is the number of 3-regular graphs (with loops and multiple edges permitted) on } 2m \text{ vertices.} \]
Plethysms and enumeration

Theorem (Read 1959)
\[ \langle s_{2m} \circ s_3, s_{3m} \circ s_2 \rangle \text{ is the number of 3-regular graphs (with loops and multiple edges permitted) on } 2m \text{ vertices.} \]

Why on earth should this be true?
Plethysms and enumeration

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Why on earth should this be true?

- \( s_{(2m)} \circ s_{(3)} \) is the cycle index of \( S_{6m} \) acting on \( \Omega^{(3^{2m})} \).
- \( s_{(3m)} \circ s_{(2)} \) is the cycle index of \( S_{6m} \) acting on \( \Omega^{(2^{3m})} \).
Plethysms and enumeration

Theorem (Read 1959)

$\langle s_{2m} \circ s(3), s_{3m} \circ s(2) \rangle$ is the number of 3-regular graphs (with loops and multiple edges permitted) on 2m vertices.

Why on earth should this be true?

- $s_{2m} \circ s(3)$ is the cycle index of $S_{6m}$ acting on $\Omega^{(3^{2m})}$.
- $s_{3m} \circ s(2)$ is the cycle index of $S_{6m}$ acting on $\Omega^{(2^{3m})}$.
- Their inner product is the number of orbits of $S_{6m}$ on $\Omega^{(3^{2m})} \times \Omega^{(2^{3m})}$ . . .
Plethysms and enumeration

Theorem (Read 1959)
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- . . .which is the number of orbits of \( S_3 \wr S_{2m} \) on \( \Omega^{(2^{3m})} \) . . .
Plethysms and enumeration

Theorem (Read 1959)

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**Plethysms and enumeration**

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- ... which is the number of 3-regular graphs:
Plethysms and enumeration

Theorem (Read 1959)

\[ \langle s(2m) \circ s(3), s(3m) \circ s(2) \rangle \] is the number of 3-regular graphs (with loops and multiple edges permitted) on 2m vertices.

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► \( s(2m) \circ s(3) \) is the cycle index of \( S_{6m} \) acting on \( \Omega^{(3^{2m})} \).

► \( s(3m) \circ s(2) \) is the cycle index of \( S_{6m} \) acting on \( \Omega^{(2^{3m})} \).

► Their inner product is the number of orbits of \( S_{6m} \) on \( \Omega^{(3^{2m})} \times \Omega^{(2^{3m})} \) . . .

► . . . which is the number of orbits of \( S_3 \ltimes S_{2m} \) on \( \Omega^{(2^{3m})} \) . . .

► . . . which is the number of 3-regular graphs:

```
1  2  3
\[ \bullet \bullet \bullet \]
4  5  6
\[ \bullet \bullet \bullet \rightarrow \]
\[ \bullet \bullet \bullet \]
```
Plethysms and enumeration

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- . . . which is the number of orbits of \( S_3 \wr S_{2m} \) on \( \Omega^{(2^{3m})} \) . . .
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Plethysms and enumeration

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Their inner product is the number of orbits of \( S_{6m} \) on \( \Omega^{(3^{2m})} \times \Omega^{(2^{3m})} \) . . .

\[ \ldots \text{which is the number of orbits of } S_3 \wr S_{2m} \text{ on } \Omega^{(2^{3m})} \ldots \]

\[ \ldots \text{which is the number of 3-regular graphs:} \]

1 2 3

\[ \begin{array} {ccc}
1 & 2 & 3 \\
\bullet & \bullet & \bullet \\
4 & 5 & 6
\end{array} \rightarrow \begin{array} {c}
\text{graph}
\end{array} \]
Plethysms and enumeration

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Plethysms and enumeration

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\[ \langle s_{(2m)} \circ s_{(3)}, s_{(3m)} \circ s_{(2)} \rangle \] is the number of 3-regular graphs (with loops and multiple edges permitted) on \(2m\) vertices.

