### BREDON COHOMOLOGICAL FINITENESS CONDITIONS FOR GENERALISATIONS OF THOMPSON GROUPS

#### C. MARTÍNEZ-PÉREZ AND B. E. A. NUCINKIS

ABSTRACT. We define a family of groups that generalises Thompson's groups T and V, and also those of Higman, Stein and Brin. For groups in this family we study centralisers and conjugacy classes of finite subgroups. We use this to show that these groups have a slightly weaker property, quasi- $\underline{F}_{\infty}$ , to that of a group possessing a finite type model for the classifying space for proper actions  $\underline{E}G$ . We also generalise some well-known properties of ordinary cohomology to Bredon cohomology.

#### 1. INTRODUCTION

Thompson's groups F, T and G (also denoted V), which can be defined as certain homeomorphism groups of the unit interval, the circle and the Cantor-set, respectively, have received a large amount of attention in recent years. There are many interesting generalisations of these groups, such as the Higman-Thompson groups  $F_{n,r}$ ,  $T_{n,r}$ ,  $G_{n,r}$  (Recall that  $T = T_{2,1}$  and G = $G_{2,1}$ ), the T- and G-groups defined by Stein [22] and the higher dimensional Thompson groups  $sV = sG_{2,1}$  defined by Brin [4]. All these groups contain every finite group, are finitely presented and with the exception of sV for  $s \geq 4$ , are known to be of type  $FP_{\infty}$  [5, 22, 10, 3]. Furthermore, they contain free abelian groups of countable rank. In this paper we consider automorphism groups of certain Cantor algebras which include Higman-Thompson, Stein and Brin groups.

As in the original exposition by Higman [8] and in Brown's proof [5] that  $F_{n,r}$ ,  $T_{n,r}$  and  $G_{n,r}$  are of type  $\operatorname{FP}_{\infty}$ , we consider a Cantor algebra  $U_r(\Sigma)$  on a so called valid set of relations  $\Sigma$  and define groups  $G_r(\Sigma)$  as follows: the elements of  $G_r(\Sigma)$  are bijections between admissible subsets of  $U_r(\Sigma)$ . One can show that these groups are finitely generated, see [16]. Provided that the relations in  $\Sigma$  are order preserving we can also define the groups  $T_r(\Sigma)$ , which are given by cyclic order preserving bijections. One can also define generalisations of  $F_{n,r}$ .

The admissible subsets of  $U_r(\Sigma)$  form a poset and the groups  $T_r(\Sigma)$  and  $G_r(\Sigma)$  act on the geometric realisation  $|\mathfrak{A}_r(\Sigma)|$  of this poset (for the original Thompson-Higman algebras this was already used by Brown in [5]).

Let G be either  $T_r(\Sigma)$  or  $G_r(\Sigma)$ . For every finite subgroup Q we consider the fixed point sets  $\mathfrak{A}_r(\Sigma)^Q$ . The Q-set structure of every admissible subset  $Y \in \mathfrak{A}_r(\Sigma)^Q$  is determined by its decomposition into transitive Q-sets. We

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show (Theorem 4.3 and Lemma 7.7) that there are finitely many conjugacy classes in G of subgroups isomorphic to Q. Furthermore we show (Theorem 4.5) that there is a central extension

$$K \mapsto C_{G_r(\Sigma)}(Q) \twoheadrightarrow G_{r_1}(\Sigma) \times \ldots \times G_{r_t}(\Sigma)$$

with finite kernel, where the  $r_1, ..., r_t$  are integers uniquely determined by Q. The analogue for  $T_r(\Sigma)$  also generalises a result of Matucci [17, Theorem 7.1.5] for the original Thompson group T. In particular, we have (Theorem 7.5) that, for a certain l also determined by Q, there is a central extension

$$K \rightarrow C_{T_r(\Sigma)}(Q) \twoheadrightarrow T_l(\Sigma)$$

with cyclic kernel of finite order.

Recently a variant of the Eilenberg-Mac Lane space, the classifying space with respect to a family of subgroups, has been the focus considerable attention. Through its connection with important conjectures such as the Baum-Connes conjecture, the classifying space for proper actions has been well researched in recent times. Let X be a G-CW-complex. It is said to be a model for  $E_{\mathfrak{X}}G$ , the classifying space with isotropy in the family  $\mathfrak{X}$  if  $X^K$ is contractible for  $K \in \mathfrak{X}$  and  $X^K$  is empty otherwise. The classifying space X for a family satisfies the following universal property: whenever there is a G-CW-complex Y with isotropy lying in the family  $\mathfrak{X}$ , there is a G-map  $Y \to X$ , which is unique up to G-homotopy. In particular,  $E_{\mathfrak{X}}G$  is unique up to G-homotopy equivalence.

For the class  $\mathcal{F}$  of finite subgroups we denote  $E_{\mathfrak{X}}G$  by  $\underline{E}G$ , the classifying space for proper actions. We say a group is of type  $\underline{F}_{\infty}$  if it admits a finite type model for  $\underline{E}G$ . We show:

#### **Theorem 3.1.** $|\mathfrak{A}_r(\Sigma)|$ is a model for $\underline{E}G_r(\Sigma)$ .

(but these groups cannot possess any finite dimensional model). The algebraic mirror to classifying spaces with isotropy in a family is given by Bredon cohomology. We shall review its properties in Section 2. Many notions from ordinary cohomology have a Bredon analogue. For example, we say a group G is of type Bredon-FP $_{\infty}$  if there is a Bredon-projective resolution of the constant Bredon-module  $\mathbb{Z}(-)$  with finitely generated Bredon-projective modules. The connection to classifying spaces and to ordinary cohomology is given by the following two results:

**Theorem 1.1.** [15, Theorem 0.1] A group G has a finite type model for a classifying space with isotropy in a family if and only if the group is of type Bredon-FP<sub> $\infty$ </sub> and there is a model for a classifying space with finite 2-skeleton.

