# Plethysms: permutations, weights and Schur functions

Mark Wildon (joint work with Rowena Paget)

Royal Holloway University of London

London Algebra Colloquium

City University, 1 December 2016

# Outline

- §1 Motivation: Examples of plethysms
- §2 Main result: Minimal and maximal constituents of $s_\nu \circ s_\mu$

# §1 Polynomial representations of $\mathrm{GL}(V)$

Let $V$ be a finite-dimensional $\mathbb{C}$-vector space.

- The natural representation of $\mathrm{GL}(V)$ is irreducible.

# §1 Polynomial representations of $\mathrm{GL}(V)$

Let $V$ be a finite-dimensional $\mathbb{C}$-vector space.

- The natural representation of $\mathrm{GL}(V)$ is irreducible.
- $V \otimes V \cong \mathrm{Sym}^2 V \oplus \wedge^2 V$.

# §1 Polynomial representations of $\mathrm{GL}(V)$

Let $V$ be a finite-dimensional $\mathbb{C}$-vector space.

- The natural representation of $\mathrm{GL}(V)$ is irreducible.
- $V \otimes V \cong \mathrm{Sym}^2 V \oplus \wedge^2 V$.
- $V^{\otimes 3} \cong \mathrm{Sym}^3 V \oplus \wedge^3 V \oplus \ldots$

# §1 Polynomial representations of $\mathrm{GL}(V)$

Let $V$ be a finite-dimensional $\mathbb{C}$-vector space.

- The natural representation of $\mathrm{GL}(V)$ is irreducible.
- $V \otimes V \cong \mathrm{Sym}^2 V \oplus \wedge^2 V$.
- $V^{\otimes 3} \cong \mathrm{Sym}^3 V \oplus \wedge^3 V \oplus \ \ldots$
  - $u = (v_1 \wedge v_2) \otimes v_1 \in \wedge^2 V \otimes V$ is highest weight, weight $(2, 1)$.

# §1 Polynomial representations of $\mathrm{GL}(V)$

Let $V$ be a finite-dimensional $\mathbb{C}$-vector space.

- The natural representation of $\mathrm{GL}(V)$ is irreducible.
- $V \otimes V \cong \mathrm{Sym}^2 V \oplus \wedge^2 V$.
- $V^{\otimes 3} \cong \mathrm{Sym}^3 V \oplus \wedge^3 V \oplus \ldots$
  - $u = (v_1 \wedge v_2) \otimes v_1 \in \wedge^2 V \otimes V$ is highest weight, weight $(2, 1)$.
  - Why highest weight? Check $u$ killed by Lie algebra action of $e \in \mathrm{gl}(V)$, defined by $e(v_2) = v_1$, $e(v_i) = 0$ if $i \neq 2$:

## §1 Polynomial representations of $\mathrm{GL}(V)$

Let $V$ be a finite-dimensional $\mathbb{C}$-vector space.

- The natural representation of $\mathrm{GL}(V)$ is irreducible.
- $V \otimes V \cong \mathrm{Sym}^2 V \oplus \wedge^2 V$.
- $V^{\otimes 3} \cong \mathrm{Sym}^3 V \oplus \wedge^3 V \oplus \ \ldots$
  - $u = (v_1 \wedge v_2) \otimes v_1 \in \wedge^2 V \otimes V$ is highest weight, weight $(2, 1)$.
  - Why highest weight? Check $u$ killed by Lie algebra action of $e \in \mathrm{gl}(V)$, defined by $e(v_2) = v_1$, $e(v_i) = 0$ if $i \neq 2$:

$$\begin{aligned} eu &= (ev_1 \wedge v_2) \otimes v_1 + (v_1 \wedge ev_2) \otimes v_1 + (v_1 \wedge v_2) \otimes ev_1 \\ &= 0 + 0 + 0. \end{aligned}$$

## §1 Polynomial representations of $\mathrm{GL}(V)$

Let $V$ be a finite-dimensional $\mathbb{C}$-vector space.

- The natural representation of $\mathrm{GL}(V)$ is irreducible.
- $V \otimes V \cong \mathrm{Sym}^2 V \oplus \wedge^2 V$.
- $V^{\otimes 3} \cong \mathrm{Sym}^3 V \oplus \wedge^3 V \oplus \ \ldots$
  - $u = (v_1 \wedge v_2) \otimes v_1 \in \wedge^2 V \otimes V$ is highest weight, weight $(2, 1)$.
  - Why highest weight? Check $u$ killed by Lie algebra action of $e \in \mathrm{gl}(V)$, defined by $e(v_2) = v_1$, $e(v_i) = 0$ if $i \neq 2$:

$$\begin{aligned} eu &= (ev_1 \wedge v_2) \otimes v_1 + (v_1 \wedge ev_2) \otimes v_1 + (v_1 \wedge v_2) \otimes ev_1 \\ &= 0 + 0 + 0. \end{aligned}$$

  - Two isomorphic complementary submodules are generated by $(v_1 \otimes v_2 - v_2 \otimes v_1) \otimes v_1$ and $v_1 \otimes (v_1 \otimes v_2 - v_2 \otimes v_1)$,

# §1 Polynomial representations of $\mathrm{GL}(V)$

Let $V$ be a finite-dimensional $\mathbb{C}$-vector space.

- The natural representation of $\mathrm{GL}(V)$ is irreducible.
- $V \otimes V \cong \mathrm{Sym}^2 V \oplus \wedge^2 V$.
- $V^{\otimes 3} \cong \mathrm{Sym}^3 V \oplus \wedge^3 V \oplus \Delta^{(2,1)} V \oplus \Delta^{(2,1)} V$
  - $u = (v_1 \wedge v_2) \otimes v_1 \in \wedge^2 V \otimes V$ is highest weight, weight $(2,1)$.
  - Why highest weight? Check $u$ killed by Lie algebra action of $e \in \mathrm{gl}(V)$, defined by $e(v_2) = v_1$, $e(v_i) = 0$ if $i \neq 2$:

$$\begin{aligned} eu &= (ev_1 \wedge v_2) \otimes v_1 + (v_1 \wedge ev_2) \otimes v_1 + (v_1 \wedge v_2) \otimes ev_1 \\ &= 0 + 0 + 0. \end{aligned}$$

  - Two isomorphic complementary submodules are generated by $(v_1 \otimes v_2 - v_2 \otimes v_1) \otimes v_1$ and $v_1 \otimes (v_1 \otimes v_2 - v_2 \otimes v_1)$,

# §1 Polynomial representations of $\mathrm{GL}(V)$

Let $V$ be a finite-dimensional $\mathbb{C}$-vector space.

- The natural representation of $\mathrm{GL}(V)$ is irreducible.
- $V \otimes V \cong \mathrm{Sym}^2 V \oplus \wedge^2 V$.
- $V^{\otimes 3} \cong \mathrm{Sym}^3 V \oplus \wedge^3 V \oplus \Delta^{(2,1)} V \oplus \Delta^{(2,1)} V$
  - $u = (v_1 \wedge v_2) \otimes v_1 \in \wedge^2 V \otimes V$ is highest weight, weight $(2,1)$.
  - Why highest weight? Check $u$ killed by Lie algebra action of $e \in \mathrm{gl}(V)$, defined by $e(v_2) = v_1$, $e(v_i) = 0$ if $i \neq 2$:

$$\begin{aligned} eu &= (ev_1 \wedge v_2) \otimes v_1 + (v_1 \wedge ev_2) \otimes v_1 + (v_1 \wedge v_2) \otimes ev_1 \\ &= 0 + 0 + 0. \end{aligned}$$

  - Two isomorphic complementary submodules are generated by $(v_1 \otimes v_2 - v_2 \otimes v_1) \otimes v_1$ and $v_1 \otimes (v_1 \otimes v_2 - v_2 \otimes v_1)$,

Generally

$$V^{\otimes r} \cong \bigoplus_{\lambda \in \mathrm{Par}(r)} (\Delta^\lambda V)^{\oplus d_\lambda}$$

where $\Delta^\lambda V$ is the unique irreducible representation of $\mathrm{GL}(V)$ of highest weight $\lambda$.

# §1 Polynomial representations of $\mathrm{GL}(V)$

Let $V$ be a finite-dimensional $\mathbb{C}$-vector space.

- The natural representation of $\mathrm{GL}(V)$ is irreducible.
- $V \otimes V \cong \mathrm{Sym}^2 V \oplus \wedge^2 V$.
- $V^{\otimes 3} \cong \mathrm{Sym}^3 V \oplus \wedge^3 V \oplus \Delta^{(2,1)} V \oplus \Delta^{(2,1)} V$
  - $u = (v_1 \wedge v_2) \otimes v_1 \in \wedge^2 V \otimes V$ is highest weight, weight $(2, 1)$.
  - Why highest weight? Check $u$ killed by Lie algebra action of $e \in \mathrm{gl}(V)$, defined by $e(v_2) = v_1$, $e(v_i) = 0$ if $i \neq 2$:

$$\begin{aligned} eu &= (ev_1 \wedge v_2) \otimes v_1 + (v_1 \wedge ev_2) \otimes v_1 + (v_1 \wedge v_2) \otimes ev_1 \\ &= 0 + 0 + 0. \end{aligned}$$

  - Two isomorphic complementary submodules are generated by $(v_1 \otimes v_2 - v_2 \otimes v_1) \otimes v_1$ and $v_1 \otimes (v_1 \otimes v_2 - v_2 \otimes v_1)$,

Generally

$$V^{\otimes r} \cong \bigoplus_{\lambda \in \mathrm{Par}(r)} (\Delta^\lambda V)^{\oplus d_\lambda}$$

where $\Delta^\lambda V$ is the unique irreducible representation of $\mathrm{GL}(V)$ of highest weight $\lambda$. For instance $\mathrm{Sym}^n V = \Delta^{(n)} V$, $\wedge^n V = \Delta^{(1^n)} V$.

# Plethysm: Composing polynomial representations

Consider $\mathrm{Sym}^2(\mathrm{Sym}^2 V) \to \mathrm{Sym}^4 V$: $(uv)(u'v') \mapsto uvu'v'$.

- Kernel is $\Delta^{(2,2)}V$. Why? $(v_1v_1)(v_2v_2) - (v_1v_2)(v_1v_2)$ is highest weight, of weight $(2,2)$.

# Plethysm: Composing polynomial representations

Consider $\mathrm{Sym}^2(\mathrm{Sym}^2 V) \to \mathrm{Sym}^4 V$: $(uv)(u'v') \mapsto uvu'v'$.

- Kernel is $\Delta^{(2,2)} V$. Why? $(v_1 v_1)(v_2 v_2) - (v_1 v_2)(v_1 v_2)$ is highest weight, of weight $(2, 2)$.
- $\mathrm{Sym}^2(\mathrm{Sym}^2 V) \cong \Delta^{(2,2)} V \oplus \Delta^{(4)} V$.

# Plethysm: Composing polynomial representations

Consider $\mathrm{Sym}^2(\mathrm{Sym}^2 V) \to \mathrm{Sym}^4 V$: $(uv)(u'v') \mapsto uvu'v'$.

- Kernel is $\Delta^{(2,2)}V$. Why? $(v_1v_1)(v_2v_2) - (v_1v_2)(v_1v_2)$ is highest weight, of weight $(2,2)$.
- $\mathrm{Sym}^2(\mathrm{Sym}^2 V) \cong \Delta^{(2,2)}V \oplus \Delta^{(4)}V$.
- Take $\dim V = 2$. Geometrically:

# Plethysm: Composing polynomial representations

Consider $\mathrm{Sym}^2(\mathrm{Sym}^2 V) \to \mathrm{Sym}^4 V$: $(uv)(u'v') \mapsto uvu'v'$.

- Kernel is $\Delta^{(2,2)}V$. Why? $(v_1v_1)(v_2v_2) - (v_1v_2)(v_1v_2)$ is highest weight, of weight $(2,2)$.
- $\mathrm{Sym}^2(\mathrm{Sym}^2 V) \cong \Delta^{(2,2)}V \oplus \Delta^{(4)}V$.
- Take $\dim V = 2$. Geometrically:
  - $\mathrm{Sym}^2 V = \langle v_1v_1, 2v_1v_2, v_2v_2\rangle_{\mathbb{C}}$

# Plethysm: Composing polynomial representations

Consider $\mathrm{Sym}^2(\mathrm{Sym}^2 V) \to \mathrm{Sym}^4 V$: $(uv)(u'v') \mapsto uvu'v'$.

- Kernel is $\Delta^{(2,2)}V$. Why? $(v_1v_1)(v_2v_2) - (v_1v_2)(v_1v_2)$ is highest weight, of weight $(2, 2)$.
- $\mathrm{Sym}^2(\mathrm{Sym}^2 V) \cong \Delta^{(2,2)}V \oplus \Delta^{(4)}V$.
- Take dim $V = 2$. Geometrically:
  - $\mathrm{Sym}^2 V = \langle v_1v_1, 2v_1v_2, v_2v_2\rangle_{\mathbb{C}}$
  - $\mathcal{O}(\mathrm{Sym}^2 V) = \mathbb{C}[Y_{11}, Y_{12}, Y_{22}]$

# Plethysm: Composing polynomial representations

Consider $\mathrm{Sym}^2(\mathrm{Sym}^2 V) \to \mathrm{Sym}^4 V$: $(uv)(u'v') \mapsto uvu'v'$.

- Kernel is $\Delta^{(2,2)}V$. Why? $(v_1v_1)(v_2v_2) - (v_1v_2)(v_1v_2)$ is highest weight, of weight $(2,2)$.
- $\mathrm{Sym}^2(\mathrm{Sym}^2 V) \cong \Delta^{(2,2)}V \oplus \Delta^{(4)}V$.
- Take $\dim V = 2$. Geometrically:
  - $\mathrm{Sym}^2 V = \langle v_1v_1, 2v_1v_2, v_2v_2\rangle_{\mathbb{C}}$
  - $\mathcal{O}(\mathrm{Sym}^2 V) = \mathbb{C}[Y_{11}, Y_{12}, Y_{22}]$
  - let $\mathcal{C}$ be the image of the squaring map $V \hookrightarrow \mathrm{Sym}^2 V$,

$$\alpha v_1 + \beta v_2 \mapsto \alpha^2 v_1v_1 + 2\alpha\beta v_1v_2 + \beta^2 v_2v_2$$

# Plethysm: Composing polynomial representations

Consider $\mathrm{Sym}^2(\mathrm{Sym}^2 V) \to \mathrm{Sym}^4 V$: $(uv)(u'v') \mapsto uvu'v'$.

- Kernel is $\Delta^{(2,2)}V$. Why? $(v_1v_1)(v_2v_2) - (v_1v_2)(v_1v_2)$ is highest weight, of weight $(2,2)$.
- $\mathrm{Sym}^2(\mathrm{Sym}^2 V) \cong \Delta^{(2,2)}V \oplus \Delta^{(4)}V$.
- Take dim $V = 2$. Geometrically:
  - $\mathrm{Sym}^2 V = \langle v_1v_1, 2v_1v_2, v_2v_2\rangle_{\mathbb{C}}$
  - $\mathcal{O}(\mathrm{Sym}^2 V) = \mathbb{C}[Y_{11}, Y_{12}, Y_{22}]$
  - let $\mathcal{C}$ be the image of the squaring map $V \hookrightarrow \mathrm{Sym}^2 V$,

$$\alpha v_1 + \beta v_2 \mapsto \alpha^2 v_1v_1 + 2\alpha\beta v_1v_2 + \beta^2 v_2v_2$$

  - $\mathcal{C} = \mathrm{Zeros}(Y_{11}Y_{22} - Y_{12}^2)$; the $\mathrm{GL}(V)$-submodule of $\mathcal{O}(\mathrm{Sym}^2 V)$ generated by $Y_{11}Y_{22} - Y_{12}^2$ is $\Delta^{(2,2)}V$.

# Plethysm: Composing polynomial representations

Consider $\mathrm{Sym}^2(\mathrm{Sym}^2 V) \to \mathrm{Sym}^4 V$: $(uv)(u'v') \mapsto uvu'v'$.

