Vertices of Specht Modules

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Outline

1. Vertices and the Brauer Correspondence for Modules
2. Vertices of Specht Modules
3. Complexity of Modules and Two Results of K. J. Lim
Let \( G \) be a finite group. Let \( F \) be a field of prime characteristic \( p \). Let \( V \) be an indecomposable \( FG \)-module.

A subgroup \( P \leq G \) is said to be a vertex of \( V \) if there is an \( FG \)-module \( U \) such that \( V \mid U \uparrow^G_P \), and \( P \) is minimal with this property.
Let $G$ be a finite group. Let $F$ be a field of prime characteristic $p$. Let $V$ be an indecomposable $FG$-module.

A subgroup $P \leq G$ is said to be a vertex of $V$ is there is an $FG$-module $U$ such that $V \uparrow_{P}^{G}$, and $P$ is minimal with this property.

Green showed in 1959 that

(i) Vertices are $p$-subgroups of $G$;
(ii) If $P, Q \leq G$ are vertices of $V$ then $P^{x} = Q$ for some $x \in G$. 
Brauer Correspondence for Modules

Let $V^Q = \{ v \in V : vg = v \text{ for all } g \in Q \}$. Given $R \leq Q \leq G$ define the trace map $\operatorname{Tr}_R^Q : V^R \to V^Q$ by

$$\operatorname{Tr}_R^Q(v) = \sum_{i=1}^{m} vg_i$$

where $Q = Rg_1 \cup \ldots \cup Rg_m$. 

The Brauer correspondent of $V$ with respect to $Q$ is $V(Q) = V^Q \sum_{R < Q} \operatorname{Tr}_R^Q V^R$. 

The Brauer correspondent is a module for $N_G(Q)$. 

Theorem (Broué 1985) If $V(Q) \neq 0$ then $Q$ is contained in a vertex of $V$. 
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**Theorem (Broué 1985)**

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Brauer Correspondence for $p$-Permutation Modules

Let $P_{\text{max}}$ be a Sylow $p$-subgroup of $G$.

An $FG$-module $V$ is $p$-permutation if it has a basis $\mathcal{B} = \{v_1, \ldots, v_n\}$ such that $v_i g \in \mathcal{B}$ for all $g \in P_{\text{max}}$.

Remark: $V$ is an indecomposable $p$-permutation module if and only if $V \mid F \uparrow_P^G$ for some $P \leq G$.

Lemma
Suppose that $V$ is $p$-permutation with respect to the basis $\mathcal{B}$. If $Q \leq P_{\text{max}}$ then $V(Q) = \langle \mathcal{B}^Q \rangle_F$.

Theorem (Broué 1985)
Let $V$ be an indecomposable $p$-permutation $FG$-module. Then $V(Q) \neq 0 \iff Q$ is contained in a vertex of $V$. If $V$ has vertex $P$ then $V(P)$ is the Green correspondent of $V$. 

Theorem

Let $n \in \mathbb{N}$ and let $p$ be a prime such that $p \nmid n$. The vertex of $S^{(n-r,1^r)}$, defined over a field of characteristic $p$, is a Sylow $p$-subgroup of $S_{n-r-1} \times S_r$.

The proof uses a $p$-permutation basis for $S^{(n-r-1,1^r)}$. 

Application: In characteristic 2 Specht modules may be decomposable. I used this theorem to give a short proof of a theorem of Murphy (1980): if $n$ is odd and $2^{\ell-1} \leq r < 2^\ell$ then $S^{(n-r,1^r)}$ is decomposable, unless $n \equiv 2^r + 1 \mod 2^\ell$. 

Remark: Suppose that $S^{\lambda}$, defined over a field of characteristic $p$, is indecomposable with vertex $Q$. It follows from a theorem of Green (1960) that if $g$ is a $p$-element such that $\chi^{\lambda}(g) \neq 0$ then there exists $x \in G$ such that $g \in Qx$. 
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Remark: Suppose that \( S^\lambda \), defined over a field of characteristic \( p \), is indecomposable with vertex \( Q \). It follows from a theorem of Green (1960) that if \( g \) is a \( p \)-element such that \( \chi^\lambda(g) \neq 0 \) then there exists \( x \in G \) such that \( g \in Q^x \).
Open Problems

Problem

Find vertices of hook Specht modules $S^{(n-r,1^r)}$ over fields of characteristic $p \geq 3$ where $p \mid n$.