In particular we say a group is of type  $\underline{FP}_{\infty}$  if it is of type Bredon- $FP_{\infty}$  for the family of finite subgroups.

**Theorem 1.2.** [14, Theorem 4.2] A group G admits a finite type model for <u>E</u>G if and only G has finitely many conjugacy classes of finite subgroups and for each finite subgroup K of G the centraliser  $C_G(K)$  is of type  $FP_{\infty}$  and finitely presented.

Equivalently, G admits a finite type model for  $\underline{E}G$  if and only it is of type  $\underline{FP}_{\infty}$  and centralizers of finite subgroups are finitely presented. Since the

groups we are considering do not have a bound on the orders of their finite subgroups, we need to weaken the condition on the number of conjugacy classes and require that for each finite subgroup Q of G there are only finitely many conjugacy classes of subgroups isomorphic to Q. Groups satisfying this slightly weaker condition are said to be of type quasi-<u>FP</u> $_{\infty}$ . Note, that for groups with a bound on the orders of its finite subgroups this definition is identical to <u>FP</u> $_{\infty}$ . In [12] it was shown that there are examples of groups of type FP $_{\infty}$ , which are not of type <u>FP} $_{\infty}$ . These examples are virtually torsion free, admit a finite dimensional model for <u>E</u>G and can be constructed to have either infinitely many conjugacy classes of finite subgroups or to have centralisers of finite subgroups not of type FP $_{\infty}$ . There are a number of classes of groups of type FP $_{\infty}$  admitting cocompact models for <u>E</u>G including Gromov hyperbolic groups [18],  $Out(F_n)$  [23] or elementary amenable groups [11].</u>

Using our results on centralizers and conjugacy classes of finite subgroups we show:

**Theorem 7.1.**  $G_r(\Sigma)$  is of type quasi-<u>FP</u> $_{\infty}$  for any r if and only if it is of type FP $_{\infty}$  for any r.

Analogously we have:

**Theorem 7.8.**  $T_r(\Sigma)$  is of type quasi-<u>FP</u> $_{\infty}$  for any r if and only if it is of type FP $_{\infty}$  for any r.

We also consider the geometric analogue, to be of type quasi- $\underline{F}_{\infty}$ , and prove the analogues to Theorems 7.1 and 7.8.

The paper is structured as follows: In Section 2 we define the Cantor algebras and the corresponding generalisations of Thompson's groups G and T. We then use this Cantor algebra to build a model for <u>E</u>G in Section 3. In Section 4 we prove the results on centralizers and conjugacy classes of finite subgroups that will be used later.

In Section 5 we collect all necessary background on Bredon cohomology with respect to an arbitrary family, and on Bredon cohomological finiteness conditions for modules. We prove an analogue to the Bieri-Eckmann criterion for property  $FP_n$  for modules. In Section 7 we specialise to the case of the family of finite subgroups and define what it means for a group to be quasi-<u>FP</u><sub> $\infty$ </sub> and quasi-<u>F</u><sub> $\infty$ </sub>. Finally, the main results are proven in Section 7.

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#### 2. Generalisations of Thompson-Higman groups

As mentioned in the introduction, the generalised Thompson-Higman groups can be viewed as certain automorphisms groups of Cantor algebras. We shall begin by defining these algebras. We use the notation of [10], Section 2. In particular, we consider a finite set  $\{1, \ldots, s\}$  whose elements are called colours. To each colour *i* we associate an integer  $n_i > 1$  which is called its arity. We say that *U* is an  $\Omega$ -algebra, if, for each colour *i*, the following operations (we let all operations act on the right) are defined in U (for detail, see [7] and [10]):

i) One  $n_i$ -ary operation  $\lambda_i$ :

$$\lambda_i: U^{n_i} \to U.$$

We call these operations ascending operations, or contractions. ii)  $n_i$  1-ary operations  $\alpha_i^1, \ldots, \alpha_i^{n_i}$ :

$$\alpha_i^j: U \to U.$$

We call these operations 1-ary descending operations.

We denote  $\Omega = \{\lambda_i, \alpha_i^j\}_{i,j}$ . For each colour *i* we also consider the map  $\alpha_i: U \to U^{n_i}$  given by

$$v\alpha_i := (v\alpha_i^1, v\alpha_i^2, \dots, v\alpha_i^{n_i})$$

for any  $v \in U$ . These maps are called descending operations, or expansions. For any subset Y of U, a simple expansion of colour i of Y is obtained by substituting some element  $y \in Y$  by the  $n_i$  elements of the tuple  $y\alpha_i$ . A simple contraction of colour i of Y is the set obtained by substituting a certain collection of  $n_i$  distinct elements of Y, say  $\{a_1, \ldots, a_{n_i}\}$ , by  $(a_1, \ldots, a_{n_i})\lambda_i$ . We also use the term operation to refer to the effect of a simple expansion, respectively contraction on a set.

For any set X there is a  $\Omega$ -algebra, free on X, which is called the  $\Omega$ -word algebra on X and is denoted by  $W_{\Omega}(X)$ . An admissible subset  $A \subseteq W_{\Omega}(X)$  is a subset that can be obtained after finitely many expansions or contractions from the set X.

Descending operations can be visualised by tree diagrams, see the following example with  $X = \{x\}, s = 1$  and  $n_1 = 2$ :



The set  $A = \{x\alpha^1\alpha^1, x\alpha^1\alpha^2, x\alpha^2\}$  is an admissible subset. In pictures we often omit the maps and label the nodes by positive integers as follows:



From now on we fix the set X and assume it is finite. We consider the variety of  $\Omega$ -algebras satisfying a certain set of identities as follows:

**Definition 2.1.** Let  $\Sigma$  be the following set of laws in the alphabet X.

i) For any  $u \in W_{\Omega}(X)$ , any colour *i*, and any  $n_i$ -tuple  $(u_1, \ldots, u_{n_i}) \in W_{\Omega}(X)^{n_i}$ ,

$$u\alpha_i\lambda_i = u_i$$

 $(u_1,\ldots,u_{n_i})\lambda_i\alpha_i=(u_1,\ldots,u_{n_i}).$ 

The set of all these relations is denoted  $\Sigma_1$ 

ii) A certain, possibly empty, set

$$\Sigma_2 := \{ r_{ij} \mid 1 \le i < j \le s \}$$

of identifications between sets of simple expansions of  $w\alpha_i$  and  $w\alpha_j$ for any  $w \in W_{\Omega}(X)$  which do not depend on w.