- Kernel is $\Delta^{(2,2)}V$. Why? $(v_1v_1)(v_2v_2) - (v_1v_2)(v_1v_2)$ is highest weight, of weight $(2,2)$.
- $\mathrm{Sym}^2(\mathrm{Sym}^2 V) \cong \Delta^{(2,2)}V \oplus \Delta^{(4)}V$.
- Take dim $V = 2$. Geometrically:
  - $\mathrm{Sym}^2 V = \langle v_1v_1, 2v_1v_2, v_2v_2\rangle_{\mathbb{C}}$
  - $\mathcal{O}(\mathrm{Sym}^2 V) = \mathbb{C}[Y_{11}, Y_{12}, Y_{22}]$
  - let $\mathcal{C}$ be the image of the squaring map $V \hookrightarrow \mathrm{Sym}^2 V$,

$$\alpha v_1 + \beta v_2 \mapsto \alpha^2 v_1v_1 + 2\alpha\beta v_1v_2 + \beta^2 v_2v_2$$

  - $\mathcal{C} = \mathrm{Zeros}(Y_{11}Y_{22} - Y_{12}^2)$; the $\mathrm{GL}(V)$-submodule of $\mathcal{O}(\mathrm{Sym}^2 V)$ generated by $Y_{11}Y_{22} - Y_{12}^2$ is $\Delta^{(2,2)}V$.

Next step up: $f \in \mathrm{Sym}^4(\mathrm{Sym}^2 V) = \mathcal{O}(\mathrm{Sym}^2 V)_4$ may

- Vanish doubly on $\mathcal{C}$: $(Y_{11}Y_{22} - Y_{12}^2)^2$
- Vanish singly on $\mathcal{C}$: $Y_{11}^2(Y_{11}Y_{22} - Y_{12}^2)$

# Plethysm: Composing polynomial representations

Consider $\mathrm{Sym}^2(\mathrm{Sym}^2 V) \to \mathrm{Sym}^4 V$: $(uv)(u'v') \mapsto uvu'v'$.

- Kernel is $\Delta^{(2,2)}V$. Why? $(v_1v_1)(v_2v_2) - (v_1v_2)(v_1v_2)$ is highest weight, of weight $(2,2)$.
- $\mathrm{Sym}^2(\mathrm{Sym}^2 V) \cong \Delta^{(2,2)}V \oplus \Delta^{(4)}V$.
- Take dim $V = 2$. Geometrically:
  - $\mathrm{Sym}^2 V = \langle v_1v_1, 2v_1v_2, v_2v_2\rangle_{\mathbb{C}}$
  - $\mathcal{O}(\mathrm{Sym}^2 V) = \mathbb{C}[Y_{11}, Y_{12}, Y_{22}]$
  - let $\mathcal{C}$ be the image of the squaring map $V \hookrightarrow \mathrm{Sym}^2 V$,

$$\alpha v_1 + \beta v_2 \mapsto \alpha^2 v_1v_1 + 2\alpha\beta v_1v_2 + \beta^2 v_2v_2$$

  - $\mathcal{C} = \mathrm{Zeros}(Y_{11}Y_{22} - Y_{12}^2)$; the $\mathrm{GL}(V)$-submodule of $\mathcal{O}(\mathrm{Sym}^2 V)$ generated by $Y_{11}Y_{22} - Y_{12}^2$ is $\Delta^{(2,2)}V$.

Next step up: $f \in \mathrm{Sym}^4(\mathrm{Sym}^2 V) = \mathcal{O}(\mathrm{Sym}^2 V)_4$ may

- Vanish doubly on $\mathcal{C}$: $(Y_{11}Y_{22} - Y_{12}^2)^2$
- Vanish singly on $\mathcal{C}$: $Y_{11}^2(Y_{11}Y_{22} - Y_{12}^2)$
- Such functions are in kernel of $\mathrm{Sym}^4(\mathrm{Sym}^2 V) \to \mathrm{Sym}^8 V$, so

$$\mathrm{Sym}^4(\mathrm{Sym}^2 V) \cong \Delta^{(4,4)}V \oplus \Delta^{(6,2)}V \oplus \Delta^{(8)}V.$$

# Plethysm: Symmetric groups and wreath products

Take dim $V \geq 4$. So $S_4 \leq \mathrm{GL}(V)$: $(13) \mapsto \begin{pmatrix} \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix}$.

# Plethysm: Symmetric groups and wreath products

Take $\dim V \geq 4$. So $S_4 \leq \mathrm{GL}(V)$: $(1234) \mapsto \begin{pmatrix} \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \end{pmatrix}$.

- Weight space $(1,1,1,1)$ inside $\mathrm{Sym}^2(\mathrm{Sym}^2 V)$ is

$$\langle (v_1 v_2)(v_3 v_4), (v_1 v_3)(v_2 v_4), (v_1 v_4)(v_2 v_3) \rangle.$$

## Plethysm: Symmetric groups and wreath products

Take $\dim V \geq 4$. So $S_4 \leq \mathrm{GL}(V)$: $(1234) \mapsto \begin{pmatrix} \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \end{pmatrix}$.

- Weight space $(1,1,1,1)$ inside $\mathrm{Sym}^2(\mathrm{Sym}^2 V)$ is
$$\langle (v_1 v_2)(v_3 v_4), (v_1 v_3)(v_2 v_4), (v_1 v_4)(v_2 v_3) \rangle.$$
- Identify $(v_1 v_2)(v_3 v_4)$ with the set partition $\big\{\{1,2\},\{3,4\}\big\}$.

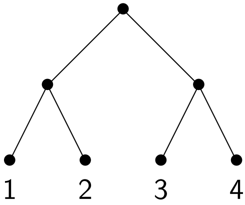
# Plethysm: Symmetric groups and wreath products

Take $\dim V \geq 4$. So $S_4 \leq \mathrm{GL}(V)$: $(1234) \mapsto \begin{pmatrix} \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \end{pmatrix}$.

- Weight space $(1,1,1,1)$ inside $\mathrm{Sym}^2(\mathrm{Sym}^2 V)$ is

$$\langle (v_1 v_2)(v_3 v_4), (v_1 v_3)(v_2 v_4), (v_1 v_4)(v_2 v_3) \rangle.$$

- Identify $(v_1 v_2)(v_3 v_4)$ with the set partition $\{\{1,2\},\{3,4\}\}$.
- Stabiliser $S_2 \wr S_2 = (S_2 \times S_2) \rtimes S_2 = \langle (12), (34) \rangle \rtimes \langle (13)(24) \rangle$.



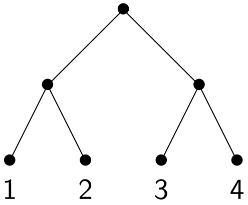
# Plethysm: Symmetric groups and wreath products

Take $\dim V \geq 4$. So $S_4 \leq \mathrm{GL}(V)$: $(1234) \mapsto \begin{pmatrix} \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \end{pmatrix}$.

- Weight space $(1,1,1,1)$ inside $\mathrm{Sym}^2(\mathrm{Sym}^2 V)$ is

$$\langle (v_1 v_2)(v_3 v_4), (v_1 v_3)(v_2 v_4), (v_1 v_4)(v_2 v_3) \rangle.$$

- Identify $(v_1 v_2)(v_3 v_4)$ with the set partition $\{\{1,2\},\{3,4\}\}$.
- Stabiliser $S_2 \wr S_2 = (S_2 \times S_2) \rtimes S_2 = \langle (12),(34) \rangle \rtimes \langle (13)(24) \rangle$.




- Weight space is permutation module $\mathbb{C}\big\uparrow_{S_2 \wr S_2}^{S_4}$

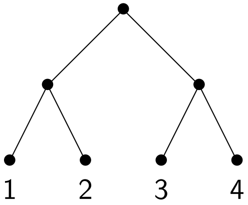
# Plethysm: Symmetric groups and wreath products

Take $\dim V \geq 4$. So $S_4 \leq \mathrm{GL}(V)$: $(1234) \mapsto \begin{pmatrix} \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \end{pmatrix}$.

- Weight space $(1,1,1,1)$ inside $\mathrm{Sym}^2(\mathrm{Sym}^2 V)$ is

$$\langle (v_1 v_2)(v_3 v_4), (v_1 v_3)(v_2 v_4), (v_1 v_4)(v_2 v_3) \rangle.$$

- Identify $(v_1 v_2)(v_3 v_4)$ with the set partition $\{\{1,2\},\{3,4\}\}$.
- Stabiliser $S_2 \wr S_2 = (S_2 \times S_2) \rtimes S_2 = \langle (12), (34) \rangle \rtimes \langle (13)(24) \rangle$.




- Weight space is permutation module $\mathbb{C}\big\uparrow_{S_2 \wr S_2}^{S_4}$
- Character $\chi^{(2,2)} + \chi^{(4)}$, corresponding to $\Delta^{(2,2)} \oplus \Delta^{(4)}$.

# Imprimitivity is surprisingly primitive!

Let $f(X) \in \mathbb{Q}[X]$ be irreducible with roots $\alpha_1, \ldots, \alpha_d \in \mathbb{C}$.

- Then $\mathrm{Gal}(f)$ acts on $K = \mathbb{Q}[\alpha_1, \ldots, \alpha_d]$, permuting the roots $\alpha_1, \ldots, \alpha_d$ transitively.

# Imprimitivity is surprisingly primitive!

Let $f(X) \in \mathbb{Q}[X]$ be irreducible with roots $\alpha_1, \dots, \alpha_d \in \mathbb{C}$.

- Then $\mathrm{Gal}(f)$ acts on $K = \mathbb{Q}[\alpha_1, \dots, \alpha_d]$, permuting the roots $\alpha_1, \dots, \alpha_d$ transitively.
- Let $L = \mathbb{Q}[\sqrt{\alpha_1}, \dots, \sqrt{\alpha_d}]$. Then $\mathrm{Gal}(L/K) \leq C_2 \times \cdots \times C_2$ and

# Imprimitivity is surprisingly primitive!

Let $f(X) \in \mathbb{Q}[X]$ be irreducible with roots $\alpha_1, \ldots, \alpha_d \in \mathbb{C}$.

- Then $\mathrm{Gal}(f)$ acts on $K = \mathbb{Q}[\alpha_1, \ldots, \alpha_d]$, permuting the roots $\alpha_1, \ldots, \alpha_d$ transitively.
- Let $L = \mathbb{Q}[\sqrt{\alpha_1}, \ldots, \sqrt{\alpha_d}]$. Then $\mathrm{Gal}(L/K) \leq C_2 \times \cdots \times C_2$ and $\ldots \mathrm{Gal}(L/\mathbb{Q}) \leq C_2 \wr \mathrm{Gal}(K/\mathbb{Q})$.

# Imprimitivity is surprisingly primitive!

Let $f(X) \in \mathbb{Q}[X]$ be irreducible with roots $\alpha_1, \dots, \alpha_d \in \mathbb{C}$.

- Then $\mathrm{Gal}(f)$ acts on $K = \mathbb{Q}[\alpha_1, \dots, \alpha_d]$, permuting the roots $\alpha_1, \dots, \alpha_d$ transitively.
- Let $L = \mathbb{Q}[\sqrt{\alpha_1}, \dots, \sqrt{\alpha_d}]$. Then $\mathrm{Gal}(L/K) \leq C_2 \times \cdots \times C_2$ and $\dots \mathrm{Gal}(L/\mathbb{Q}) \leq C_2 \wr \mathrm{Gal}(K/\mathbb{Q})$.

For example, $X^3 - 12X - 4 = (X - \alpha)(X - \beta)(X - \gamma)$ has Galois group $S_{\{\alpha,\beta,\gamma\}}$. Since $\alpha\beta\gamma = 4 \in \mathbb{Q}^{\times 2}$, $\mathrm{Gal}(X^6 - 12X^2 - 4)$ is a proper subgroup of $C_2 \wr S_3$:

$$
\begin{tikzcd}[row sep=large]
\mathrm{Gal}(L/\mathbb{Q}) = \left\langle \begin{matrix} (\sqrt{\alpha},-\sqrt{\alpha})(\sqrt{\beta},-\sqrt{\beta} \\ (\sqrt{\beta},-\sqrt{\beta})(\sqrt{\gamma},-\sqrt{\gamma}) \end{matrix} \right\rangle \rtimes \left\langle \begin{matrix} (\sqrt{\alpha},\sqrt{\beta})(-\sqrt{\alpha},-\sqrt{\beta}) \\ (\sqrt{\alpha},\sqrt{\beta},\sqrt{\gamma})(-\sqrt{\alpha},-\sqrt{\beta},-\sqrt{\gamma}) \end{matrix} \right\rangle \leq C_2 \wr S_3 \arrow[d, dash] \\
\mathrm{Gal}(L/K) = \left\langle \begin{matrix} (\sqrt{\alpha},-\sqrt{\alpha})(\sqrt{\beta},-\sqrt{\beta} \\ (\sqrt{\beta},-\sqrt{\beta})(\sqrt{\gamma},-\sqrt{\gamma}) \end{matrix} \right\rangle \arrow[d, dash] \\
1
\end{tikzcd}
$$

# Foulkes' Conjecture

Let $\Omega^{(m^n)}$ be the set of all set partitions of $\{1, 2, \ldots, mn\}$ into $n$ sets each of size $m$.

## Conjecture (Foulkes)

*If $m \leq n$ then there is an injective map of $S_{mn}$-representations $\langle \Omega^{(n^m)} \rangle_{\mathbb{C}} \to \langle \Omega^{(m^n)} \rangle_{\mathbb{C}}$.*

# Foulkes' Conjecture

Let $\Omega^{(m^n)}$ be the set of all set partitions of $\{1, 2, \ldots, mn\}$ into $n$ sets each of size $m$.

### Conjecture (Foulkes)

*If $m \leq n$ then there is an injective map of $S_{mn}$-representations $\langle \Omega^{(n^m)} \rangle_{\mathbb{C}} \to \langle \Omega^{(m^n)} \rangle_{\mathbb{C}}$.*

Equivalently, there is an injective map of $\mathrm{GL}(V)$-representations

$$\mathrm{Sym}^m(\mathrm{Sym}^n V) \to \mathrm{Sym}^n(\mathrm{Sym}^m V).$$

# Foulkes' Conjecture

Let $\Omega^{(m^n)}$ be the set of all set partitions of $\{1, 2, \ldots, mn\}$ into $n$ sets each of size $m$.

## Conjecture (Foulkes)

*If $m \leq n$ then there is an injective map of $S_{mn}$-representations $\langle \Omega^{(n^m)} \rangle_{\mathbb{C}} \to \langle \Omega^{(m^n)} \rangle_{\mathbb{C}}$.*

Equivalently, if $\phi^{(m^n)}$ is the character of $\langle \Omega^{(m^n)} \rangle_{\mathbb{C}}$, then $\langle \phi^{(n^m)}, \chi^\lambda \rangle \leq \langle \phi^{(m^n)}, \chi^\lambda \rangle$ for all $\lambda \in \mathrm{Par}(mn)$.

# Foulkes' Conjecture

Let $\Omega^{(m^n)}$ be the set of all set partitions of $\{1, 2, \ldots, mn\}$ into $n$ sets each of size $m$.

## Conjecture (Foulkes)

*If $m \leq n$ then there is an injective map of $S_{mn}$-representations $\langle \Omega^{(n^m)} \rangle_{\mathbb{C}} \to \langle \Omega^{(m^n)} \rangle_{\mathbb{C}}$.*

Equivalently, if $\phi^{(m^n)}$ is the character of $\langle \Omega^{(m^n)} \rangle_{\mathbb{C}}$, then $\langle \phi^{(n^m)}, \chi^\lambda \rangle \leq \langle \phi^{(m^n)}, \chi^\lambda \rangle$ for all $\lambda \in \mathrm{Par}(mn)$.

$$\phi^{(n^2)} = \chi^{(2n)} + \chi^{(2n-2,2)} + \chi^{(2n-4,4)} + \cdots$$

$$\phi^{(2^n)} = \sum_{\lambda \in \mathrm{Par}(n)} \chi^{2\lambda}$$

- Hence FC holds when $m = 2$.