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  - ...which is the number of orbits of \(S_3 \wr S_{2m}\) on \(\Omega^{(2^{3m})} \)
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Plethysms and enumeration

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- Their inner product is the number of orbits of \( S_{6m} \) on \( \Omega^{(3^{2m})} \times \Omega^{(2^{3m})} \) ....
- ...which is the number of orbits of \( S_3 \wr S_{2m} \) on \( \Omega^{(2^{3m})} \) ....
- ...which is the number of 3-regular graphs:

\[
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{example_graph.png}
\end{array}
\]
Plethysms and enumeration

Theorem (Read 1959)

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- \( s_{(2m)} \circ s_{(3)} \) is the cycle index of \( S_{6m} \) acting on \( \Omega^{(32m)} \).
- \( s_{(3m)} \circ s_{(2)} \) is the cycle index of \( S_{6m} \) acting on \( \Omega^{(23m)} \).
- Their inner product is the number of orbits of \( S_{6m} \) on \( \Omega^{(32m)} \times \Omega^{(23m)} \) ...
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Plethysms and enumeration

Theorem (Read 1959)

⟨s_{(2m)} ∘ s_{(3)}, s_{(3m)} ∘ s_{(2)}⟩ is the number of 3-regular graphs (with loops and multiple edges permitted) on 2m vertices.

Why on earth should this be true?

▶ s_{(2m)} ∘ s_{(3)} is the cycle index of $S_{6m}$ acting on $Ω^{(3^{2m})}$.
▶ s_{(3m)} ∘ s_{(2)} is the cycle index of $S_{6m}$ acting on $Ω^{(2^{3m})}$.
▶ Their inner product is the number of orbits of $S_{6m}$ on $Ω^{(3^{2m})} × Ω^{(2^{3m})}$ ...
▶ ... which is the number of orbits of $S_3 ∩ S_{2m}$ on $Ω^{(2^{3m})}$ ...
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§2: Minimal and maximal constituents of plethysms

Let $\lambda, \lambda^* \in \text{Par}(r)$. We say $\lambda$ dominates $\lambda^*$, and write $\lambda \trianglerighteq \lambda^*$, if

$$\lambda_1 + \cdots + \lambda_j \geq \lambda^*_1 + \cdots + \lambda^*_j.$$

for all $j$. For example

- $(4, 2, 2) \trianglerighteq (3, 3, 1, 1)$,
§2: Minimal and maximal constituents of plethysms

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- $(4, 2, 2) \trianglerighteq (3, 3, 1, 1)$,
- $(4, 1, 1)$ and $(3, 3)$ are incomparable.
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Quiz: Choose partitions $\lambda$ and $\lambda^*$ of $n$ (a large number) uniformly at random. What is the chance that $\lambda$ and $\lambda^*$ are comparable?
§2: Minimal and maximal constituents of plethysms

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Our main theorem gives a combinatorial characterization of all maximal and minimal partitions $\lambda$ in the dominance order on $\text{Par}(mn)$ such that $s_{\lambda}$ has non-zero coefficient in $s_{\nu} \circ s_{\mu}$. 

This solves a special case of Stanley's Problem 9.
§2: Minimal and maximal constituents of plethysms

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This solves a special case of Stanley’s Problem 9.
Special case $\mu = (m)$ for minimals

Let $A = \{a_1, \ldots, a_m\}$ and $B = \{b_1, \ldots, b_m\}$ be $m$-subsets of $\mathbb{N}$, written so that $a_1 < \ldots < a_m$ and $b_1 < \ldots < b_m$. We say that $A$ majorizes $B$, and write $A \preceq B$, if

$$a_1 \leq b_1, \ldots, a_m \leq b_m.$$
Special case $\mu = (m)$ for minimals

Let $A = \{a_1, \ldots, a_m\}$ and $B = \{b_1, \ldots, b_m\}$ be $m$-subsets of $\mathbb{N}$, written so that $a_1 < \ldots < a_m$ and $b_1 < \ldots < b_m$. We say that $A$ majorizes $B$, and write $A \preceq B$, if

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A closed set family of size $r$ is a family $\mathcal{P}$ of $m$-subsets of $\mathbb{N}$ such that $|\mathcal{P}| = r$ and if $B \in \mathcal{P}$ and $A \preceq B$ then $A \in \mathcal{P}$.
Special case $\mu = (m)$ for minimals