When factoring out the fully invariant congruence  $\mathfrak{q}$  generated by  $\Sigma$ , we obtain an  $\Omega$ -algebra  $W_{\Omega}(X)/\mathfrak{q}$  satisfying the identities in  $\Sigma$ . For detail of the construction the reader is referred to [10, Section 2].

**Definition 2.2.** Let r = |X| and  $\Sigma$  as in Definition 2.1. Then the algebra  $W_{\Omega}(X)/\mathfrak{q} = U_r(\Sigma)$  is called a Cantor-Algebra.

Moreover, there is an epimorphism of  $\Omega$ -algebras

$$W_{\Omega}(X) \twoheadrightarrow U_r(\Sigma)$$
$$A \mapsto \bar{A}.$$

As in [10] we say that  $\Sigma$  is valid if for any admissible  $Y \subseteq W_{\Omega}(X)$ , we have  $|Y| = |\overline{Y}|$ . This condition implies that  $U_r(\Sigma)$  is a free object on X in the class of those  $\Omega$ -algebras which satisfy the identities  $\Sigma$  above.

If the set  $\Sigma$  used to define  $U_r(\Sigma)$  is valid, we also say that  $U_r(\Sigma)$  is valid.

**Example 2.3.** Higman [8] defined an algebra  $V_{n,r}$  with |X| = r, s = 1 and arity n as above with  $\Sigma_2$  being empty. This algebra, which we call Higman algebra, is used in the original construction of the Higman-Thompson-groups  $G_{n,r}$ . For detail see also [5]. In particular, these algebras are valid [8, Section 2].

**Example 2.4.** Higman's construction for arity n = 2 can be generalised as follows [10, Section 2]: Let  $s \ge 1$  and  $n_i = 2$  for all  $1 \le i \le s$ . Hence we consider the set of s colours  $\{1, \ldots, s\}$ , all of which have arity 2, together with the relations:  $\Sigma := \Sigma_1 \cup \Sigma_2$  with

$$\Sigma_2 := \{ \alpha_i^l \alpha_j^t = \alpha_j^t \alpha_i^l \mid 1 \le i \ne j \le s; l, t = 1, 2 \}.$$

Then  $\Sigma$  is valid (see [10] Lemma 2.9).

Furthermore one can also consider s colours, all of arity  $n_i = n$ , for all  $1 \le i \le s$ . Let

$$\Sigma_2 := \{ \alpha_i^l \alpha_j^t = \alpha_j^t \alpha_i^l \mid 1 \le i \ne j \le s; 1 \le l, t \le n \}.$$

Using the same arguments as in [10, Section 2] one can show that the  $\Sigma$  obtained in this way is also valid.

We call the resulting Cantor algebras  $U_r(\Sigma)$  Brin algebras.

The following tree-diagram visualises the relations in  $\Sigma_2$ . Here r = 1, s = 2 and n = 2. We express an expansion of colour 1 with dotted lines and an expansion of colour 2 by straight lines.



**Definition 2.5.** Let  $\Sigma$  be valid and consider  $Y, Z \subseteq U_r(\Sigma)$ . If Z can be obtained from Y by a finite number of simple expansions then we say that Z is a descendant of Y and denote

 $Y \leq Z$ .

Conversely, Y is called an *ascendant* of Z and can be obtained after a finite number of simple contractions. Note that this implies that if either of the sets Y or Z is admissible, then so is the other. In fact, the set of admissible subsets of  $U_r(\Sigma)$  is a poset with respect to the partial order  $\leq$ . This poset is denoted by  $\mathfrak{A}_r(\Sigma)$ .

It is easy to prove that any admissible subset is a basis of  $U_r(\Sigma)$  (see [10] Lemma 2.5).

Remark 2.6. Let  $\Sigma$  be valid and assume that we have s colours of arities  $\{n_1, \ldots, n_s\}$ . Let r be a positive integer. Observe that the cardinality of any admissible subset of  $U_r(\Sigma)$  must be of the form  $m \equiv r \mod d$  for

$$d := \gcd\{n_i - 1 \mid i = 1, \dots, s\}.$$

Moreover, for any  $m \equiv r \mod d$ , there is some admissible subset of cardinality m. And as admissible subsets are bases, we get  $U_r(\Sigma) = U_m(\Sigma)$ .

**Definition 2.7.** Let B, C be admissible subsets of  $U_r(\Sigma)$ . We say that T is the unique least upper bound of B and C if  $B \leq T$ ,  $C \leq T$  and for all admissible sets S such that  $B \leq S$  and  $C \leq S$  we have  $T \leq S$ .

We say, by abusing notation a little, that  $U_r(\Sigma)$  is bounded if for all admissible subsets B, C such that there is some admissible A with  $A \leq B, C$  there is a unique least upper bound of B and C.

One can also define greatest lower bounds, but this places a stronger restriction on the algebra, see [10]. Moreover, note that a priori we require the existence of an upper bound only when our sets have a lower bound (A) but this turns out to be not too restrictive:

**Lemma 2.8.** Let  $U_r(\Sigma)$  be valid and bounded. Then any two admissible subsets have some (possibly not unique) common upper bound.

*Proof.* Use the same proof as in [10, Proposition 3.4].