# Foulkes' Conjecture

Let $\Omega^{(m^n)}$ be the set of all set partitions of $\{1, 2, \ldots, mn\}$ into $n$ sets each of size $m$.

## Conjecture (Foulkes)

*If $m \leq n$ then there is an injective map of $S_{mn}$-representations $\langle \Omega^{(n^m)} \rangle_{\mathbb{C}} \to \langle \Omega^{(m^n)} \rangle_{\mathbb{C}}$.*

Equivalently, if $\phi^{(m^n)}$ is the character of $\langle \Omega^{(m^n)} \rangle_{\mathbb{C}}$, then $\langle \phi^{(n^m)}, \chi^{\lambda} \rangle \leq \langle \phi^{(m^n)}, \chi^{\lambda} \rangle$ for all $\lambda \in \mathrm{Par}(mn)$.

$$\phi^{(n^2)} = \chi^{(2n)} + \chi^{(2n-2,2)} + \chi^{(2n-4,4)} + \cdots$$

$$\phi^{(2^n)} = \sum_{\lambda \in \mathrm{Par}(n)} \chi^{2\lambda}$$

- Hence FC holds when $m = 2$.
- These are the only multiplicity-free Foulkes characters for $mn \geq 18$ (Saxl, 1980).

# Foulkes' Conjecture

Let $\Omega^{(m^n)}$ be the set of all set partitions of $\{1, 2, \ldots, mn\}$ into $n$ sets each of size $m$.

## Conjecture (Foulkes)

*If $m \leq n$ then there is an injective map of $S_{mn}$-representations $\langle \Omega^{(n^m)} \rangle_{\mathbb{C}} \to \langle \Omega^{(m^n)} \rangle_{\mathbb{C}}$.*

Equivalently, if $\phi^{(m^n)}$ is the character of $\langle \Omega^{(m^n)} \rangle_{\mathbb{C}}$, then $\langle \phi^{(n^m)}, \chi^\lambda \rangle \leq \langle \phi^{(m^n)}, \chi^\lambda \rangle$ for all $\lambda \in \mathrm{Par}(mn)$.

$$\phi^{(n^2)} = \chi^{(2n)} + \chi^{(2n-2,2)} + \chi^{(2n-4,4)} + \cdots$$

$$\phi^{(2^n)} = \sum_{\lambda \in \mathrm{Par}(n)} \chi^{2\lambda}$$

- Hence FC holds when $m = 2$.
- These are the only multiplicity-free Foulkes characters for $mn \geq 18$ (Saxl, 1980).

# Decomposition Numbers

- Giannelli, MW 2014: results on decomposition numbers of symmetric groups obtained from local structure of $\langle \Omega^{(2^n)} \rangle$ over fields of prime characteristic.

$$(1,2,3,4) \mapsto \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{pmatrix} \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \end{pmatrix} \end{array}$$

# Decomposition Numbers

- Giannelli, MW 2014: results on decomposition numbers of symmetric groups obtained from local structure of $\langle \Omega^{(2^n)} \rangle$ over fields of prime characteristic.

$$
\begin{array}{cc}
 & \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\
(1,4,3) \quad \mapsto & \begin{pmatrix} \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot \end{pmatrix}
\end{array}
$$

# Decomposition Numbers

- Giannelli, MW 2014: results on decomposition numbers of symmetric groups obtained from local structure of $\langle \Omega^{(2^n)} \rangle$ over fields of prime characteristic.

$$
\begin{array}{cc}
 & \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\
(1,2)(3,4) \mapsto & \begin{pmatrix} \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \end{pmatrix}
\end{array}
$$

# Decomposition Numbers

- Giannelli, MW 2014: results on decomposition numbers of symmetric groups obtained from local structure of $\langle \Omega^{(2^n)} \rangle$ over fields of prime characteristic.

$$(1,2)(3,4) \mapsto \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{pmatrix} \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \end{pmatrix} \end{array}$$

In the new basis

$$\begin{aligned} w_1 &= v_1 + v_2 + v_3 + v_4 \\ w_2 &= v_2 - v_1 \\ w_3 &= v_3 - v_1 \\ w_4 &= v_4 - v_1 \end{aligned}$$

# Decomposition Numbers

- Giannelli, MW 2014: results on decomposition numbers of symmetric groups obtained from local structure of $\langle \Omega^{(2^n)} \rangle$ over fields of prime characteristic.

$$
(1,2)(3,4) \mapsto \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{pmatrix} \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \end{pmatrix} \end{array} \qquad \mapsto \begin{array}{c} \begin{array}{cccc} w_1 & w_2 & w_3 & w_4 \end{array} \\ \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & -1 & -1 & -1 \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \end{pmatrix} \end{array}
$$

In the new basis

$$
\begin{aligned}
w_1 &= v_1 + v_2 + v_3 + v_4 \\
w_2 &= v_2 - v_1 \\
w_3 &= v_3 - v_1 \\
w_4 &= v_4 - v_1
\end{aligned}
$$

# Decomposition Numbers

- Giannelli, MW 2014: results on decomposition numbers of symmetric groups obtained from local structure of $\langle \Omega^{(2^n)} \rangle$ over fields of prime characteristic.

$$(1,2,3,4) \mapsto \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{pmatrix} \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \end{pmatrix} \end{array} \qquad \mapsto \begin{array}{c} \begin{array}{cccc} w_1 & w_2 & w_3 & w_4 \end{array} \\ \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & -1 & -1 & -1 \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \end{pmatrix} \end{array}$$

In the new basis

$$\begin{aligned} w_1 &= v_1 + v_2 + v_3 + v_4 \\ w_2 &= v_2 - v_1 \\ w_3 &= v_3 - v_1 \\ w_4 &= v_4 - v_1 \end{aligned}$$

# Decomposition Numbers

- Giannelli, MW 2014: results on decomposition numbers of symmetric groups obtained from local structure of $\langle \Omega^{(2^n)} \rangle$ over fields of prime characteristic.

$$(1,2)(3,4) \mapsto \begin{array}{c} \begin{array}{ccc} w_2 & w_3 & w_4 \end{array} \\ \begin{pmatrix} -1 & -1 & -1 \\ \cdot & \cdot & 1 \\ \cdot & 1 & \cdot \end{pmatrix} \end{array} \qquad \mapsto \begin{array}{c} \begin{array}{ccc} z & w_3 & w_4 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \end{pmatrix} \end{array}$$

In the rational basis

$$w_2 = v_2 - v_1$$
$$w_3 = v_3 - v_1$$
$$w_4 = v_4 - v_1$$

In the $\mathbb{F}_2$-basis

$$z = v_1 + v_2 + v_3 + v_4$$
$$w_3 = v_3 - v_1$$
$$w_4 = v_4 - v_1$$

# Decomposition Numbers

- Giannelli, MW 2014: results on decomposition numbers of symmetric groups obtained from local structure of $\langle\Omega^{(2^n)}\rangle$ over fields of prime characteristic.

$$(1,2,3,4) \mapsto \begin{array}{c} \begin{array}{ccc} w_2 & w_3 & w_4 \end{array} \\ \begin{pmatrix} -1 & -1 & -1 \\ 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \end{pmatrix} \end{array} \qquad \mapsto \begin{array}{c} \begin{array}{ccc} z & w_3 & w_4 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 \\ \cdot & 1 & 1 \\ \cdot & \cdot & 1 \end{pmatrix} \end{array}$$

In the rational basis

$$w_2 = v_2 - v_1$$
$$w_3 = v_3 - v_1$$
$$w_4 = v_4 - v_1$$

In the $\mathbb{F}_2$-basis

$$z = v_1 + v_2 + v_3 + v_4$$
$$w_3 = v_3 - v_1$$
$$w_4 = v_4 - v_1$$

# Decomposition Numbers

- Giannelli, MW 2014: results on decomposition numbers of symmetric groups obtained from local structure of $\langle \Omega^{(2^n)} \rangle$ over fields of prime characteristic.

$$(1,2,3,4) \mapsto \begin{array}{c} \begin{array}{ccc} w_2 & w_3 & w_4 \end{array} \\ \begin{pmatrix} -1 & -1 & -1 \\ 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \end{pmatrix} \end{array} \qquad \mapsto \begin{array}{c} \begin{array}{ccc} z & w_3 & w_4 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 \\ \cdot & 1 & 1 \\ \cdot & \cdot & 1 \end{pmatrix} \end{array}$$

In the rational basis

$$w_2 = v_2 - v_1$$
$$w_3 = v_3 - v_1$$
$$w_4 = v_4 - v_1$$

In the $\mathbb{F}_2$-basis

$$z = v_1 + v_2 + v_3 + v_4$$
$$w_3 = v_3 - v_1$$
$$w_4 = v_4 - v_1$$

Hence $S^{(3,1)}_{\mathbb{F}_2}$ has a trivial submodule.
The quotient is a 2-dimensional simple $\mathbb{F}_2 S_4$-module

# Decomposition matrix of $\mathbb{F}_3 S_6$

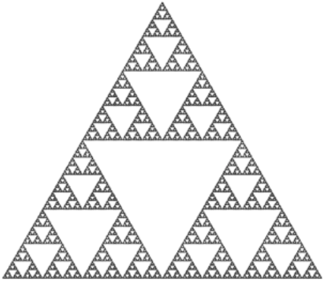
| | (6) | (5,1) | (4,2) | (3,3) | (4,1,1) | (3,2,1) | (2,2,1,1) |
|---|---|---|---|---|---|---|---|
| $(6)$ | 1 | | | | | | |
| $(5,1)$ | 1 | 1 | | | | | |
| $(4,2)$ | · | · | 1 | | | | |
| $(3,3)$ | · | 1 | · | 1 | | | |
| $(4,1,1)$ | · | 1 | · | · | 1 | | |
| $(3,2,1)$ | 1 | 1 | · | 1 | 1 | 1 | |
| $(2,2,1,1)$ | · | · | · | · | · | · | 1 |
| $(2,2,2)$ | 1 | · | · | · | · | 1 | · |
| $(3,1,1,1)$ | · | · | · | · | 1 | 1 | · |
| $(2,1,1,1,1)$ | · | · | · | 1 | · | 1 | · |
| $(1,1,1,1,1,1)$ | · | · | · | 1 | · | · | · |

# Decomposition matrix of $\mathbb{F}_3 S_6$: two-row partitions

| | **(6)** | **(5,1)** | **(4,2)** | **(3,3)** | (4,1,1) | (3,2,1) | (2,2,1,1) |
|---|---|---|---|---|---|---|---|
| **(6)** | 1 | | | | | | |
| **(5,1)** | 1 | 1 | | | | | |
| **(4,2)** | · | · | 1 | | | | |
| **(3,3)** | · | 1 | · | 1 | | | |
| (4,1,1) | · | 1 | · | · | 1 | | |
| (3,2,1) | 1 | 1 | · | 1 | 1 | 1 | |
| (2,2,1,1) | · | · | · | · | · | · | 1 |
| (2,2,2) | 1 | · | · | · | · | 1 | · |
| (3,1,1,1) | · | · | · | · | 1 | 1 | · |
| (2,1,1,1,1) | · | · | · | 1 | · | 1 | · |
| (1,1,1,1,1,1) | · | · | · | 1 | · | · | · |

# General form of the two-row decomposition matrix

$$
\begin{array}{ccccccccccc|}
1 &&&&&&&&&& \\
1 & 1 &&&&&&&&& \\
 & 1 & 1 &&&&&&&& \\
 & 1 & 1 & 1 &&&&&&& \\
1 & 1 & & 1 & 1 &&&&&& \\
1 & & & 1 & 1 & 1 &&&&& \\
 & & & 1 & & 1 & 1 &&&& \\
 & & & 1 & & 1 & 1 & 1 &&& \\
 & & & 1 & 1 & 1 & & 1 & 1 && \\
 & & & 1 & 1 & & & 1 & 1 & 1 & \\
 & & & 1 & & & & 1 & & 1 & (1) \\
\hline
\end{array}
$$



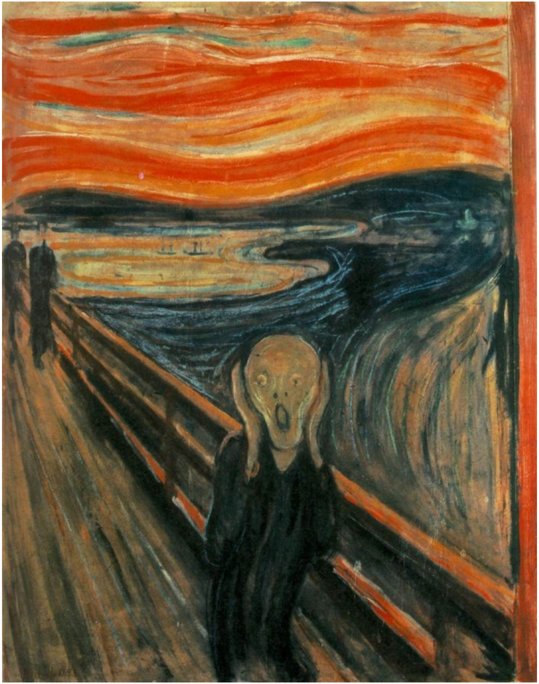
# Decomposition matrix of $\mathbb{F}_3 S_6$: separated into blocks

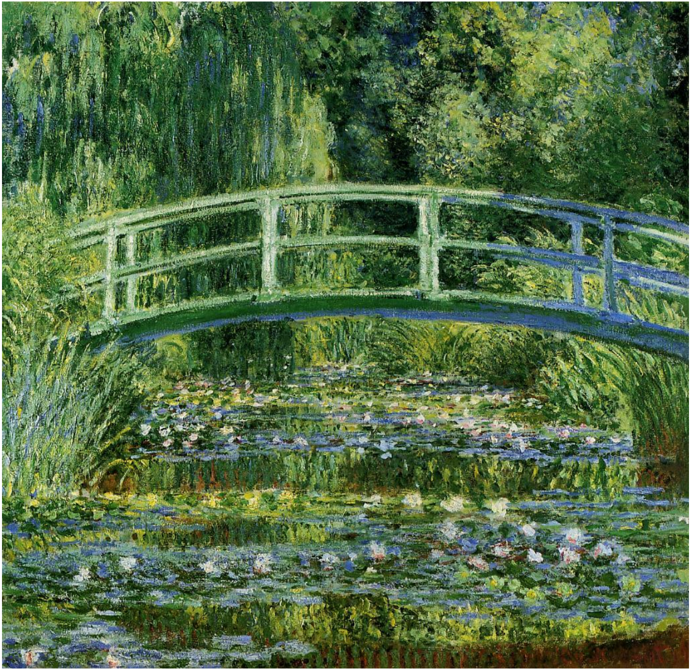
$$
\begin{array}{r|ccccc}
 & (6) & (5,1) & (3,3) & (4,1,1) & (3,2,1) \\
\hline
(6) & 1 & & & & \\
(5,1) & 1 & 1 & & & \\
(3,3) & \cdot & 1 & 1 & & \\
(4,1,1) & \cdot & 1 & \cdot & 1 & \\
(3,2,1) & 1 & 1 & 1 & 1 & 1 \\
(2,2,2) & 1 & \cdot & \cdot & \cdot & 1 \\
(3,1,1,1) & \cdot & \cdot & \cdot & 1 & 1 \\
(2,1,1,1,1) & \cdot & \cdot & 1 & \cdot & 1 \\
(1,1,1,1,1,1) & \cdot & \cdot & 1 & \cdot & \cdot
\end{array}
$$

$$
\begin{array}{r|c}
 & (4,2) \\
\hline
(4,2) & 1
\end{array}
\qquad
\begin{array}{r|c}
 & (2,2,1,1) \\
\hline
(2,2,1,1) & 1
\end{array}
$$