Let $A = \{a_1, \ldots, a_m\}$ and $B = \{b_1, \ldots, b_m\}$ be $m$-subsets of $\mathbb{N}$, written so that $a_1 < \ldots < a_m$ and $b_1 < \ldots < b_m$. We say that $A$ majorizes $B$, and write $A \preceq B$, if

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Special case \( \mu = (m) \) for minimals

- A closed set family of size \( r \) is a family \( \mathcal{P} \) of \( m \)-subsets of \( \mathbb{N} \) such that \( |\mathcal{P}| = r \) and if \( B \in \mathcal{P} \) and \( A \subseteq B \) then \( A \in \mathcal{P} \).
- A closed set family tuple of size \( \nu \) is a tuple \((\mathcal{P}_1, \ldots, \mathcal{P}_e)\) where \( \mathcal{P}_j \) is a closed set family of size \( \nu_j \) for each \( j \).
Special case $\mu = (m)$ for minimals

- A *closed set family of size* $r$ is a family $\mathcal{P}$ of $m$-subsets of $\mathbb{N}$ such that $|\mathcal{P}| = r$ and if $B \in \mathcal{P}$ and $A \leq B$ then $A \in \mathcal{P}$.
- A *closed set family tuple of size* $\nu$ is a tuple $(\mathcal{P}_1, \ldots, \mathcal{P}_e)$ where $\mathcal{P}_j$ is a closed set family of size $\nu_j$ for each $j$.
- The *weight* of $(\mathcal{P}_1, \ldots, \mathcal{P}_e)$ is the partition $\lambda$ such that each $i \in \mathbb{N}$ appears in exactly $\lambda_i$ sets in the $\mathcal{P}_j$.
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- The type of $(\mathcal{P}_1, \ldots, \mathcal{P}_e)$ is the conjugate partition $\lambda'$.
- For example,

$$(\{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}\}, \{\{1, 2, 3\}\})$$

is a closed set family tuple of size $(3, 1)$, weight $(4, 3, 3, 2)$ and type $(4, 4, 3, 1)$. 

Theorem (Paget, MW, 2014)

Let $m$ be odd. The minimal partitions $\lambda$ such that $s_\lambda$ has non-zero coefficient in $s_\nu \circ s_{(m)}$ are precisely the minimal types of the closed set family tuples of size $\nu$. 
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Special case $\nu = (n)$ for minimals

- A $\mu$-tableau is *conjugate-semistandard* if its rows are strictly increasing and its columns are non-decreasing. When $\mu = (m)$ such tableaux correspond to $m$-subsets: $\{1, 3, 4\} \leftrightarrow \begin{array}{ccc} 1 & 3 & 4 \end{array}$.
- The majorization order generalizes to a partial order on conjugate-semistandard $\mu$-tableaux.
- We define closed $\mu$-tableau families and their weights and types analogously.

Theorem (Paget, MW, 2016)

Let $m$ be odd and let $\mu \in \text{Par}(n)$. The minimal partitions $\lambda$ such that $s_\lambda$ has non-zero coefficient in $s(n) \circ s\mu$ are precisely the minimal types of the closed $\mu$-tableau families of size $n$. This determines all minimal $\lambda$ such that $\Delta_\lambda V$ appears in the coordinate ring of $\Delta_\mu V$.
Special case $\nu = (n)$ for minimals

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- The majorization order generalizes to a partial order on conjugate-semistandard $\mu$-tableaux.
- We define closed $\mu$-tableau families and their weights and types analogously. For example
  \[
  \left\{ \begin{array}{c} 1 \\ 2 \end{array}, \quad \begin{array}{c} 1 \\ 2 \\ 1 \end{array}, \quad \begin{array}{c} 1 \\ 3 \end{array} \right\}
  \]
is a closed $(2, 1)$-tableau family of size 3, weight $(5, 3, 1)$ and type $(3, 2, 2, 1, 1)$. 

\[ \text{Theorem (Paget, MW, 2016)} \]

Let $m$ be odd and let $\mu \in \text{Par}(n)$. The minimal partitions $\lambda$ such that $s_\lambda$ has non-zero coefficient in $s_n \circ s_\mu$ are precisely the minimal types of the closed $\mu$-tableau families of size $n$. This determines all minimal $\lambda$ such that $\Delta_\lambda V$ appears in the coordinate ring of $\Delta_\mu V$. 
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**Theorem (Paget, MW, 2016)**

*Let $m$ be odd and let $\mu \in \operatorname{Par}(n)$. The minimal partitions $\lambda$ such that $s_\lambda$ has non-zero coefficient in $s_{(n)} \circ s_\mu$ are precisely the minimal types of the closed $\mu$-tableau families of size $n$. This determines all minimal $\lambda$ such that $\Delta^\lambda V$ appears in the coordinate ring of $\Delta^\mu V$.***
Application to invariants of Riemann curvature tensor

A question on invariant theory of $GL_n(\mathbb{C})$.