**Example 2.9.** The Brin algebras defined in Example 2.4 are valid and bounded. The existence of a unique least upper bound for n = 2 is shown in [10, Lemma 3.2]. The general case is analogous.

**Example 2.10.** Let  $P \subseteq \mathbb{Q}_{>0}$  be a finitely generated multiplicative group. Then by a result of Brown, see [22, Proposition 1.1], P has a basis of the form  $\{n_1, \ldots, n_s\}$  with all  $n_i \ge 0$   $(i = 1, \ldots, s)$ . Now consider  $\Omega$ -algebras on s colours of arities  $\{n_1, \ldots, n_s\}$  and let  $\Sigma = \Sigma_1 \cup \Sigma_2$  with  $\Sigma_2$  the set of identities given by the following order preserving identification:

$$\{\alpha_i^1 \alpha_j^1, \dots, \alpha_i^1 \alpha_j^{n_j}, \alpha_i^2 \alpha_j^1, \dots, \alpha_i^2 \alpha_j^{n_j}, \dots, \alpha_i^{n_i} \alpha_j^1, \dots, \alpha_i^{n_i} \alpha_j^{n_j}\} = \\ \{\alpha_j^1 \alpha_i^1, \dots, \alpha_j^1 \alpha_i^{n_i}, \alpha_j^2 \alpha_i^1, \dots, \alpha_j^2 \alpha_i^{n_i}, \dots, \alpha_j^{n_j} \alpha_i^1, \dots, \alpha_j^{n_j} \alpha_i^{n_i}\},$$

where  $i \neq j$  and  $i, j \in \{1, ..., s\}$ .

The Cantor algebras  $U_r(\Sigma)$  thus obtained will be called Brown-Stein algebras.

Note that, as  $\{n_1, \ldots, n_s\}$  is a basis for P, the  $n_i$  are all distinct. Hence, when visualising the identities in  $\Sigma_2$  for the Brown-Stein algebra, it suffices to only use one colour, as the arity of an expansion already determines the colour. In the following example let  $r = 1, s = 2, n_1 = 2$  and  $n_2 = 3$ .



Lemma 2.11. The Brown-Stein algebras are valid and bounded.

*Proof.* This is Proposition 1.2 (due to K. Brown) in [22].

In fact, in [22] Lemma 2.8 is proven directly, i.e that any two admissible subsets have some common upper bound.

We can now define the generalised Thompson-Higman groups. Recall, that in a valid Cantor algebra  $U_r(\Sigma)$ , admissible subsets are bases.

**Definition 2.12.** Let  $U_r(\Sigma)$  be a valid Cantor algebra. We define  $G_r(\Sigma)$  to be the group of those  $\Omega$ -algebra automorphisms of  $U_r(\Sigma)$ , which are induced by a map  $V \to W$ , where V and W are admissible subsets of the same cardinality.

**Example 2.13.** If  $U_r(\Sigma)$  is a Higman algebra as in Example 2.3, we retrieve the original Higman-Thompson-groups  $G_{n,r}$ . Let  $U_r(\Sigma)$  be a Brin algebra on s colours of arity 2 as in Example 2.4. Then the groups constructed are Brin's [4] generalisations sV of Thompson's group  $V = G_{2,1}$ . The description of sVas automorphism groups of a Cantor algebra can be found in [10]. Finally, the groups  $G_r(\Sigma)$ , when  $U_r(\Sigma)$  is a Brown-Stein algebra as in Example 2.10, were considered in [22].

Suppose now that the elements of the set X are ordered. It can be seen that this order is inherited by certain elements in  $W_{\Omega}(X)$  including all admissible subsets, see for example [5] or [8]. If the relations in  $\Sigma$  preserve that ordering, then we also have an inherited order on the admissible subsets of  $U_r(\Sigma)$ . We shall call this the induced ordering. **Definition 2.14.** Suppose we have a Cantor algebra  $U_r(\Sigma)$  where  $\Sigma$  preserves the induced ordering. We may define subgroups  $F_r(\Sigma)$  and  $T_r(\Sigma)$  of  $G_r(\Sigma)$  as follows. We let  $F_r(\Sigma)$  be the group of order preserving automorphisms between ordered admissible subsets of the same cardinality and  $T_r(\Sigma)$  the group of cyclic order preserving automorphisms between ordered admissible subsets of the same cardinality.

**Example 2.15.** For  $U_r(\Sigma)$  a Higman algebra of Example 2.3 the definition above yields the groups  $F_{n,r}$  and  $T_{n,r}$  as in [5]. Recall that Thompson's groups are  $F = F_{2,1}$  and  $T = T_{2,1}$ .

Let  $U_r(\Sigma)$  be a Brown-Stein algebra as in Example 2.10 In this case,  $\Sigma$  is order preserving, so we may define the groups  $F_r(\Sigma)$  and  $T_r(\Sigma)$ , which are considered in [22].

Since  $\Sigma_2$  in the definition of the Brin algebra of Example 2.4 is not orderpreserving, it makes no sense to define the groups  $F_r(\Sigma)$  or  $T_r(\Sigma)$  for this algebra.

Remark 2.16. Note, that if definable, the groups  $F_r(\Sigma)$  are torsion-free. In both cases mentioned in Example 2.15, the resulting groups  $F_r(\Sigma)$  are known to be of type FP<sub> $\infty$ </sub> and finitely presented [5, 22].

3. A model for  $\underline{E}G$  for generalised Thompson groups

From now on we fix a valid  $\Sigma$  and a finite positive integer r. Also assume that the Cantor algebra  $U_r(\Sigma)$  is bounded. We shall use the poset of admissible subsets of  $U_r(\Sigma)$  to construct a model for  $\underline{E}G_r(\Sigma)$ . In this section we give a quite elementary proof of the following result.

**Theorem 3.1.** The geometric realisation of the poset of admissible subsets is a model for  $\underline{E}G_r(\Sigma)$ .