# Decomposition matrix of $\mathbb{F}_2 S_{10}$: separated into blocks

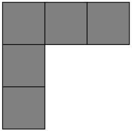
| | (10) | (9,1) | (8,2) | (7,3) | (6,4) | (6,3,1) | (5,3,2) | (7,2,1) | (5,4,1) | (4,3,2,1) |
|---|---|---|---|---|---|---|---|---|---|---|
| $(10)$ | 1 | | | | | | | | | |
| $(9,1)$ | 1 | 1 | | | | | | | | |
| $(8,2)$ | 1 | 1 | 1 | | | | | | | |
| $(7,3)$ | 1 | · | 1 | 1 | | | | | | |
| $(6,4)$ | · | · | 1 | 1 | 1 | | | | | |
| $(6,3,1)$ | 1 | · | 2 | 1 | 1 | 1 | | | | |
| $(5,3,2)$ | 2 | 1 | 1 | · | 1 | 1 | 1 | | | |
| $(5,5)$ | · | · | 1 | · | 1 | · | · | | | |
| $(8,1,1)$ | 2 | 1 | 1 | · | · | · | · | | | |
| $(6,2,2)$ | 1 | · | 1 | · | · | 1 | · | | | |
| $(4,4,2)$ | 2 | 1 | 1 | · | 1 | · | 1 | | | |
| $(4,3,3)$ | 2 | 1 | · | · | · | · | 1 | | | |
| $(7,1,1,1)$ | 2 | 1 | 1 | 1 | · | · | · | | | |
| $(6,2,1)$ | 2 | 1 | 3 | 1 | 1 | 1 | · | | | |
| $(5,3,1,1)$ | 3 | 1 | 3 | 1 | 2 | 1 | 1 | | | |
| $(4,4,1,1)$ | 2 | 1 | 1 | 1 | 1 | · | 1 | | | |
| $(5,2,2,1)$ | 3 | 1 | 2 | 1 | 1 | 1 | 1 | | | |
| $(6,1,1,1,1)$ | 2 | 1 | 2 | 1 | 1 | · | · | | | |
| $(7,2,1)$ | | | | | | | | 1 | · | |
| $(5,4,1)$ | | | | | | | | 1 | 1 | |
| $(4,3,2,1)$ | | | | | | | | | | 1 |





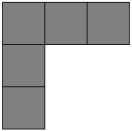
# Decomposition Numbers: 3-block of $S_{12}$ with core $(3,1,1)$

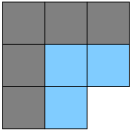
| | $(12,1^2)$ | $(9,4,1)$ | $(9,3,2)$ | $(8,4,2)$ | $(6^2,2)$ | $(6,4^4)$ | $(6,4,2^2)$ | $(6,3,2^2,1)$ | $(5,4,2^2,1)$ | $(4^2,2^2,1^2)$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $(12,1^2)=\langle 2\rangle$ | 1 | | | | | | | | | |
| $(9,4,1)=\langle 2,2\rangle$ | 1 | 1 | | | | | | | | |
| $(9,3,2)=\langle 2,1\rangle$ | 2 | 1 | 1 | | | | | | | |
| $(8,4,2)=\langle 1\rangle$ | 1 | 1 | 1 | 1 | | | | | | |
| $(6^2,2)=\langle 1,2\rangle$ | | | | 1 | 1 | | | | | |
| $(6,4^4)=\langle 1,2,2\rangle$ | | | 1 | 1 | 1 | 1 | | | | |
| $(6,4,2^2)=\langle 2,2,2\rangle$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | | | |
| $(6,3,2^2,1)=\langle 1,1,2\rangle$ | 2 | 1 | 1 | | | | 1 | 1 | | |
| $(5,4,2^2,1)=\langle 1,1\rangle$ | 1 | 1 | 1 | | 1 | 1 | 1 | 1 | 1 | |
| $(4^2,2^2,1^2)=\langle 3\rangle$ | 1 | | 1 | | 1 | 1 | | 1 | 1 | 1 |
| $(9,1^5)=\langle 2,3\rangle$ | | | 1 | | | | | | | |
| $(6,4,1^4)=\langle 2,2,3\rangle$ | | | | | | | 1 | | | |
| $(6,3,2,1^3)=\langle 1,2,3\rangle$ | | | 1 | | | 1 | 1 | 1 | | |
| $(6,2^3,1^2)=\langle 3,2\rangle$ | | | | | | | | 1 | | |
| $(6,1^8)=\langle 2,3,3\rangle$ | | | | | | 1 | | | | |
| $(5,4,2,1^3)=\langle 1,3\rangle$ | | | | | 2 | 1 | 1 | 1 | 1 | |
| $(3^4,1^2)=\langle 3,1\rangle$ | 1 | | 1 | | | 1 | | | | 1 |
| $(3^2,2^4)=\langle 1,1,3\rangle$ | 1 | | | | | | | | | 1 |
| $(3^2,2^2,1^4)=\langle 1,1,1\rangle$ | | | | | 1 | 1 | | | 1 | 1 |
| $(3^2,2,1^6)=\langle 1,3,3\rangle$ | | | | | 2 | 1 | | | 1 | |
| $(3,2^3,1^5)=\langle 3,3\rangle$ | | | | | 1 | | | | 1 | |
| $(3,1^{11})=\langle 3,3,3\rangle$ | | | | | 1 | | | | | |



# Decomposition Numbers: 3-block of $S_{12}$ with core $(3,1,1)$

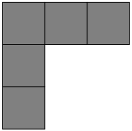
| | $(12,1^2)$ | $(9,4,1)$ | $(9,3,2)$ | $(8,4,2)$ | $(6^2,2)$ | $(6,4^4)$ | $(6,4,2^2)$ | $(6,3,2^2,1)$ | $(5,4,2^2,1)$ | $(4^2,2^2,1^2)$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $(12,1^2)=\langle 2\rangle$ | 1 | | | | | | | | | |
| $(9,4,1)=\langle 2,2\rangle$ | 1 | 1 | | | | | | | | |
| $(9,3,2)=\langle 2,1\rangle$ | 2 | 1 | 1 | | | | | | | |
| $(8,4,2)=\langle 1\rangle$ | 1 | 1 | 1 | 1 | | | | | | |
| $(6^2,2)=\langle 1,2\rangle$ | | | | 1 | 1 | | | | | |
| $(6,4^4)=\langle 1,2,2\rangle$ | | | 1 | 1 | 1 | 1 | | | | |
| $(6,4,2^2)=\langle 2,2,2\rangle$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | | | |
| $(6,3,2^2,1)=\langle 1,1,2\rangle$ | 2 | 1 | 1 | | | | 1 | 1 | | |
| $(5,4,2^2,1)=\langle 1,1\rangle$ | 1 | 1 | 1 | | 1 | 1 | 1 | 1 | 1 | |
| $(4^2,2^2,1^2)=\langle 3\rangle$ | 1 | | 1 | | 1 | 1 | | 1 | 1 | 1 |
| $(9,1^5)=\langle 2,3\rangle$ | | | 1 | | | | | | | |
| $(6,4,1^4)=\langle 2,2,3\rangle$ | | | | | | | 1 | | | |
| $(6,3,2,1^3)=\langle 1,2,3\rangle$ | | | 1 | | | 1 | 1 | 1 | | |
| $(6,2^3,1^2)=\langle 3,2\rangle$ | | | | | | | | 1 | | |
| $(6,1^8)=\langle 2,3,3\rangle$ | | | | | | 1 | | | | |
| $(5,4,2,1^3)=\langle 1,3\rangle$ | | | | | 2 | 1 | 1 | 1 | 1 | |
| $(3^4,1^2)=\langle 3,1\rangle$ | 1 | | 1 | | | 1 | | | | 1 |
| $(3^2,2^4)=\langle 1,1,3\rangle$ | 1 | | | | | | | | | 1 |
| $(3^2,2^2,1^4)=\langle 1,1,1\rangle$ | | | | | 1 | 1 | | | 1 | 1 |
| $(3^2,2,1^6)=\langle 1,3,3\rangle$ | | | | | 2 | 1 | | | 1 | |
| $(3,2^3,1^5)=\langle 3,3\rangle$ | | | | | 1 | | | | 1 | |
| $(3,1^{11})=\langle 3,3,3\rangle$ | | | | | 1 | | | | | |

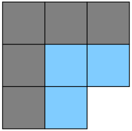


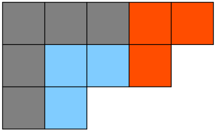


# Decomposition Numbers: 3-block of $S_{12}$ with core $(3,1,1)$

| | $(12,1^2)$ | $(9,4,1)$ | $(9,3,2)$ | $(8,4,2)$ | $(6^2,2)$ | $(6,4^4)$ | $(6,4,2^2)$ | $(6,3,2^2,1)$ | $(5,4,2^2,1)$ | $(4^2,2^2,1^2)$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $(12,1^2)=\langle 2\rangle$ | 1 | | | | | | | | | |
| $(9,4,1)=\langle 2,2\rangle$ | 1 | 1 | | | | | | | | |
| $(9,3,2)=\langle 2,1\rangle$ | 2 | 1 | 1 | | | | | | | |
| $(8,4,2)=\langle 1\rangle$ | 1 | 1 | 1 | 1 | | | | | | |
| $(6^2,2)=\langle 1,2\rangle$ | | | | 1 | 1 | | | | | |
| $(6,4^4)=\langle 1,2,2\rangle$ | | | 1 | 1 | 1 | 1 | | | | |
| $(6,4,2^2)=\langle 2,2,2\rangle$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | | | |
| $(6,3,2^2,1)=\langle 1,1,2\rangle$ | 2 | 1 | 1 | | | | 1 | 1 | | |
| $(5,4,2^2,1)=\langle 1,1\rangle$ | 1 | 1 | 1 | | 1 | 1 | 1 | 1 | 1 | |
| $(4^2,2^2,1^2)=\langle 3\rangle$ | 1 | | 1 | | 1 | 1 | | 1 | 1 | 1 |
| $(9,1^5)=\langle 2,3\rangle$ | | | 1 | | | | | | | |
| $(6,4,1^4)=\langle 2,2,3\rangle$ | | | | | | | 1 | | | |
| $(6,3,2,1^3)=\langle 1,2,3\rangle$ | | | 1 | | | 1 | 1 | 1 | | |
| $(6,2^3,1^2)=\langle 3,2\rangle$ | | | | | | | | 1 | | |
| $(6,1^8)=\langle 2,3,3\rangle$ | | | | | | 1 | | | | |
| $(5,4,2,1^3)=\langle 1,3\rangle$ | | | | | 2 | 1 | 1 | 1 | 1 | |
| $(3^4,1^2)=\langle 3,1\rangle$ | 1 | | 1 | | | 1 | | | | 1 |
| $(3^2,2^4)=\langle 1,1,3\rangle$ | 1 | | | | | | | | | 1 |
| $(3^2,2^2,1^4)=\langle 1,1,1\rangle$ | | | | | 1 | 1 | | | 1 | 1 |
| $(3^2,2,1^6)=\langle 1,3,3\rangle$ | | | | | 2 | 1 | | | 1 | |
| $(3,2^3,1^5)=\langle 3,3\rangle$ | | | | | 1 | | | | 1 | |
| $(3,1^{11})=\langle 3,3,3\rangle$ | | | | | 1 | | | | | |

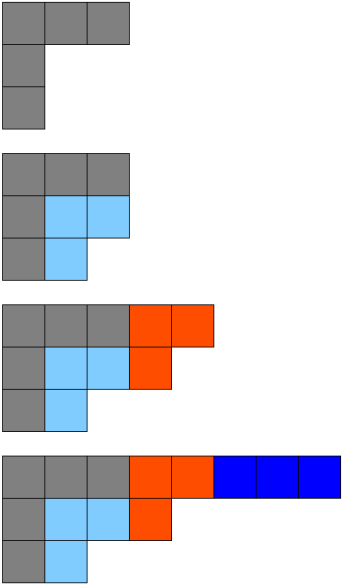






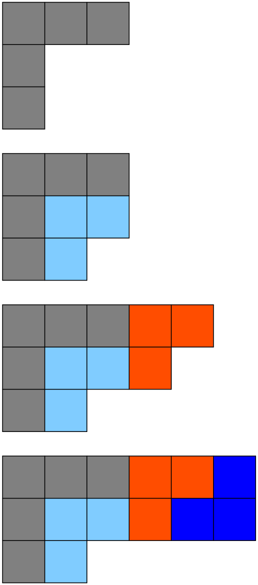
# Decomposition Numbers: 3-block of $S_{12}$ with core $(3,1,1)$

| | $(12,1^2)$ | $(9,4,1)$ | $(9,3,2)$ | $(8,4,2)$ | $(6^2,2)$ | $(6,4^4)$ | $(6,4,2^2)$ | $(6,3,2^2,1)$ | $(5,4,2^2,1)$ | $(4^2,2^2,1^2)$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $(12,1^2)=\langle 2\rangle$ | 1 | | | | | | | | | |
| $(9,4,1)=\langle 2,2\rangle$ | 1 | 1 | | | | | | | | |
| $(9,3,2)=\langle 2,1\rangle$ | 2 | 1 | 1 | | | | | | | |
| $(8,4,2)=\langle 1\rangle$ | 1 | 1 | 1 | 1 | | | | | | |
| $(6^2,2)=\langle 1,2\rangle$ | | | | 1 | 1 | | | | | |
| $(6,4^4)=\langle 1,2,2\rangle$ | | | 1 | 1 | 1 | 1 | | | | |
| $(6,4,2^2)=\langle 2,2,2\rangle$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | | | |
| $(6,3,2^2,1)=\langle 1,1,2\rangle$ | 2 | 1 | 1 | | | | 1 | 1 | | |
| $(5,4,2^2,1)=\langle 1,1\rangle$ | 1 | 1 | 1 | | 1 | 1 | 1 | 1 | 1 | |
| $(4^2,2^2,1^2)=\langle 3\rangle$ | 1 | | 1 | | 1 | 1 | | 1 | 1 | 1 |
| $(9,1^5)=\langle 2,3\rangle$ | | | 1 | | | | | | | |
| $(6,4,1^4)=\langle 2,2,3\rangle$ | | | | | | | 1 | | | |
| $(6,3,2,1^3)=\langle 1,2,3\rangle$ | | | 1 | | | 1 | 1 | 1 | | |
| $(6,2^3,1^2)=\langle 3,2\rangle$ | | | | | | | | 1 | | |
| $(6,1^8)=\langle 2,3,3\rangle$ | | | | | | 1 | | | | |
| $(5,4,2,1^3)=\langle 1,3\rangle$ | | | | | 2 | 1 | 1 | 1 | 1 | |
| $(3^4,1^2)=\langle 3,1\rangle$ | 1 | | 1 | | | 1 | | | | 1 |
| $(3^2,2^4)=\langle 1,1,3\rangle$ | 1 | | | | | | | | | 1 |
| $(3^2,2^2,1^4)=\langle 1,1,1\rangle$ | | | | | 1 | 1 | | | 1 | 1 |
| $(3^2,2,1^6)=\langle 1,3,3\rangle$ | | | | | 2 | 1 | | | 1 | |
| $(3,2^3,1^5)=\langle 3,3\rangle$ | | | | | 1 | | | | 1 | |
| $(3,1^{11})=\langle 3,3,3\rangle$ | | | | | 1 | | | | | |



# Decomposition Numbers: 3-block of $S_{12}$ with core $(3,1,1)$

| | $(12,1^2)$ | $(9,4,1)$ | $(9,3,2)$ | $(8,4,2)$ | $(6^2,2)$ | $(6,4^4)$ | $(6,4,2^2)$ | $(6,3,2^2,1)$ | $(5,4,2^2,1)$ | $(4^2,2^2,1^2)$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $(12,1^2)=\langle 2\rangle$ | 1 | | | | | | | | | |
| $(9,4,1)=\langle 2,2\rangle$ | 1 | 1 | | | | | | | | |
| $(9,3,2)=\langle 2,1\rangle$ | 2 | 1 | 1 | | | | | | | |
| $(8,4,2)=\langle 1\rangle$ | 1 | 1 | 1 | 1 | | | | | | |
| $(6^2,2)=\langle 1,2\rangle$ | | | | 1 | 1 | | | | | |
| $(6,4^4)=\langle 1,2,2\rangle$ | | | 1 | 1 | 1 | 1 | | | | |
| $(6,4,2^2)=\langle 2,2,2\rangle$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | | | |
| $(6,3,2^2,1)=\langle 1,1,2\rangle$ | 2 | 1 | 1 | | | | 1 | 1 | | |
| $(5,4,2^2,1)=\langle 1,1\rangle$ | 1 | 1 | 1 | | 1 | 1 | 1 | 1 | 1 | |
| $(4^2,2^2,1^2)=\langle 3\rangle$ | 1 | | 1 | | 1 | 1 | | 1 | 1 | 1 |
| $(9,1^5)=\langle 2,3\rangle$ | | | 1 | | | | | | | |
| $(6,4,1^4)=\langle 2,2,3\rangle$ | | | | | | | 1 | | | |
| $(6,3,2,1^3)=\langle 1,2,3\rangle$ | | | 1 | | | 1 | 1 | 1 | | |
| $(6,2^3,1^2)=\langle 3,2\rangle$ | | | | | | | | 1 | | |
| $(6,1^8)=\langle 2,3,3\rangle$ | | | | | | 1 | | | | |
| $(5,4,2,1^3)=\langle 1,3\rangle$ | | | | | 2 | 1 | 1 | 1 | 1 | |
| $(3^4,1^2)=\langle 3,1\rangle$ | 1 | | 1 | | | 1 | | | | 1 |
| $(3^2,2^4)=\langle 1,1,3\rangle$ | 1 | | | | | | | | | 1 |
| $(3^2,2^2,1^4)=\langle 1,1,1\rangle$ | | | | | 1 | 1 | | | 1 | 1 |
| $(3^2,2,1^6)=\langle 1,3,3\rangle$ | | | | | 2 | 1 | | | 1 | |
| $(3,2^3,1^5)=\langle 3,3\rangle$ | | | | | 1 | | | | 1 | |
| $(3,1^{11})=\langle 3,3,3\rangle$ | | | | | 1 | | | | | |



# Foulkes' Conjecture and Howe's Conjecture

Let $\Omega^{(m^n)}$ be the set of all set partitions of $\{1, 2, \ldots, mn\}$ into $n$ sets each of size $m$.