Solved when $p = 2$ by Murphy and Peel (1984).

Work is in progress with Susanne Danz and Karin Erdmann on $S^{(n-3,1,1,1)}$ in characteristic 3.

Problem

Clarify the relationship between character values on $p$-elements and vertices in characteristic $p$. 
Let $\lambda$ be a partition and let $t$ be a $\lambda$-tableau. Let $H(t)$ be the subgroup of the row stabilising group of $t$ that permutes, as blocks for its action, the entries of columns of $t$ of equal length.

For example if $\lambda = (8, 4, 1)$ and

$$
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
9 & 10 & 11 & 12 \\
13 \\
\end{array}
$$

then $H(t)$ is generated by

$$(2, 3, 4)(10, 11, 12), (2, 3)(10, 11), (5, 6, 7, 8), (5, 6).$$
A Subgroup Bound on Vertices

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then $H(t)$ is generated by

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**Theorem**

*If $S^\lambda$ is indecomposable then it has a vertex containing a Sylow $p$-subgroup of $H(t)$.***
We assume w.l.o.g. $t$ is the greatest tableau under $\triangleright$. Let $Q$ be a Sylow $p$-subgroup of $H(t)$.

(1) Show that $e_t$ is fixed by every permutation in $Q$. So for instance, we need

$$
\begin{array}{cccccccc}
  & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
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\end{array}
$$

to be fixed by $(2, 3, 4)(10, 11, 12)$ and $(5, 6, 7)$. 
Outline Proof

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$$
e_{12345678}
9101112
13$$

to be fixed by $(2,3,4)(10,11,12)$ and $(5,6,7)$.

(2) Then show that $e_t \notin \sum_{R \subset Q} \text{Tr}_R^Q(S^\lambda)^R$. Hence $S^\lambda(Q) \neq 0$, so by Broué’s theorem, $S^\lambda$ has a vertex containing $Q$. 

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For (2) it suffices to show that if $u$ is a $\lambda$-tableau and $g \in H(t)$ is a $p$-element then, when $e_u + e_ug + \cdots + e_u g^{p-1}$ is written as a linear combination of standard polytabloids, the coefficient of $e_t$ is 0.
§3 Complexity of Modules and Two Results of K. J. Lim

Definition
Let $G$ be a finite group and let $V$ and an $FG$-module. Let

\[ \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow V \]

be a minimal projective resolution of $V$. The complexity of $V$ is the least non-negative integer $c$ such that

\[
\lim_{n \to \infty} \frac{\dim_F P_n}{n^c} = 0.
\]

Theorem (Lim 2011, Theorem 3.2)
Suppose that the Specht module $S^\mu$, defined over a field of odd characteristic, has an abelian vertex. Let $m$ be the $p$-rank of $Q$. If $c$ is the complexity of $S^\mu$ and $w$ is the weight of $\mu$ then $c = w = m$ and $Q$ is conjugate to the elementary abelian subgroup

\[
\langle (1, \ldots, p) \rangle \times \cdots \times (wp - p + 1, \ldots, wp) \rangle \leq S_{wp}.
\]
Abelian Vertices

In 2003 I proved:

**Theorem**

*The Specht module $S^\lambda$, defined over a field of characteristic $p$, has a non-trivial cyclic vertex if and only if $\lambda$ has $p$-weight 1.*
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**Theorem (Lim 2011, Corollary 5.1)**
Let $p$ be an odd prime and let $1 \leq m \leq p - 1$. The Specht module $S^\lambda$, defined over a field of characteristic $p$ has an abelian vertex of $p$-rank $m$ if and only if the $p$-weight of $\mu$ is $m$. 
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**Problem**
Classify all Specht modules with abelian vertex.