We fix an admissible subset  $X \subseteq U_r(\Sigma)$  of cardinality r.

**Lemma 3.2.** For any finite  $Q \leq G_r(\Sigma)$  there exists some admissible subset Z such that ZQ = Z. Moreover we may assume  $X \leq Z$ .

*Proof.* For every  $q \in Q$  choose a common upper bound  $T_q$  of X and Xq. Then put  $Z_q := T_q q^{-1}$  and let Y be an upper bound of

 $\{Z_q \mid q \in Q\}.$ 

Note that  $X \leq Z_1 = T_1$  and for any  $q \in Q$ ,

$$X \le T_q = Z_q q \le Y q.$$

Therefore we may choose Z the unique least upper bound of  $\{Yq \mid q \in Q\}$ . By definition of least upperbound we get ZQ = Z.

**Proposition 3.3.** Any two elements in  $\mathfrak{A}_r(\Sigma)^Q$  have an upper bound in  $\mathfrak{A}_r(\Sigma)^Q$ .

*Proof.* Let  $Y, Z \in \mathfrak{A}_r(\Sigma)^Q$ . We begin by showing that there are admissible sets  $Y_1, Z_1 \in \mathfrak{A}_r(\Sigma)^Q$  such that  $Y_1$  is an upper bound of X and Y and  $Z_1$  is an upper bound of X and Z. It suffices to prove the existence of  $Y_1$ . Take an upper bound  $Y_2 \in \mathfrak{A}_r(\Sigma)$  of X and Y and consider

$$\{Y_2q^{-1} \mid q \in Q\}.$$

Let  $Y_3 \in \mathfrak{A}_r(\Sigma)$  be an upper bound of this set. Then, for any  $q \in Q$ ,

$$Y_2 \le Y_3 q.$$

Therefore  $X \leq Y_3 q$ . This implies that we may choose  $Y_1$  to be the least upper bound of

$$\{Y_3q \mid q \in Q\}.$$

Clearly,  $Y, X \leq Y_1$ . Again, the definition of least upper bound implies that  $Y_1 \in \mathfrak{A}_r(\Sigma)^Q$ .

Now, let T be the least upper bound of  $Y_1$  and  $Z_1$ . Then for any  $q \in Q$ 

$$Y_1 = Y_1 q \le T q$$

$$Z_1 = Z_1 q \le T q$$

so we get  $T \in \mathfrak{A}_r(\Sigma)^Q$ .

*Proof.* (of Theorem 3.1) Lemmas 3.2 and 3.3 imply that for any finite subgroup  $Q \leq G_r(\Sigma)$  the poset  $\mathfrak{A}_r(\Sigma)^Q$  is non-empty and directed, thus  $|\mathfrak{A}_r(\Sigma)|^Q = |\mathfrak{A}_r(\Sigma)^Q| \simeq *$ . Moreover for any  $V \in \mathfrak{A}_r(\Sigma)$ ,

$$\operatorname{Stab}_{G_r(\Sigma)}(V) = \{g \in G_r(\Sigma) \mid Vg = V\}$$

is contained in the group of permutations of the finite set V, thus it is finite. This implies that for any  $H \leq G_r(\Sigma)$ ,  $\mathfrak{A}_r(\Sigma)^H = \emptyset$  unless H is finite.  $\Box$ 

This model is not of finite type, but there is a filtration of  $|\mathfrak{A}_r(\Sigma)^Q|$  by finite type subcomplexes, exactly as in the construction in [5, Theorem 4.17]:

**Proposition 3.4.** For any finite  $Q \leq G_r(\Sigma)$  there is a filtration of  $|\mathfrak{A}_r(\Sigma)^Q|$ 

$$\ldots \subset |\mathfrak{A}_r(\Sigma)^Q|_{n-1} \subset |\mathfrak{A}_r(\Sigma)^Q|_n \subset |\mathfrak{A}_r(\Sigma)^Q|_{n+1} \subset \ldots$$

such that each  $|\mathfrak{A}_r(\Sigma)^Q|_n/C_{G_r(\Sigma)}(Q)$  is finite.

*Proof.* Let

$$|\mathfrak{A}_r(\Sigma)^Q|_n := \{ Y \in \mathfrak{A}_r(\Sigma)^Q \mid |Y| \le n \}.$$

Consider  $Y, Z \in \mathfrak{A}_r(\Sigma)^Q$  with |Y| = |Z| and isomorphic as Q-sets. This means that there is a Q-bijection

$$\sigma: Y \to Z.$$

Let  $g \in G_r(\Sigma)$  be the element given by  $yg = y\sigma$  for each  $y \in Y$ . Then for any  $q \in Q$ ,  $(yq)g = (yq)\sigma = y\sigma q = ygq$ . This means that the commutator [g,q] acts as the identity on the admissible set Y and therefore [g,q] = 1. Hence  $g \in C_{G_r(\Sigma)}(Q)$ . As for any  $m \leq n$  there are finitely many possible Q-sets of cardinality m, the result follows.

Remark 3.5. Provided that  $\Sigma$  is order-preserving, Theorem 3.1 and Proposition 3.4 can be restated replacing  $G_r(\Sigma)$  with  $T_r(\Sigma)$ .

# 4. Centralisers and conjugacy classes of finite subgroups for $G_r(\Sigma)$ and $T_r(\Sigma)$ .

Let  $Q \leq G_r(\Sigma)$  be a finite subgroup. In this section we give a more detailed analysis of the poset  $\mathfrak{A}_r(\Sigma)^Q$  to determine  $C_{G_r(\Sigma)}(Q)$  and the number of conjugacy classes of finite subgroups. This will be useful later to prove our main result on the kind of cohomological finiteness properties that these groups may satisfy. Let  $\{w_1, \ldots, w_t\}$  be the set of lengths of all the possible transitive permutation representations of Q. Any  $Y \in \mathfrak{A}_r(\Sigma)^Q$  is a finite Q-set so it is determined by its decomposition in transitive Q-sets. If we take one of those sets and apply the operation  $\alpha_i$  for a fixed colour i to each of its elements, we obtain a new admissible subset which is also fixed by Q. We say that this is a simple Q-expansion of Y. More explicitly, the admissible set obtained from Y is:

$$Y \smallsetminus \{yq \mid q \in Q\} \cup \{yq\alpha_i^j \mid q \in Q, 1 \le j \le n_i\}$$

for a certain  $y \in Y$ .