## Conjecture (Howe 1987)

*The $\mathbb{C}S_{mn}$-homomorphism $\theta^{(m^n)} : \langle \Omega^{(n^m)} \rangle_{\mathbb{C}} \to \langle \Omega^{(m^n)} \rangle_{\mathbb{C}}$ defined by*

$$\{A_1, \ldots, A_m\} \mapsto \sum \{B_1, \ldots, B_n\},$$

*where the sum is over all $\{B_1, \ldots, B_n\} \in \Omega^{(m^n)}$ such that $|A_i \cap B_j| = 1$ for all $i$ and $j$, is injective.*

# Foulkes' Conjecture and Howe's Conjecture

Let $\Omega^{(m^n)}$ be the set of all set partitions of $\{1, 2, \ldots, mn\}$ into $n$ sets each of size $m$.

## Conjecture (Howe 1987)

*The $\mathbb{C}S_{mn}$-homomorphism $\theta^{(m^n)} : \langle \Omega^{(n^m)} \rangle_{\mathbb{C}} \to \langle \Omega^{(m^n)} \rangle_{\mathbb{C}}$ defined by*

$$\{A_1, \ldots, A_m\} \mapsto \sum \{B_1, \ldots, B_n\},$$

*where the sum is over all $\{B_1, \ldots, B_n\} \in \Omega^{(m^n)}$ such that $|A_i \cap B_j| = 1$ for all $i$ and $j$, is injective.*

- Dent, Siemons 2000: FC is true for $m = 3$.

# Foulkes' Conjecture and Howe's Conjecture

Let $\Omega^{(m^n)}$ be the set of all set partitions of $\{1, 2, \ldots, mn\}$ into $n$ sets each of size $m$.

## Conjecture (Howe 1987)

*The $\mathbb{C}S_{mn}$-homomorphism $\theta^{(m^n)} : \langle \Omega^{(n^m)} \rangle_{\mathbb{C}} \to \langle \Omega^{(m^n)} \rangle_{\mathbb{C}}$ defined by*

$$\{A_1, \ldots, A_m\} \mapsto \sum \{B_1, \ldots, B_n\},$$

*where the sum is over all $\{B_1, \ldots, B_n\} \in \Omega^{(m^n)}$ such that $|A_i \cap B_j| = 1$ for all $i$ and $j$, is injective.*

- Dent, Siemons 2000: FC is true for $m = 3$.
- McKay 2007: if $\theta^{(m^n)}$ is injective then so is $\theta^{(m^{n'})}$ for all $n' \geq n$. Hence HC and FC hold for $m = 4$.

# Foulkes' Conjecture and Howe's Conjecture

Let $\Omega^{(m^n)}$ be the set of all set partitions of $\{1, 2, \ldots, mn\}$ into $n$ sets each of size $m$.

## Conjecture (Howe 1987)

*The $\mathbb{C}S_{mn}$-homomorphism $\theta^{(m^n)} : \langle \Omega^{(n^m)} \rangle_{\mathbb{C}} \to \langle \Omega^{(m^n)} \rangle_{\mathbb{C}}$ defined by*

$$\{A_1, \ldots, A_m\} \mapsto \sum \{B_1, \ldots, B_n\},$$

*where the sum is over all $\{B_1, \ldots, B_n\} \in \Omega^{(m^n)}$ such that $|A_i \cap B_j| = 1$ for all $i$ and $j$, is injective.*

- Dent, Siemons 2000: FC is true for $m = 3$.
- McKay 2007: if $\theta^{(m^n)}$ is injective then so is $\theta^{(m^{n'})}$ for all $n' \geq n$. Hence HC and FC hold for $m = 4$.
- Müller, Neunhöffer 2005: $\theta^{(5^5)}$ is not injective.

# Foulkes' Conjecture and Howe's Conjecture

Let $\Omega^{(m^n)}$ be the set of all set partitions of $\{1, 2, \ldots, mn\}$ into $n$ sets each of size $m$.

## Conjecture (Howe 1987)

*The $\mathbb{C}S_{mn}$-homomorphism $\theta^{(m^n)} : \langle \Omega^{(n^m)} \rangle_{\mathbb{C}} \to \langle \Omega^{(m^n)} \rangle_{\mathbb{C}}$ defined by*

$$\{A_1, \ldots, A_m\} \mapsto \sum \{B_1, \ldots, B_n\},$$

*where the sum is over all $\{B_1, \ldots, B_n\} \in \Omega^{(m^n)}$ such that $|A_i \cap B_j| = 1$ for all $i$ and $j$, is injective.*

- Dent, Siemons 2000: FC is true for $m = 3$.
- McKay 2007: if $\theta^{(m^n)}$ is injective then so is $\theta^{(m^{n'})}$ for all $n' \geq n$. Hence HC and FC hold for $m = 4$.
- Müller, Neunhöffer 2005: $\theta^{(5^5)}$ is not injective.
- Cheung, Ikenmeyer, Mkrtchyan 2015: $\theta^{(5^6)}$ is injective, hence FC is true for $m = 5$.

# Open problem

### Problem

*Decompose* $\phi^{(3^n)}$ *into irreducible characters of* $S_{3n}$*.*

Equivalently, decompose $\mathrm{Sym}^n(\mathrm{Sym}^3 V)$ into irreducible representations of $\mathrm{GL}(V)$.

# Open problem

### Problem

*Decompose $\phi^{(3^n)}$ into irreducible characters of $S_{3n}$.*

Equivalently, decompose $\mathrm{Sym}^n(\mathrm{Sym}^3 V)$ into irreducible representations of $\mathrm{GL}(V)$.

It is not hard to show that

$$\phi^{(3^n)}\big\downarrow_{S_{3n-1}} = \left(\phi^{(3^{n-1})} \times 1_{S_2}\right)\Big\uparrow^{S_{3n-1}}.$$

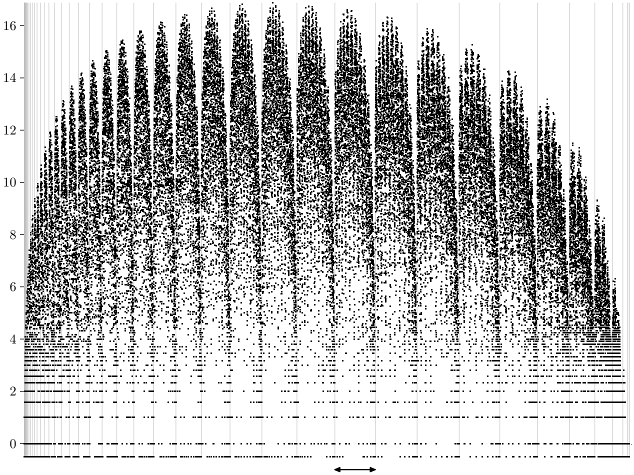
Computational evidence suggests that this property, together with $\langle \phi^{(3^n)}, 1_{S_{3n}} \rangle = 1$, determines $\phi^{(3^n)}$ uniquely.

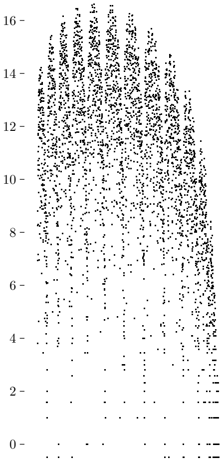
# Foulkes' Conjecture: computational results

- Müller, Neunhöffer 2005: FC is true if $m + n \leq 17$.
- Evseev, Paget, MW 2014: FC is true if $m + n \leq 19$.

# Foulkes' Conjecture: computational results

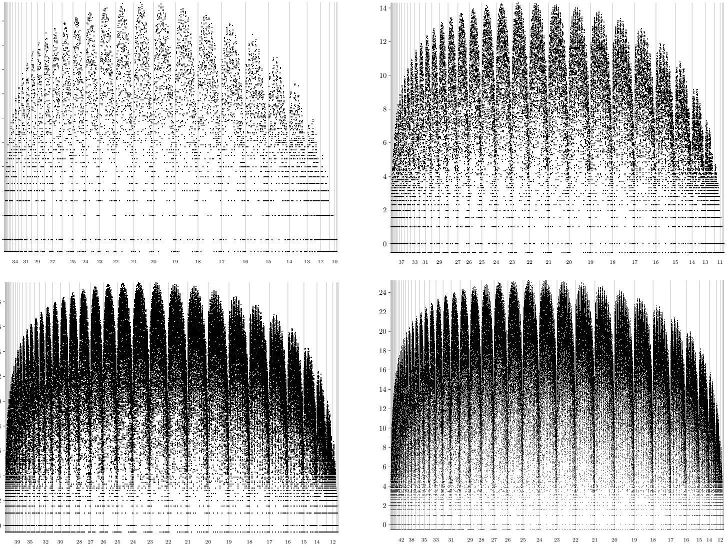
- Müller, Neunhöffer 2005: FC is true if $m + n \leq 17$.
- Evseev, Paget, MW 2014: FC is true if $m + n \leq 19$.



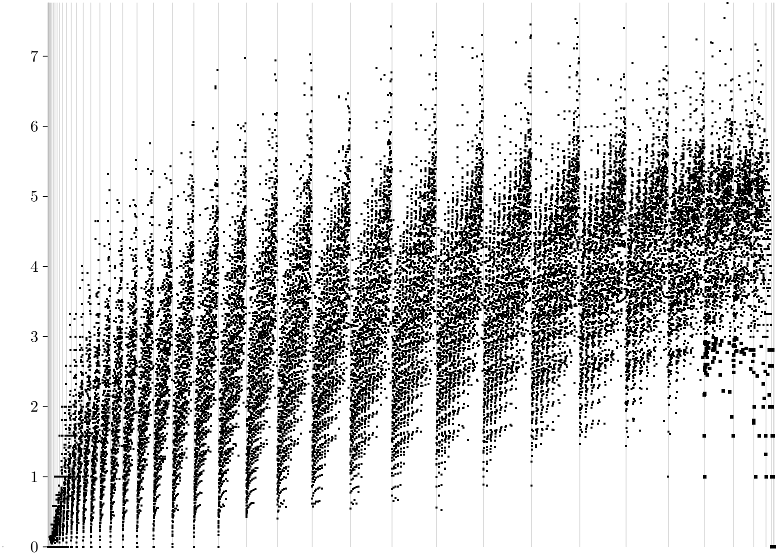
# Foulkes' Conjecture: computational results

- Müller, Neunhöffer 2005: FC is true if $m + n \leq 17$.
- Evseev, Paget, MW 2014: FC is true if $m + n \leq 19$.



# Foulkes' Conjecture: computational results

- Müller, Neunhöffer 2005: FC is true if $m + n \leq 17$.
- Evseev, Paget, MW 2014: FC is true if $m + n \leq 19$.



# Plethysm: Symmetric polynomials.

Suppose $\dim V = d$.

- A basis of weight vectors for $\mathrm{Sym}^2 V$ is

$$\begin{array}{cccccc} v_1 v_1, & v_1 v_2, & v_2 v_2, & v_1 v_3, & \dots & v_d v_d \\ x_1^2 & x_1 x_2, & x_2^2, & x_1 x_3, & \dots & x_d^2 \end{array}$$

- The *formal character* of $\mathrm{Sym}^2 V$ is

$$s_{(2)}(x_1, \dots, x_d) = x_1^2 + x_1 x_2 + x_2^2 + x_1 x_3 + \cdots + x_d^2.$$

  Formal characters are symmetric polynomials.

# Plethysm: Symmetric polynomials.

Suppose $\dim V = d$.

- A basis of weight vectors for $\mathrm{Sym}^2 V$ is

$$\begin{array}{cccccc} v_1 v_1, & v_1 v_2, & v_2 v_2, & v_1 v_3, & \dots & v_d v_d \\ x_1^2 & x_1 x_2, & x_2^2, & x_1 x_3, & \dots & x_d^2 \end{array}$$

- The *formal character* of $\mathrm{Sym}^2 V$ is

$$s_{(2)}(x_1, \dots, x_d) = x_1^2 + x_1 x_2 + x_2^2 + x_1 x_3 + \cdots + x_d^2.$$

  Formal characters are symmetric polynomials.

- The formal character $h$ of $\mathrm{Sym}^2(\mathrm{Sym}^2 V)$ is obtained by evaluating $s_{(2)}$ at the monomials $x_1^2, x_1 x_2, \dots$

$$\begin{array}{ccccc} (v_1 v_1)(v_1 v_1), & (v_1 v_1)(v_1 v_2), & (v_1 v_2)(v_1 v_2), & (v_1 v_1)(v_2 v_2), & \dots \\ x_1^2 x_1^2 & x_1^2 x_1 x_2, & x_1 x_2 x_1 x_2 & x_1^2 x_2^2, & \dots \end{array}$$

# Plethysm: Symmetric polynomials.

Suppose $\dim V = d$.

- A basis of weight vectors for $\mathrm{Sym}^2 V$ is

$$\begin{array}{cccccc} v_1 v_1, & v_1 v_2, & v_2 v_2, & v_1 v_3, & \dots & v_d v_d \\ x_1^2 & x_1 x_2, & x_2^2, & x_1 x_3, & \dots & x_d^2 \end{array}$$

- The *formal character* of $\mathrm{Sym}^2 V$ is

$$s_{(2)}(x_1, \dots, x_d) = x_1^2 + x_1 x_2 + x_2^2 + x_1 x_3 + \cdots + x_d^2.$$

  Formal characters are symmetric polynomials.