Let $\rho$ denote the irreducible algebraic representation of $GL_n(\mathbb{C})$ with the highest weight $(2, 2, \underbrace{0, \ldots, 0}_{n-2})$.

Let $k \leq n/2$ be a non-negative integer. How to decompose into irreducible representations the representation $\text{Sym}^k(\rho)$?

More specifically, I am interested whether $\text{Sym}^k(\rho)$ contains the representation with the highest weight $(\underbrace{2, \ldots, 2}_{2k}, \underbrace{0, \ldots, 0}_{n-2k})$, and if yes, whether the multiplicity is equal to one.

A side remark, the representation $\rho$ has a geometric interpretation important for me: it is the space of curvature tensors, namely the curvature tensor of any Riemannian metric on $\mathbb{R}^n$ lies in $\rho$. 
The plethysm $\text{Sym}^k \rho$ contains the irreducible representation with highest weight $(2, \ldots, 2, 0, \ldots, 0)$ exactly once. It looks like a tricky problem to say much about its other irreducible constituents.

Let $\Delta^1$ denote the Schur functor corresponding to the partition $\lambda$, and let $E$ be an $n$-dimensional complex vector space. Using symmetric polynomials (or other methods) one finds

$$\text{Sym}^2(\text{Sym}^2 E) = \Delta^{(2,2)} E \oplus \text{Sym}^4 E.$$  

Therefore

$$\text{Sym}^k \text{Sym}^2 \text{Sym}^2 E \cong \sum_{r=0}^k \text{Sym}^r(\Delta^{(2,2)} E) \otimes \text{Sym}^{k-r}(\text{Sym}^4 E).$$

The irreducible representations contained in the $r$th summand are labelled by partitions with at most $2r + (k - r) = k + r$ parts. So to show that $\text{Sym}^k(\Delta^{(2,2)}(E))$ contains $\Delta^{(2k)} E$, it suffices to show that $\Delta^{(2k)} E$ appears in $\text{Sym}^k \text{Sym}^2 \text{Sym}^2 E$.

Let $U = \text{Sym}^2 E$. There is a canonical surjection

$$\text{Sym}^k(\text{Sym}^2 U) \to \text{Sym}^{2k} U.$$  

given by mapping $(u_1 u'_1 \ldots (u_k u'_k) \in \text{Sym}^k(\text{Sym}^2 U)$ to $u_1 u'_1 \ldots u_k u'_k \in \text{Sym}^{2k} U$. Therefore $\text{Sym}^k(\text{Sym}^2 U)$ contains $\text{Sym}^{2k} U = \text{Sym}^{2k}(\text{Sym}^2 E)$. It is well known that

$$\text{Sym}^{2k}(\text{Sym}^2 E) = \sum_{\lambda} \Delta^{2\lambda}(E)$$

where the sum is over all partitions $\lambda$ of $2k$ and $2(\lambda_1, \ldots, \lambda_m) = (2\lambda_1, \ldots, 2\lambda_m)$. Taking $\lambda = (1^k)$ we see that $\Delta^{(2k)} E$ appears.

It remains to show that the multiplicity of $\Delta^{(2k)} E$ in $\text{Sym}^k(\Delta^{(2,2)} E)$ is 1. We work over $\mathbb{C}$, so there is a chain of inclusions

$$\text{Sym}^k(\Delta^{(2,2)}(E)) \subseteq \text{Sym}^k(\text{Sym}^2 E \otimes \text{Sym}^2 E) \subseteq (\text{Sym}^2 E)^{\otimes 2k}.$$  

By the Littlewood–Richardson rule (or the easier Young's rule), the multiplicity of $\Delta^{(2k)} E$ in the right-hand side is 1.