Conversely, if we choose  $n_i$  different orbits of the same type (i.e., corresponding to the same permutation representation) in Y, then we may contract them to a single orbit (of the same type), we call this a simple Q-contraction. Note that the admissible subset obtained this way, again lies in  $\mathfrak{A}_r(\Sigma)^Q$ .

Large parts of the next three results can be found in [8, Section 6]. We shall, for the reader's convenience, recall the arguments.

**Lemma 4.1.** Let  $Y, Z \in \mathfrak{A}_r(\Sigma)^Q$  with Y < Z and assume there is no admissible subset  $C \in \mathfrak{A}_r(\Sigma)^Q$  with  $Y \leq C \leq Z$ . Then Z is a simple Q-expansion of Y. Hence Y is a simple Q-contraction of Z.

*Proof.* We may choose a chain of simple expansions

$$Y < Y_1 < \ldots < Y_r < Z.$$

Let  $w \in Y$  be the vertex expanded in the first simple expansion  $Y < Y_1$  and  $W \subseteq Y$  be the Q-orbit with  $w \in W$ . Assume also that this first expansion corresponds to the colour i. Then as Z contains certain descendants of  $\{w\alpha_i\}$  and it is Q-invariant it must also contain the analogous descendants of  $\{u\alpha_i \mid u \in W\}$ . Therefore if C denotes the simple Q-expansion consisting of expanding W by  $\alpha_i$ , then  $Y < C \leq Z$ . As  $C \in \mathfrak{A}_r(\Sigma)^Q$ , we deduce by the hypothesis that C = Z.

**Proposition 4.2.** For any finite subgroup  $Q \leq G(\Sigma)$ , there is a uniquely determined set of integers  $\pi(Q) := \{r_1, \ldots, r_t\}$  with  $0 \leq r_j \leq d$  and

$$\sum_{j=1}^{t} r_j w_j \equiv r \mod d$$

such that there is an admissible subset  $Y \in \mathfrak{A}_r(\Sigma)^Q$  with  $|Y| = \sum_{j=1}^t r_j w_j$ . Moreover, any other element in  $\mathfrak{A}_r(\Sigma)^Q$  can be obtained from Y by a finite sequence of simple Q-expansions or Q-contractions.

*Proof.* First, note that by 3.2,  $\mathfrak{A}_r(\Sigma)^Q \neq \emptyset$ . Now choose some  $Z \in \mathfrak{A}_r(\Sigma)^Q$  and decompose it as a disjoint union of transitive Q-sets. Let  $k_j$  be the

number of transitive sets in this decomposition which are of type j, i.e which correspond to the same permutation representation. Observe that whenever we apply simple Q-contractions or Q-expansions to Z, if the set thus obtained has  $m_j$  transitive Q-sets of type j, then  $m_j \equiv k_j \mod d$ . Note also that

$$r \equiv |Z| = \sum_{j=1}^{t} k_j w_j \mod d.$$

Let

 $r_j = \begin{cases} 0, \text{ if } k_j = 0\\ d, \text{ if } 0 \neq k_j \equiv 0 \mod d\\ l \text{ with } 0 < l < d \text{ and } l \equiv k_j \mod d, \text{ otherwise.} \end{cases}$ 

By successively performing simple Q-contractions or Q-expansions of Z we may get an admissible set Y such that the number of transitive Q-sets of type j in Y is exactly  $r_j$ . Observe that the  $r_j$  are uniquely determined, whereas Y is not. Finally, 3.3 implies that for any other  $C \in \mathfrak{A}_r(\Sigma)^Q$ , there is an upper bound, say D, of Y and C with  $D \in \mathfrak{A}_r(\Sigma)^Q$  which means that

$$Y \le D \ge C.$$

By Lemma 4.1 we may choose chains

$$Y = D_0 < D_1 < \ldots < D_{l_1} = D = C_0 > C_1 > \ldots > C_{l_2} = C$$

such that each step consits of a simple Q-expansion/contraction and we are done.  $\hfill \Box$ 

**Theorem 4.3.** Let  $Q_1, Q_2 \leq G_r(\Sigma)$  be finite subgroups with  $Q_1 \cong Q_2$ . Then  $Q_1$  and  $Q_2$  are conjugate in  $G_r(\Sigma)$  if and only if  $\pi(Q_1) = \pi(Q_2)$ .

In particular, there are only finitely many conjugacy classes of subgroups isomorphic to  $Q_1$ .

Proof. Fix an isomorphism  $\alpha : Q_1 \to Q_2$ . Assume first that  $\pi(Q_1) = \pi(Q_2)$ . Then there are admissible subsets  $V_1, V_2$  with  $V_i \in \mathfrak{A}_r(\Sigma)^{Q_i}$  having the same number of elements and moreover the same structure as  $Q_i$ -sets. This means that there is a bijection  $\psi : V_1 \to V_2$  such that for any  $q \in Q_1$  and  $v \in V_1$ ,  $(vq)^{\psi} = v^{\psi}q^{\alpha}$ . Thus  $\psi$  yields an element  $g \in G_r(\Sigma)$  with  $g^{-1}qg = q^{\alpha}$ .