- The formal character $h$ of $\mathrm{Sym}^2(\mathrm{Sym}^2 V)$ is obtained by evaluating $s_{(2)}$ at the monomials $x_1^2, x_1 x_2, \dots$

$$\begin{array}{ccccc} (v_1 v_1)(v_1 v_1), & (v_1 v_1)(v_1 v_2), & (v_1 v_2)(v_1 v_2), & (v_1 v_1)(v_2 v_2), & \dots \\ x_1^2 x_1^2 & x_1^2 x_1 x_2, & x_1 x_2 x_1 x_2 & x_1^2 x_2^2, & \dots \end{array}$$

$$\begin{aligned} h(x_1, \dots, x_d) &= x_1^4 + x_1^3 x_2 + 2x_1^2 x_2^2 + 2x_1^2 x_2 x_3 + 3x_1 x_2 x_3 x_4 \\ &= \left(x_1^4 + x_1^3 x_2 + x_1^2 x_2^2 + x_1^2 x_2 x_3 + x_1 x_2 x_3 x_4 + \cdots\right) \\ &\qquad + \left(x_1^2 x_2^2 + x_1^2 x_2 x_3 + 2x_1 x_2 x_3 x_4 + \cdots\right) \\ &= s_{(4)}(x_1, \dots, x_d) + s_{(2,2)}(x_1, \dots, x_d) \end{aligned}$$

# $\pi\lambda\eta\theta\upsilon\sigma\mu o\sigma$: Stanley's Problem 9

Let $f$ and $g$ be symmetric polynomials. Assume $g$ has coefficients in $\mathbb{N}_0$ when expressed in the monomial basis. The *plethysm* $f \circ g$ is defined by evaluating $f$ at the monomials of $g$.

- The formal character of $\Delta^\nu(\Delta^\mu V)$ is $s_\nu \circ s_\mu$.
- The corresponding character of $S_{mn}$ is

$$\Big(\widetilde{(\chi^\mu)^{\times n}} \mathrm{Inf}_{S_n}^{S_m \wr S_n} \chi^\nu\Big)\Big\uparrow_{S_m \wr S_n}^{S_{mn}}$$

# $\pi\lambda\eta\theta\upsilon\sigma\mu o\sigma$: Stanley's Problem 9

Let $f$ and $g$ be symmetric polynomials. Assume $g$ has coefficients in $\mathbb{N}_0$ when expressed in the monomial basis. The *plethysm* $f \circ g$ is defined by evaluating $f$ at the monomials of $g$.

- The formal character of $\Delta^\nu(\Delta^\mu V)$ is $s_\nu \circ s_\mu$.
- The corresponding character of $S_{mn}$ is

$$\left(\widetilde{(\chi^\mu)^{\times n}} \operatorname{Inf}_{S_n}^{S_m \wr S_n} \chi^\nu\right)\Big\uparrow_{S_m \wr S_n}^{S_{mn}}$$

## Problem (Weak Foulkes' Conjecture)

*Show that if $m \leq n$ then $s_{(n)} \circ s_{(m)} - s_{(m)} \circ s_{(n)}$ has non-negative coefficients.*

Equivalently, $S_m \wr S_n$ has at least as many orbits as $S_n \wr S_m$ on the coset space $S_{mn}/S_{\lambda_1} \times S_{\lambda_2} \times \cdots$, for each $\lambda \in \mathrm{Par}(mn)$.

# $\pi\lambda\eta\theta\upsilon\sigma\mu o\sigma$: Stanley's Problem 9

Let $f$ and $g$ be symmetric polynomials. Assume $g$ has coefficients in $\mathbb{N}_0$ when expressed in the monomial basis. The *plethysm* $f \circ g$ is defined by evaluating $f$ at the monomials of $g$.

- The formal character of $\Delta^\nu(\Delta^\mu V)$ is $s_\nu \circ s_\mu$.
- The corresponding character of $S_{mn}$ is

$$\big(\widetilde{(\chi^\mu)^{\times n}} \mathrm{Inf}_{S_n}^{S_m \wr S_n} \chi^\nu\big)\big\uparrow_{S_m \wr S_n}^{S_{mn}}$$

## Problem (Weak Foulkes' Conjecture)

*Show that if $m \leq n$ then $s_{(n)} \circ s_{(m)} - s_{(m)} \circ s_{(n)}$ has non-negative coefficients.*

Equivalently, $S_m \wr S_n$ has at least as many orbits as $S_n \wr S_m$ on the coset space $S_{mn}/S_{\lambda_1} \times S_{\lambda_2} \times \cdots$, for each $\lambda \in \mathrm{Par}(mn)$.

## Problem (Stanley, 2000)

*Let $\mu \in \mathrm{Par}(m)$, $\nu \in \mathrm{Par}(n)$, $\lambda \in \mathrm{Par}(mn)$. Find a combinatorial interpretation of the coefficient of $s_\lambda$ in $s_\nu \circ s_\mu$.*

# Plethysms and enumeration

## Theorem (Read 1959)

$\langle s_{(2m)} \circ s_{(3)}, s_{(3m)} \circ s_{(2)} \rangle$ *is the number of* 3*-regular graphs (with loops and multiple edges permitted) on* $2m$ *vertices.*

# Plethysms and enumeration

## Theorem (Read 1959)

*$\langle s_{(2m)} \circ s_{(3)}, s_{(3m)} \circ s_{(2)} \rangle$ is the number of 3-regular graphs (with loops and multiple edges permitted) on $2m$ vertices.*

Why on earth should this be true?

# Plethysms and enumeration

## Theorem (Read 1959)

$\langle s_{(2m)} \circ s_{(3)}, s_{(3m)} \circ s_{(2)} \rangle$ *is the number of* 3*-regular graphs (with loops and multiple edges permitted) on* $2m$ *vertices.*

Why on earth should this be true?

- $s_{(2m)} \circ s_{(3)}$ is the cycle index of $S_{6m}$ acting on $\Omega^{(3^{2m})}$.
- $s_{(3m)} \circ s_{(2)}$ is the cycle index of $S_{6m}$ acting on $\Omega^{(2^{3m})}$.

# Plethysms and enumeration

## Theorem (Read 1959)

$\langle s_{(2m)} \circ s_{(3)}, s_{(3m)} \circ s_{(2)} \rangle$ *is the number of* 3*-regular graphs (with loops and multiple edges permitted) on* $2m$ *vertices.*

Why on earth should this be true?

- $s_{(2m)} \circ s_{(3)}$ is the cycle index of $S_{6m}$ acting on $\Omega^{(3^{2m})}$.
- $s_{(3m)} \circ s_{(2)}$ is the cycle index of $S_{6m}$ acting on $\Omega^{(2^{3m})}$.
- Their inner product is the number of orbits of $S_{6m}$ on $\Omega^{(3^{2m})} \times \Omega^{(2^{3m})} \ldots$

# Plethysms and enumeration

## Theorem (Read 1959)

$\langle s_{(2m)} \circ s_{(3)}, s_{(3m)} \circ s_{(2)} \rangle$ *is the number of* 3*-regular graphs (with loops and multiple edges permitted) on* $2m$ *vertices.*

Why on earth should this be true?

- $s_{(2m)} \circ s_{(3)}$ is the cycle index of $S_{6m}$ acting on $\Omega^{(3^{2m})}$.
- $s_{(3m)} \circ s_{(2)}$ is the cycle index of $S_{6m}$ acting on $\Omega^{(2^{3m})}$.
- Their inner product is the number of orbits of $S_{6m}$ on $\Omega^{(3^{2m})} \times \Omega^{(2^{3m})}$ ...
- ...which is the number of orbits of $S_3 \wr S_{2m}$ on $\Omega^{(2^{3m})}$ ...

# Plethysms and enumeration

## Theorem (Read 1959)

*$\langle s_{(2m)} \circ s_{(3)}, s_{(3m)} \circ s_{(2)} \rangle$ is the number of 3-regular graphs (with loops and multiple edges permitted) on $2m$ vertices.*

Why on earth should this be true?

- $s_{(2m)} \circ s_{(3)}$ is the cycle index of $S_{6m}$ acting on $\Omega^{(3^{2m})}$.
- $s_{(3m)} \circ s_{(2)}$ is the cycle index of $S_{6m}$ acting on $\Omega^{(2^{3m})}$.
- Their inner product is the number of orbits of $S_{6m}$ on $\Omega^{(3^{2m})} \times \Omega^{(2^{3m})}$ ...
- ... which is the number of orbits of $S_3 \wr S_{2m}$ on $\Omega^{(2^{3m})}$ ...
- ... which is the number of 3-regular graphs:

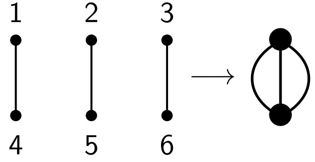
# Plethysms and enumeration

## Theorem (Read 1959)

*$\langle s_{(2m)} \circ s_{(3)}, s_{(3m)} \circ s_{(2)} \rangle$ is the number of 3-regular graphs (with loops and multiple edges permitted) on $2m$ vertices.*

Why on earth should this be true?

- $s_{(2m)} \circ s_{(3)}$ is the cycle index of $S_{6m}$ acting on $\Omega^{(3^{2m})}$.
- $s_{(3m)} \circ s_{(2)}$ is the cycle index of $S_{6m}$ acting on $\Omega^{(2^{3m})}$.
- Their inner product is the number of orbits of $S_{6m}$ on $\Omega^{(3^{2m})} \times \Omega^{(2^{3m})}$ ...
- ...which is the number of orbits of $S_3 \wr S_{2m}$ on $\Omega^{(2^{3m})}$ ...
- ...which is the number of 3-regular graphs:



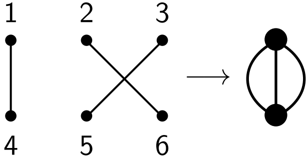
# Plethysms and enumeration

## Theorem (Read 1959)

*$\langle s_{(2m)} \circ s_{(3)}, s_{(3m)} \circ s_{(2)} \rangle$ is the number of 3-regular graphs (with loops and multiple edges permitted) on $2m$ vertices.*

Why on earth should this be true?

- $s_{(2m)} \circ s_{(3)}$ is the cycle index of $S_{6m}$ acting on $\Omega^{(3^{2m})}$.
- $s_{(3m)} \circ s_{(2)}$ is the cycle index of $S_{6m}$ acting on $\Omega^{(2^{3m})}$.
- Their inner product is the number of orbits of $S_{6m}$ on $\Omega^{(3^{2m})} \times \Omega^{(2^{3m})}$ ...
- ...which is the number of orbits of $S_3 \wr S_{2m}$ on $\Omega^{(2^{3m})}$ ...
- ...which is the number of 3-regular graphs:



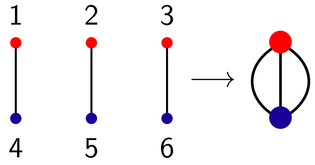
# Plethysms and enumeration

## Theorem (Read 1959)

*$\langle s_{(2m)} \circ s_{(3)}, s_{(3m)} \circ s_{(2)} \rangle$ is the number of 3-regular graphs (with loops and multiple edges permitted) on $2m$ vertices.*

Why on earth should this be true?

- $s_{(2m)} \circ s_{(3)}$ is the cycle index of $S_{6m}$ acting on $\Omega^{(3^{2m})}$.
- $s_{(3m)} \circ s_{(2)}$ is the cycle index of $S_{6m}$ acting on $\Omega^{(2^{3m})}$.
- Their inner product is the number of orbits of $S_{6m}$ on $\Omega^{(3^{2m})} \times \Omega^{(2^{3m})}$ ...
- ... which is the number of orbits of $S_3 \wr S_{2m}$ on $\Omega^{(2^{3m})}$ ...
- ... which is the number of 3-regular graphs:



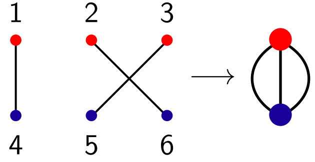
# Plethysms and enumeration

## Theorem (Read 1959)

*$\langle s_{(2m)} \circ s_{(3)}, s_{(3m)} \circ s_{(2)} \rangle$ is the number of 3-regular graphs (with loops and multiple edges permitted) on $2m$ vertices.*

Why on earth should this be true?

- $s_{(2m)} \circ s_{(3)}$ is the cycle index of $S_{6m}$ acting on $\Omega^{(3^{2m})}$.
- $s_{(3m)} \circ s_{(2)}$ is the cycle index of $S_{6m}$ acting on $\Omega^{(2^{3m})}$.
- Their inner product is the number of orbits of $S_{6m}$ on $\Omega^{(3^{2m})} \times \Omega^{(2^{3m})}$ ...
- ...which is the number of orbits of $S_3 \wr S_{2m}$ on $\Omega^{(2^{3m})}$ ...
- ...which is the number of 3-regular graphs:



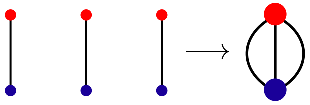
# Plethysms and enumeration

## Theorem (Read 1959)

*$\langle s_{(2m)} \circ s_{(3)}, s_{(3m)} \circ s_{(2)} \rangle$ is the number of 3-regular graphs (with loops and multiple edges permitted) on $2m$ vertices.*

Why on earth should this be true?

- $s_{(2m)} \circ s_{(3)}$ is the cycle index of $S_{6m}$ acting on $\Omega^{(3^{2m})}$.
- $s_{(3m)} \circ s_{(2)}$ is the cycle index of $S_{6m}$ acting on $\Omega^{(2^{3m})}$.
- Their inner product is the number of orbits of $S_{6m}$ on $\Omega^{(3^{2m})} \times \Omega^{(2^{3m})}$ ...
- ...which is the number of orbits of $S_3 \wr S_{2m}$ on $\Omega^{(2^{3m})}$ ...
- ...which is the number of 3-regular graphs:



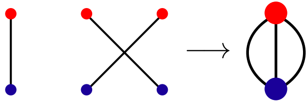
# Plethysms and enumeration

## Theorem (Read 1959)

$\langle s_{(2m)} \circ s_{(3)}, s_{(3m)} \circ s_{(2)} \rangle$ *is the number of* 3*-regular graphs (with loops and multiple edges permitted) on* $2m$ *vertices.*

Why on earth should this be true?

- $s_{(2m)} \circ s_{(3)}$ is the cycle index of $S_{6m}$ acting on $\Omega^{(3^{2m})}$.
- $s_{(3m)} \circ s_{(2)}$ is the cycle index of $S_{6m}$ acting on $\Omega^{(2^{3m})}$.
- Their inner product is the number of orbits of $S_{6m}$ on $\Omega^{(3^{2m})} \times \Omega^{(2^{3m})}$ ...
- ... which is the number of orbits of $S_3 \wr S_{2m}$ on $\Omega^{(2^{3m})}$ ...
- ... which is the number of 3-regular graphs:



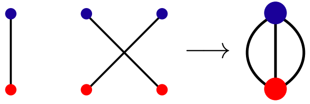
# Plethysms and enumeration

## Theorem (Read 1959)

*$\langle s_{(2m)} \circ s_{(3)}, s_{(3m)} \circ s_{(2)} \rangle$ is the number of 3-regular graphs (with loops and multiple edges permitted) on $2m$ vertices.*

Why on earth should this be true?

- $s_{(2m)} \circ s_{(3)}$ is the cycle index of $S_{6m}$ acting on $\Omega^{(3^{2m})}$.
- $s_{(3m)} \circ s_{(2)}$ is the cycle index of $S_{6m}$ acting on $\Omega^{(2^{3m})}$.
- Their inner product is the number of orbits of $S_{6m}$ on $\Omega^{(3^{2m})} \times \Omega^{(2^{3m})}$ ...
- ... which is the number of orbits of $S_3 \wr S_{2m}$ on $\Omega^{(2^{3m})}$ ...
- ... which is the number of 3-regular graphs:



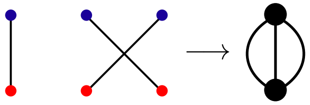
# Plethysms and enumeration

## Theorem (Read 1959)

*$\langle s_{(2m)} \circ s_{(3)}, s_{(3m)} \circ s_{(2)} \rangle$ is the number of 3-regular graphs (with loops and multiple edges permitted) on $2m$ vertices.*

Why on earth should this be true?

- $s_{(2m)} \circ s_{(3)}$ is the cycle index of $S_{6m}$ acting on $\Omega^{(3^{2m})}$.
- $s_{(3m)} \circ s_{(2)}$ is the cycle index of $S_{6m}$ acting on $\Omega^{(2^{3m})}$.
- Their inner product is the number of orbits of $S_{6m}$ on $\Omega^{(3^{2m})} \times \Omega^{(2^{3m})}$ ...
- ...which is the number of orbits of $S_3 \wr S_{2m}$ on $\Omega^{(2^{3m})}$ ...
- ...which is the number of 3-regular graphs:



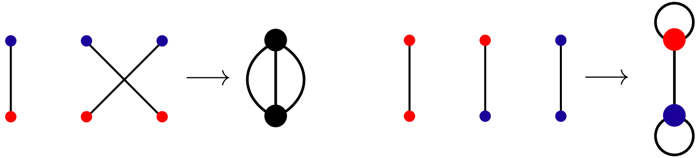
# Plethysms and enumeration

## Theorem (Read 1959)

*$\langle s_{(2m)} \circ s_{(3)}, s_{(3m)} \circ s_{(2)} \rangle$ is the number of 3-regular graphs (with loops and multiple edges permitted) on $2m$ vertices.*

Why on earth should this be true?

- $s_{(2m)} \circ s_{(3)}$ is the cycle index of $S_{6m}$ acting on $\Omega^{(3^{2m})}$.
- $s_{(3m)} \circ s_{(2)}$ is the cycle index of $S_{6m}$ acting on $\Omega^{(2^{3m})}$.
- Their inner product is the number of orbits of $S_{6m}$ on $\Omega^{(3^{2m})} \times \Omega^{(2^{3m})}$ ...
- ...which is the number of orbits of $S_3 \wr S_{2m}$ on $\Omega^{(2^{3m})}$ ...
- ...which is the number of 3-regular graphs:



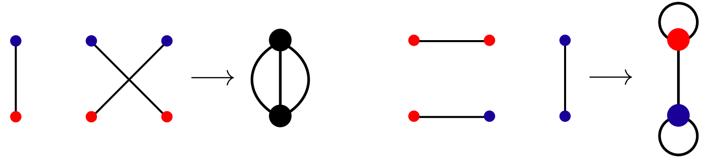
# Plethysms and enumeration

## Theorem (Read 1959)

$\langle s_{(2m)} \circ s_{(3)}, s_{(3m)} \circ s_{(2)} \rangle$ *is the number of* 3*-regular graphs (with loops and multiple edges permitted) on* $2m$ *vertices.*

Why on earth should this be true?

- $s_{(2m)} \circ s_{(3)}$ is the cycle index of $S_{6m}$ acting on $\Omega^{(3^{2m})}$.
- $s_{(3m)} \circ s_{(2)}$ is the cycle index of $S_{6m}$ acting on $\Omega^{(2^{3m})}$.
- Their inner product is the number of orbits of $S_{6m}$ on $\Omega^{(3^{2m})} \times \Omega^{(2^{3m})}$ ...
- ...which is the number of orbits of $S_3 \wr S_{2m}$ on $\Omega^{(2^{3m})}$ ...
- ...which is the number of 3-regular graphs:



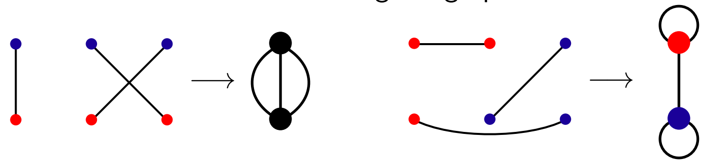
# Plethysms and enumeration

## Theorem (Read 1959)

*$\langle s_{(2m)} \circ s_{(3)}, s_{(3m)} \circ s_{(2)} \rangle$ is the number of 3-regular graphs (with loops and multiple edges permitted) on $2m$ vertices.*

Why on earth should this be true?

- $s_{(2m)} \circ s_{(3)}$ is the cycle index of $S_{6m}$ acting on $\Omega^{(3^{2m})}$.
- $s_{(3m)} \circ s_{(2)}$ is the cycle index of $S_{6m}$ acting on $\Omega^{(2^{3m})}$.
- Their inner product is the number of orbits of $S_{6m}$ on $\Omega^{(3^{2m})} \times \Omega^{(2^{3m})}$ ...
- ...which is the number of orbits of $S_3 \wr S_{2m}$ on $\Omega^{(2^{3m})}$ ...
- ...which is the number of 3-regular graphs:



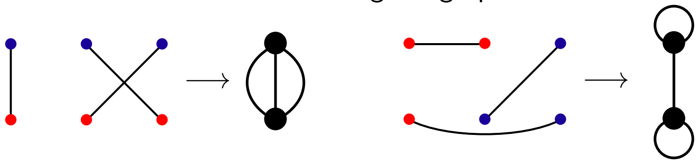
# Plethysms and enumeration

## Theorem (Read 1959)

*$\langle s_{(2m)} \circ s_{(3)}, s_{(3m)} \circ s_{(2)} \rangle$ is the number of 3-regular graphs (with loops and multiple edges permitted) on $2m$ vertices.*

Why on earth should this be true?

- $s_{(2m)} \circ s_{(3)}$ is the cycle index of $S_{6m}$ acting on $\Omega^{(3^{2m})}$.
- $s_{(3m)} \circ s_{(2)}$ is the cycle index of $S_{6m}$ acting on $\Omega^{(2^{3m})}$.
- Their inner product is the number of orbits of $S_{6m}$ on $\Omega^{(3^{2m})} \times \Omega^{(2^{3m})}$ ...
- ...which is the number of orbits of $S_3 \wr S_{2m}$ on $\Omega^{(2^{3m})}$ ...
- ...which is the number of 3-regular graphs:



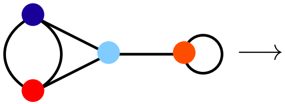
# Plethysms and enumeration

## Theorem (Read 1959)

$\langle s_{(2m)} \circ s_{(3)}, s_{(3m)} \circ s_{(2)} \rangle$ *is the number of 3-regular graphs (with loops and multiple edges permitted) on $2m$ vertices.*

Why on earth should this be true?

- $s_{(2m)} \circ s_{(3)}$ is the cycle index of $S_{6m}$ acting on $\Omega^{(3^{2m})}$.
- $s_{(3m)} \circ s_{(2)}$ is the cycle index of $S_{6m}$ acting on $\Omega^{(2^{3m})}$.
- Their inner product is the number of orbits of $S_{6m}$ on $\Omega^{(3^{2m})} \times \Omega^{(2^{3m})}$ …
- …which is the number of orbits of $S_3 \wr S_{2m}$ on $\Omega^{(2^{3m})}$ …
- …which is the number of 3-regular graphs:



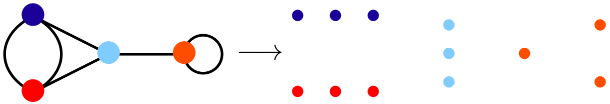
# Plethysms and enumeration

## Theorem (Read 1959)

*$\langle s_{(2m)} \circ s_{(3)}, s_{(3m)} \circ s_{(2)} \rangle$ is the number of 3-regular graphs (with loops and multiple edges permitted) on $2m$ vertices.*

Why on earth should this be true?

- $s_{(2m)} \circ s_{(3)}$ is the cycle index of $S_{6m}$ acting on $\Omega^{(3^{2m})}$.
- $s_{(3m)} \circ s_{(2)}$ is the cycle index of $S_{6m}$ acting on $\Omega^{(2^{3m})}$.
- Their inner product is the number of orbits of $S_{6m}$ on $\Omega^{(3^{2m})} \times \Omega^{(2^{3m})}$ ...
- ...which is the number of orbits of $S_3 \wr S_{2m}$ on $\Omega^{(2^{3m})}$ ...
- ...which is the number of 3-regular graphs:



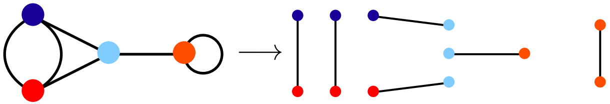
# Plethysms and enumeration

## Theorem (Read 1959)

*$\langle s_{(2m)} \circ s_{(3)}, s_{(3m)} \circ s_{(2)} \rangle$ is the number of 3-regular graphs (with loops and multiple edges permitted) on $2m$ vertices.*

Why on earth should this be true?

- $s_{(2m)} \circ s_{(3)}$ is the cycle index of $S_{6m}$ acting on $\Omega^{(3^{2m})}$.
- $s_{(3m)} \circ s_{(2)}$ is the cycle index of $S_{6m}$ acting on $\Omega^{(2^{3m})}$.
- Their inner product is the number of orbits of $S_{6m}$ on $\Omega^{(3^{2m})} \times \Omega^{(2^{3m})}$ ...
- ...which is the number of orbits of $S_3 \wr S_{2m}$ on $\Omega^{(2^{3m})}$ ...
- ...which is the number of 3-regular graphs:



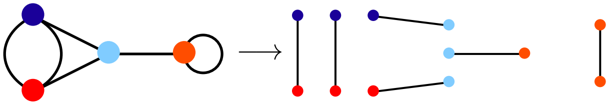
# Plethysms and enumeration

## Theorem (Read 1959)

*$\langle s_{(2m)} \circ s_{(3)}, s_{(3m)} \circ s_{(2)} \rangle$ is the number of 3-regular graphs (with loops and multiple edges permitted) on $2m$ vertices.*

Why on earth should this be true?

- $s_{(2m)} \circ s_{(3)}$ is the cycle index of $S_{6m}$ acting on $\Omega^{(3^{2m})}$.
- $s_{(3m)} \circ s_{(2)}$ is the cycle index of $S_{6m}$ acting on $\Omega^{(2^{3m})}$.
- Their inner product is the number of orbits of $S_{6m}$ on $\Omega^{(3^{2m})} \times \Omega^{(2^{3m})}$ ...
- ... which is the number of orbits of $S_3 \wr S_{2m}$ on $\Omega^{(2^{3m})}$ ...
- ... which is the number of 3-regular graphs:



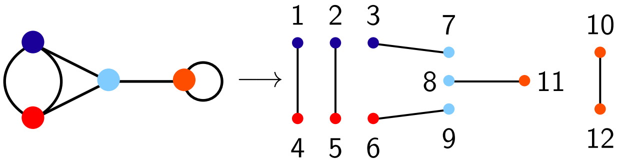
# Plethysms and enumeration

## Theorem (Read 1959)

*$\langle s_{(2m)} \circ s_{(3)}, s_{(3m)} \circ s_{(2)} \rangle$ is the number of 3-regular graphs (with loops and multiple edges permitted) on $2m$ vertices.*

Why on earth should this be true?

- $s_{(2m)} \circ s_{(3)}$ is the cycle index of $S_{6m}$ acting on $\Omega^{(3^{2m})}$.
- $s_{(3m)} \circ s_{(2)}$ is the cycle index of $S_{6m}$ acting on $\Omega^{(2^{3m})}$.
- Their inner product is the number of orbits of $S_{6m}$ on $\Omega^{(3^{2m})} \times \Omega^{(2^{3m})}$ ...
- ...which is the number of orbits of $S_3 \wr S_{2m}$ on $\Omega^{(2^{3m})}$ ...
- ...which is the number of 3-regular graphs:



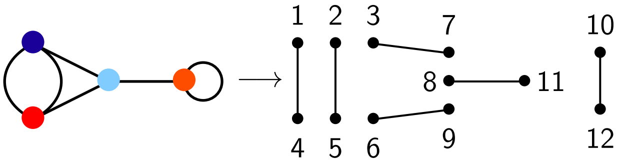
# Plethysms and enumeration

## Theorem (Read 1959)

*$\langle s_{(2m)} \circ s_{(3)}, s_{(3m)} \circ s_{(2)} \rangle$ is the number of 3-regular graphs (with loops and multiple edges permitted) on $2m$ vertices.*

Why on earth should this be true?

- $s_{(2m)} \circ s_{(3)}$ is the cycle index of $S_{6m}$ acting on $\Omega^{(3^{2m})}$.
- $s_{(3m)} \circ s_{(2)}$ is the cycle index of $S_{6m}$ acting on $\Omega^{(2^{3m})}$.
- Their inner product is the number of orbits of $S_{6m}$ on $\Omega^{(3^{2m})} \times \Omega^{(2^{3m})}$ ...
- ...which is the number of orbits of $S_3 \wr S_{2m}$ on $\Omega^{(2^{3m})}$ ...
- ...which is the number of 3-regular graphs:



# §2: Minimal and maximal constituents of plethysms

Let $\lambda, \lambda^\star \in \mathrm{Par}(r)$. We say $\lambda$ *dominates* $\lambda^\star$, and write $\lambda \trianglerighteq \lambda^\star$, if

$$\lambda_1 + \cdots + \lambda_j \geq \lambda^\star_1 + \cdots + \lambda^\star_j.$$

for all $j$. For example

- $(4, 2, 2) \trianglerighteq (3, 3, 1, 1)$,

# §2: Minimal and maximal constituents of plethysms

Let $\lambda, \lambda^\star \in \mathrm{Par}(r)$. We say $\lambda$ *dominates* $\lambda^\star$, and write $\lambda \trianglerighteq \lambda^\star$, if

$$\lambda_1 + \cdots + \lambda_j \geq \lambda^\star_1 + \cdots + \lambda^\star_j.$$

for all $j$. For example

- $(4, 2, 2) \trianglerighteq (3, 3, 1, 1)$,
- $(4, 1, 1)$ and $(3, 3)$ are incomparable.

# §2: Minimal and maximal constituents of plethysms

Let $\lambda, \lambda^\star \in \mathrm{Par}(r)$. We say $\lambda$ *dominates* $\lambda^\star$, and write $\lambda \trianglerighteq \lambda^\star$, if

$$\lambda_1 + \cdots + \lambda_j \geq \lambda^\star_1 + \cdots + \lambda^\star_j.$$

for all $j$. For example

- $(4, 2, 2) \trianglerighteq (3, 3, 1, 1)$,
- $(4, 1, 1)$ and $(3, 3)$ are incomparable.

Quiz: Choose partitions $\lambda$ and $\lambda^\star$ of $n$ (a large number) uniformly at random. What is the chance that $\lambda$ and $\lambda^\star$ are comparable?

# §2: Minimal and maximal constituents of plethysms

Let $\lambda, \lambda^\star \in \mathrm{Par}(r)$. We say $\lambda$ *dominates* $\lambda^\star$, and write $\lambda \trianglerighteq \lambda^\star$, if

$$\lambda_1 + \cdots + \lambda_j \geq \lambda^\star_1 + \cdots + \lambda^\star_j.$$

for all $j$. For example

- $(4, 2, 2) \trianglerighteq (3, 3, 1, 1)$,
- $(4, 1, 1)$ and $(3, 3)$ are incomparable.

Quiz: Choose partitions $\lambda$ and $\lambda^\star$ of $n$ (a large number) uniformly at random. What is the chance that $\lambda$ and $\lambda^\star$ are comparable?

Our main theorem gives a combinatorial characterization of all maximal and minimal partitions $\lambda$ in the dominance order on $\mathrm{Par}(mn)$ such that $s_\lambda$ has non-zero coefficient in $s_\nu \circ s_\mu$.

# §2: Minimal and maximal constituents of plethysms

Let $\lambda, \lambda^\star \in \mathrm{Par}(r)$. We say $\lambda$ *dominates* $\lambda^\star$, and write $\lambda \trianglerighteq \lambda^\star$, if

$$\lambda_1 + \cdots + \lambda_j \geq \lambda^\star_1 + \cdots + \lambda^\star_j.$$

for all $j$. For example

- $(4,2,2) \trianglerighteq (3,3,1,1)$,
- $(4,1,1)$ and $(3,3)$ are incomparable.

Quiz: Choose partitions $\lambda$ and $\lambda^\star$ of $n$ (a large number) uniformly at random. What is the chance that $\lambda$ and $\lambda^\star$ are comparable?