Conversely, assume  $Q_2 = g^{-1}Q_1g$  with  $g \in G_r(\Sigma)$ . Then for any  $V_1 \in \mathfrak{A}_r(\Sigma)^{Q_1}$ ,  $V_1g \in \mathfrak{A}_r(\Sigma)^{Q_2}$ . Moreover, g induces an isomorphism as  $Q_i$ -sets so the orbit structure of the minimal elements of  $\mathfrak{A}_r(\Sigma)^{Q_1}$  and  $\mathfrak{A}_r(\Sigma)^{Q_1}$  has to be the same.

**Lemma 4.4.** Let  $Q \leq G_r(\Sigma)$  be a finite subgroup and  $\pi(Q) = \{r_1, \ldots, r_t\}$ . Then there is a poset isomorphism

$$\mathfrak{A}_r(\Sigma)^Q \cong \mathfrak{A}_{r_1}(\Sigma) \times \ldots \times \mathfrak{A}_{r_t}(\Sigma).$$

Proof. Choose an admissible  $Y \in \mathfrak{A}_r(\Sigma)^Q$  as in Proposition 4.2. Given  $y_1, y_2 \in Y$  we put  $y_1 \sim y_2$  if  $y_2 = y_1 q$  for some  $q \in Q$ . Then  $Y/\sim$  can be seen as an element of  $\mathfrak{A}_{r_1}(\Sigma) \times \ldots \times \mathfrak{A}_{r_t}(\Sigma)$ . Observe that we may extend the definition of  $\sim$  to any  $V \in \mathfrak{A}_r(\Sigma)^Q$ . If A < B is a simple Q-expansion in  $\mathfrak{A}_r(\Sigma)^Q$  then  $(A/\sim) < (B/\sim)$  is also a simple expansion in the right hand poset, moreover the expanded vertex in A maps to the expanded vertex in

 $A/\sim$ . It is easy to see that this implies that  $V \mapsto V/\sim$  is in fact a poset isomorphism using the last assertion of Proposition 4.2.

**Theorem 4.5.** Let  $Q \leq G_r(\Sigma)$  be a finite subgroup and  $\pi(Q) = \{r_1, \ldots, r_t\}$ . Then there is a central group extension

$$K \rightarrowtail C_{G_r(\Sigma)}(Q) \twoheadrightarrow G_{r_1}(\Sigma) \times \ldots \times G_{r_t}(\Sigma)$$

such that K is finite.

Proof. Let  $g \in C_{G_r(\Sigma)}(Q)$ . Then for any  $V \in \mathfrak{A}_r(\Sigma)^Q$  and  $q \in Q$ , we have Vgq = Vqg = Vg and therefore  $Vg \in \mathfrak{A}_r(\Sigma)^Q$ . Recall that g is determined by its action on any admissible subset. Thus, in particular, it is determined by its action on V. Now, let  $v_1 \sim v_2$  with  $\sim$  as in the proof of Lemma 4.4. Then  $v_1 = v_2q$  for some  $q \in Q$  thus  $v_1g = v_2qg = v_2gq$ . This means that g yields a map which for simplicity we denote  $g^{\mu}$ 

$$g^{\mu}: (V/\sim) \to (Vg)/\sim$$
.

Note also that the fact that g commutes with Q implies that if we split Vand Vg into Q-orbits, then the elements v of V and vg of Vg belong to Q-orbits of the same type. This means that  $g^{\mu}$  yields in fact an element  $g^{\mu} \in G_{r_1}(\Sigma) \times \ldots \times G_{r_t}(\Sigma)$ . Hence there is a group homomorphism:

$$\mu: C_{G_r(\Sigma)}(Q) \to G_{r_1}(\Sigma) \times \ldots \times G_{r_t}(\Sigma).$$

Next we show that  $\mu$  is surjective. To see this, let  $(g_1, \ldots, g_t) \in G_{r_1}(\Sigma) \times \ldots \times G_{r_t}(\Sigma)$  and choose some  $(\bar{V}_1, \ldots, \bar{V}_t) \in \mathfrak{A}_{r_1}(\Sigma) \times \ldots \times \mathfrak{A}_{r_t}(\Sigma)$ . When we apply the element  $(g_1, \ldots, g_t)$  this yields  $(\bar{V}_1g_1, \ldots, \bar{V}_tg_t)$ . Then the preimages in  $\mathfrak{A}_r(\Sigma)^Q$  of  $(\bar{V}_1, \ldots, \bar{V}_t)$  and  $(\bar{V}_1g_1, \ldots, \bar{V}_tg_t)$  can be denoted

$$V = \bigcup_{j=1}^{l} \bar{V}_j \times \Omega_j$$

$$V' = \bigcup_{j=1}^{\circ} \bar{V}_j g_j \times \Omega_j$$

where  $\Omega_j$  is a *Q*-set of type *j*. Define  $g: V \to V'$  via  $(v_j, w_j)g = (v_jg_j, w_j)$ for each  $(v_j, w_j) \in \overline{V_j} \times \Omega_j$ . Clearly, this yields an element  $g \in G_r(\Omega)$ . Moreover, for any  $q \in Q$ ,  $g^{-1}qg: V \to V$  is the identity thus  $g \in C_{G_r(\Omega)}(Q)$ . And the construction above also implies that  $\mu(g) = (g_1, \ldots, g_t)$ .

Finally choose any  $V \in \mathfrak{A}_r(\Sigma)^Q$  and note that for any  $h \in \operatorname{Ker}\mu$ , Vh = V. Thus  $K = \operatorname{Ker}\mu \leq \operatorname{Stab}_{G_r(\Sigma)}(V)$  which is finite. Furthermore, for every  $g \in C_{G_r(\Sigma)}(Q)$ , the commutator [h, g] acts as the identity on V and hence every  $h \in K$  is central in  $C_{G_r(\Sigma)}(Q)$ .