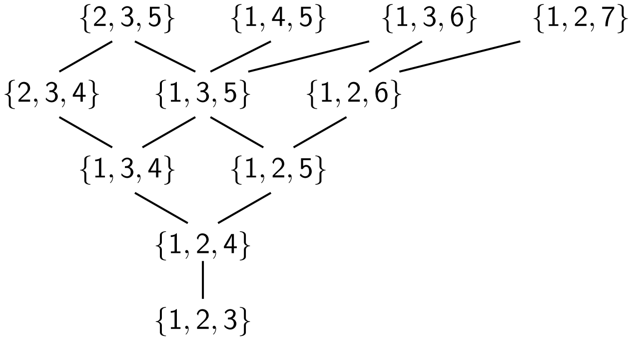
Our main theorem gives a combinatorial characterization of all maximal and minimal partitions $\lambda$ in the dominance order on $\mathrm{Par}(mn)$ such that $s_\lambda$ has non-zero coefficient in $s_\nu \circ s_\mu$.

This solves a special case of Stanley's Problem 9.

# Special case $\mu = (m)$ for minimals

Let $A = \{a_1, \ldots, a_m\}$ and $B = \{b_1, \ldots, b_m\}$ be $m$-subsets of $\mathbb{N}$, written so that $a_1 < \ldots < a_m$ and $b_1 < \ldots < b_m$. We say that $A$ *majorizes* $B$, and write $A \preceq B$, if

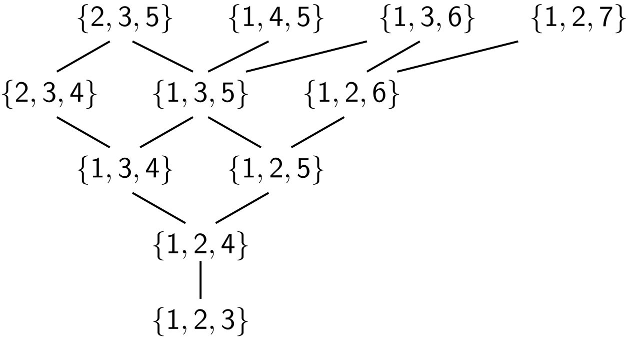
$$a_1 \leq b_1, \ldots, a_m \leq b_m.$$



# Special case $\mu = (m)$ for minimals

Let $A = \{a_1, \ldots, a_m\}$ and $B = \{b_1, \ldots, b_m\}$ be $m$-subsets of $\mathbb{N}$, written so that $a_1 < \ldots < a_m$ and $b_1 < \ldots < b_m$. We say that $A$ *majorizes* $B$, and write $A \preceq B$, if

$$a_1 \leq b_1, \ldots, a_m \leq b_m.$$

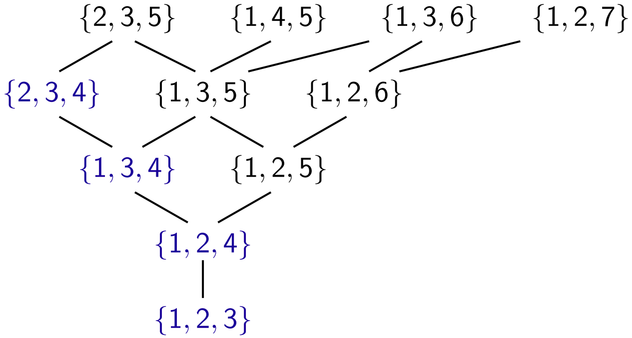



- A *closed set family of size* $r$ is a family $\mathcal{P}$ of $m$-subsets of $\mathbb{N}$ such that $|\mathcal{P}| = r$ and if $B \in \mathcal{P}$ and $A \preceq B$ then $A \in \mathcal{P}$.

# Special case $\mu = (m)$ for minimals

Let $A = \{a_1, \ldots, a_m\}$ and $B = \{b_1, \ldots, b_m\}$ be $m$-subsets of $\mathbb{N}$, written so that $a_1 < \ldots < a_m$ and $b_1 < \ldots < b_m$. We say that $A$ *majorizes* $B$, and write $A \preceq B$, if

$$a_1 \leq b_1, \ldots, a_m \leq b_m.$$

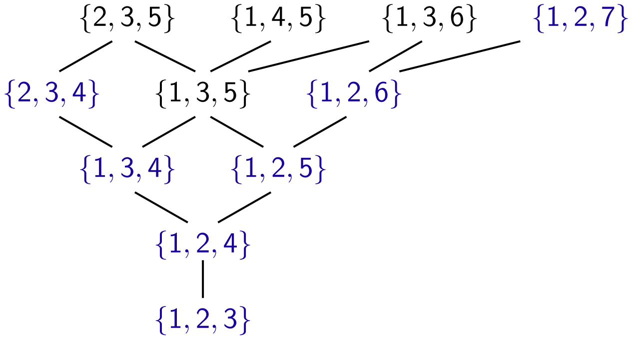



- A *closed set family of size* $r$ is a family $\mathcal{P}$ of $m$-subsets of $\mathbb{N}$ such that $|\mathcal{P}| = r$ and if $B \in \mathcal{P}$ and $A \preceq B$ then $A \in \mathcal{P}$.

# Special case $\mu = (m)$ for minimals

Let $A = \{a_1, \ldots, a_m\}$ and $B = \{b_1, \ldots, b_m\}$ be $m$-subsets of $\mathbb{N}$, written so that $a_1 < \ldots < a_m$ and $b_1 < \ldots < b_m$. We say that $A$ *majorizes* $B$, and write $A \preceq B$, if

$$a_1 \leq b_1, \ldots, a_m \leq b_m.$$




- A *closed set family of size $r$* is a family $\mathcal{P}$ of $m$-subsets of $\mathbb{N}$ such that $|\mathcal{P}| = r$ and if $B \in \mathcal{P}$ and $A \preceq B$ then $A \in \mathcal{P}$.

# Special case $\mu = (m)$ for minimals

- A *closed set family of size* $r$ is a family $\mathcal{P}$ of $m$-subsets of $\mathbb{N}$ such that $|\mathcal{P}| = r$ and if $B \in \mathcal{P}$ and $A \preceq B$ then $A \in \mathcal{P}$.
- A *closed set family tuple of size* $\nu$ is a tuple $(\mathcal{P}_1, \ldots, \mathcal{P}_e)$ where $\mathcal{P}_j$ is a closed set family of size $\nu_j$ for each $j$.

# Special case $\mu = (m)$ for minimals

- A *closed set family of size* $r$ is a family $\mathcal{P}$ of $m$-subsets of $\mathbb{N}$ such that $|\mathcal{P}| = r$ and if $B \in \mathcal{P}$ and $A \preceq B$ then $A \in \mathcal{P}$.
- A *closed set family tuple of size* $\nu$ is a tuple $(\mathcal{P}_1, \ldots, \mathcal{P}_e)$ where $\mathcal{P}_j$ is a closed set family of size $\nu_j$ for each $j$.
- The *weight* of $(\mathcal{P}_1, \ldots, \mathcal{P}_e)$ is the partition $\lambda$ such that each $i \in \mathbb{N}$ appears in exactly $\lambda_i$ sets in the $\mathcal{P}_j$.

# Special case $\mu = (m)$ for minimals

- A *closed set family of size* $r$ is a family $\mathcal{P}$ of $m$-subsets of $\mathbb{N}$ such that $|\mathcal{P}| = r$ and if $B \in \mathcal{P}$ and $A \preceq B$ then $A \in \mathcal{P}$.
- A *closed set family tuple of size* $\nu$ is a tuple $(\mathcal{P}_1, \ldots, \mathcal{P}_e)$ where $\mathcal{P}_j$ is a closed set family of size $\nu_j$ for each $j$.
- The *weight* of $(\mathcal{P}_1, \ldots, \mathcal{P}_e)$ is the partition $\lambda$ such that each $i \in \mathbb{N}$ appears in exactly $\lambda_i$ sets in the $\mathcal{P}_j$.
- The *type* of $(\mathcal{P}_1, \ldots, \mathcal{P}_e)$ is the conjugate partition $\lambda'$.
- For example,

$$\Big(\big\{\{1,2,3\},\{1,2,4\},\{1,3,4\}\big\},\big\{\{1,2,3\}\big\}\Big)$$

is a closed set family tuple of size $(3,1)$, weight $(4,3,3,2)$ and type $(4,4,3,1)$.

# Special case $\mu = (m)$ for minimals

- A *closed set family of size $r$* is a family $\mathcal{P}$ of $m$-subsets of $\mathbb{N}$ such that $|\mathcal{P}| = r$ and if $B \in \mathcal{P}$ and $A \preceq B$ then $A \in \mathcal{P}$.
- A *closed set family tuple of size $\nu$* is a tuple $(\mathcal{P}_1, \ldots, \mathcal{P}_e)$ where $\mathcal{P}_j$ is a closed set family of size $\nu_j$ for each $j$.
- The *weight* of $(\mathcal{P}_1, \ldots, \mathcal{P}_e)$ is the partition $\lambda$ such that each $i \in \mathbb{N}$ appears in exactly $\lambda_i$ sets in the $\mathcal{P}_j$.
- The *type* of $(\mathcal{P}_1, \ldots, \mathcal{P}_e)$ is the conjugate partition $\lambda'$.
- For example,

$$\Big(\big\{\{1,2,3\},\{1,2,4\},\{1,3,4\}\big\},\big\{\{1,2,3\}\big\}\Big)$$

  is a closed set family tuple of size $(3,1)$, weight $(4,3,3,2)$ and type $(4,4,3,1)$.

### Theorem (Paget, MW, 2014)

*Let $m$ be odd. The minimal partitions $\lambda$ such that $s_\lambda$ has non-zero coefficient in $s_\nu \circ s_{(m)}$ are precisely the minimal types of the closed set family tuples of size $\nu$.*

# Special case $\nu = (n)$ for minimals

- A $\mu$-tableau is *conjugate-semistandard* if its rows are strictly increasing and its columns are non-decreasing. When $\mu = (m)$ such tableaux correspond to $m$-subsets: $\{1,3,4\} \leftrightarrow \boxed{1 \mid 3 \mid 4}$.
- The majorization order generalizes to a partial order on conjugate-semistandard $\mu$-tableaux.
- We define closed $\mu$-tableau families and their weights and types analogously.

# Special case $\nu = (n)$ for minimals

- A $\mu$-tableau is *conjugate-semistandard* if its rows are strictly increasing and its columns are non-decreasing. When $\mu = (m)$ such tableaux correspond to $m$-subsets: $\{1, 3, 4\} \leftrightarrow \boxed{1 \mid 3 \mid 4}$.
- The majorization order generalizes to a partial order on conjugate-semistandard $\mu$-tableaux.
- We define closed $\mu$-tableau families and their weights and types analogously. For example

$$\left\{ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}, \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}, \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \right\}$$

is a closed $(2, 1)$-tableau family of size 3, weight $(5, 3, 1)$ and type $(3, 2, 2, 1, 1)$.

## Special case $\nu = (n)$ for minimals

- A $\mu$-tableau is *conjugate-semistandard* if its rows are strictly increasing and its columns are non-decreasing. When $\mu = (m)$ such tableaux correspond to $m$-subsets: $\{1,3,4\} \leftrightarrow \boxed{1\,|\,3\,|\,4}$.
- The majorization order generalizes to a partial order on conjugate-semistandard $\mu$-tableaux.
- We define closed $\mu$-tableau families and their weights and types analogously. For example

$$\left\{ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}\, , \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}\, , \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \right\}$$

  is a closed $(2,1)$-tableau family of size 3, weight $(5,3,1)$ and type $(3,2,2,1,1)$.

### Theorem (Paget, MW, 2016)

*Let $m$ be odd and let $\mu \in \mathrm{Par}(n)$. The minimal partitions $\lambda$ such that $s_\lambda$ has non-zero coefficient in $s_{(n)} \circ s_\mu$ are precisely the minimal types of the closed $\mu$-tableau families of size $n$.*

This determines all minimal $\lambda$ such that $\Delta^\lambda V$ appears in the coordinate ring of $\Delta^\mu V$.

# Application to invariants of Riemann curvature tensor

mathoverflow

Questions Tags Users Badges Ur

## A question on invariant theory of $GL_n(\mathbb{C})$.

12

1

Let $\rho$ denote the irreducible algebraic representation of $GL_n(\mathbb{C})$ with the highest weight $(2, 2, \underbrace{0, \ldots, 0}_{n-2})$.

Let $k \leq n/2$ be a non-negative integer. How to decompose into irreducible representations the representation $Sym^k(\rho)$?

More specifically, I am interested whether $Sym^k(\rho)$ contains the representation with the highest weight $(\underbrace{2, \ldots, 2}_{2k}, \underbrace{0, \ldots, 0}_{n-2k})$, and if yes, whether the mutiplicity is equal to one.

A a side remark, the representation $\rho$ has a geometric interpretation important for me: it is the space of curvature tensors, namely the curvature tensor of any Riemannian metric on $\mathbb{R}^n$ lies in $\rho$.

invariant-theory classical-invariant-theor dg.differential-geometry rt.representation-theory plethysm

share cite edit close flag

edited Oct 3 '12 at 19:28

asked Oct 3 '12 at 17:31

sva

4,239 • 18 • 43

# Application to invariants of Riemann curvature tensor

14

The plethysm $\mathrm{Sym}^k\rho$ contains the irreducible representation with highest weight $(2,\ldots,2,0,\ldots,0)$ exactly once. It looks like a tricky problem to say much about its other irreducible constituents.

Let $\Delta^\lambda$ denote the Schur functor corresponding to the partition $\lambda$, and let $E$ be an $n$-dimensional complex vector space. Using symmetric polynomials (or other methods) one finds

$$\mathrm{Sym}^2(\mathrm{Sym}^2 E) = \Delta^{(2,2)}E \oplus \mathrm{Sym}^4 E.$$

Therefore

$$\mathrm{Sym}^k\mathrm{Sym}^2\mathrm{Sym}^2 E \cong \sum_{r=0}^{k} \mathrm{Sym}^r(\Delta^{(2,2)}E) \otimes \mathrm{Sym}^{k-r}(\mathrm{Sym}^4 E).$$

The irreducible representations contained in the $r$th summand are labelled by partitions with at most $2r+(k-r)=k+r$ parts. So to show that $\mathrm{Sym}^k(\Delta^{(2,2)}(E))$ contains $\Delta^{(2^{2k})}E$, it suffices to show that $\Delta^{(2^{2k})}E$ appears in $\mathrm{Sym}^k\mathrm{Sym}^2\mathrm{Sym}^2 E$.

Let $U=\mathrm{Sym}^2 E$. There is a canonical surjection

$$\mathrm{Sym}^k(\mathrm{Sym}^2 U) \to \mathrm{Sym}^{2k} U.$$

given by mapping $(u_1u_1')\ldots(u_ku_k') \in \mathrm{Sym}^k(\mathrm{Sym}^2 U)$ to $u_1u_1'\ldots u_ku_k' \in \mathrm{Sym}^{2k}U$. Therefore $\mathrm{Sym}^k(\mathrm{Sym}^2 U)$ contains $\mathrm{Sym}^{2k}U = \mathrm{Sym}^{2k}(\mathrm{Sym}^2 E)$. It is well known that

$$\mathrm{Sym}^{2k}(\mathrm{Sym}^2 E) = \sum_\lambda \Delta^{2\lambda}(E)$$

where the sum is over all partitions $\lambda$ of $2k$ and $2(\lambda_1,\ldots,\lambda_m)=(2\lambda_1,\ldots,2\lambda_m)$. Taking $\lambda=(1^{2k})$ we see that $\Delta^{(2^{2k})}E$ appears.

It remains to show that the multiplicity of $\Delta^{(2^{2k})}E$ in $\mathrm{Sym}^k(\Delta^{(2,2)}E)$ is $1$. We work over $\mathbb{C}$, so there is a chain of inclusions

$$\mathrm{Sym}^k(\Delta^{(2,2)}(E)) \subseteq \mathrm{Sym}^k(\mathrm{Sym}^2 E \otimes \mathrm{Sym}^2 E) \subseteq (\mathrm{Sym}^2 E)^{\otimes 2k}.$$

By the Littlewood–Richardson rule (or the easier Young's rule), the multiplicity of $\Delta^{(2^k)}E$ in the right-hand side is $1$.

share cite edit delete flag

answered Oct 4 '12 at 0:42
Mark Wildon
3,958 • 1 • 19 • 30

This is nice. – Dan Petersen Oct 4 '12 at 6:55