Remark 4.6. In an analogous way, one can prove that there is also a group epimorphism

$$N_{G_r(\Sigma)}(Q) \twoheadrightarrow G_{r_1}(\Sigma) \times \ldots \times G_{r_t}(\Sigma)$$

with finite kernel.

and

#### 5. FINITENESS CONDITIONS IN BREDON COHOMOLOGY

In this section we collect all necessary background on Bredon cohomological finiteness conditions and also prove an analogue to Bieri's criterion for  $FP_n$ . Recall that Bredon cohomology with respect to the family of finite subgroups provides the algebraic mirror to classifying spaces for proper actions.

Let  $\mathcal{X}$  denote a family of subgroups of a given group G. In Bredon cohomology, the group G is replaced by the orbit category  $\mathcal{O}_{\mathcal{X}}G$ . The category  $\mathcal{O}_{\mathcal{X}}G$  has as objects the transitive G-sets with stabilisers in  $\mathcal{X}$ . Morphisms in  $\mathcal{O}_{\mathcal{X}}G$  are G-maps between those G-sets. Modules over the orbit category, called  $\mathcal{O}_{\mathcal{X}}G$ -modules are contravariant functors from the orbit category to the category of abelian groups. Exactness is defined pointwise: a sequence

$$A \to B \to C$$

of  $\mathcal{O}_{\mathcal{X}}G$ -modules is exact at B if and only if

$$A(\Delta) \to B(\Delta) \to C(\Delta)$$

is exact at  $B(\Delta)$  for every transitive G-set  $\Delta$ .

The category  $\mathcal{O}_{\mathcal{X}}G$ - $\mathfrak{Mod}$  of  $\mathcal{O}_{\mathcal{X}}G$ -modules has enough projectives, which are constructed as follows: For any G-sets  $\Delta$  and  $\Omega$ , denote by  $[\Delta, \Omega]$  the set of G-maps from  $\Delta$  to  $\Omega$ . Let  $\mathbb{Z}[\Delta, \Omega]$  be the free abelian group on  $[\Delta, \Omega]$ . One now obtains a  $\mathcal{O}_{\mathcal{X}}G$ -module  $\mathbb{Z}[-, \Omega]$  by fixing  $\Omega$  and letting  $\Delta$  range over the transitive G-sets with stabilisers in  $\mathcal{X}$ . A Yoneda-type argument, see [19], yields that these modules are free. In particular, the modules  $P^{K}(-) = \mathbb{Z}[-, G/K]$  for  $K \in \mathcal{X}$  are free and can be viewed as the building blocks for free  $\mathcal{O}_{\mathcal{X}}G$ -modules. Projective modules are now defined analogously to the ordinary case. The trivial  $\mathcal{O}_{\mathcal{X}}G$ -module, denoted  $\mathbb{Z}(-)$ , is the constant functor  $\mathbb{Z}$  from  $\mathcal{O}_{\mathcal{X}}G$  to the category of abelian groups.

Bieri [1] gives criteria for a  $\mathbb{Z}G$ -module to be of type  $\operatorname{FP}_n$  involving certain Ext- and Tor-functors to commute with exact colimits and direct products respectively. In this section we prove that those criteria can also be used for Bredon cohomology. The Bredon cohomology functors  $\operatorname{Ext}_{\mathfrak{X}}^*(M, -)$ are defined as derived functors of  $\operatorname{Hom}_{\mathfrak{X}}(M, -)$ . In particular, let  $M(-) \in \mathcal{O}_{\mathcal{X}}G$ - $\mathfrak{Mod}$  be a contravariant  $\mathcal{O}_{\mathcal{F}}G$ -module admitting a projective resolution  $P_*(-) \twoheadrightarrow M(-)$ . Then, for each  $N(-) \in \mathcal{O}_{\mathcal{X}}G$ - $\mathfrak{Mod}$ ,

$$\operatorname{Ext}_{\mathfrak{X}}^{*}(M,N) = H_{*}(mor(P_{*},N)).$$

One can also define Bredon homology functors  $\operatorname{Tor}^{\mathfrak{X}}_{*}(-, M)$ . In particular, analogously to the contravariant case, one can define covariant  $\mathcal{O}_{\mathcal{X}}G$ modules, or just comodules for short. The category of covariant  $\mathcal{O}_{\mathcal{X}}G$ modules, denoted  $\mathfrak{Mod}$ - $\mathcal{O}_{\mathcal{X}}G$ , behaves just as expected. For example, we have short exact sequences and enough projectives as above. In particular, the building blocks for projective modules in  $\mathfrak{Mod}$ - $\mathcal{O}_{\mathcal{X}}G$  are the covariant functors  $P_K(-) = \mathbb{Z}[G/K, -]$  for subgroups  $K \in \mathfrak{X}$ . Let  $M(-) \in \mathcal{O}_{\mathcal{X}}G$ - $\mathfrak{Mod}$ be as above. Then Bredon homology functors are the derived functors of  $-\otimes_{\mathfrak{X}} M$ , i.e., for any  $L(-) \in \mathfrak{Mod}$ - $\mathcal{O}_{\mathcal{X}}G$ ,

$$\operatorname{Tor}_{*}^{\mathfrak{t}}(L, M) = H_{*}(L \otimes_{\mathfrak{X}} P_{*}).$$

For detail on these definitions including the categorical tensor product and Yoneda-type isomorphism the reader is referred to [20]. In particular,  $\operatorname{Tor}_*^{\mathfrak{X}}(-, M)$  can be calculated using flat resolutions of M(-).

The category of  $\mathcal{O}_{\mathcal{X}}G$ -modules, as an abelian category, has well defined colimits and limits and in particular coproducts and products. We say a functor

 $T: \mathcal{O}_{\mathcal{X}}G\operatorname{-\mathfrak{Mod}} \to \mathcal{A}b$ 

commutes with exact colimits, denoted here by  $\varinjlim$ , if, for every directed system  $(M_{\lambda})_{\lambda \in \Lambda}$  of  $\mathcal{O}_{\mathcal{X}}G$ -modules, the natural map

 $\lim T(M_{\lambda}) \to T(\lim M_{\lambda})$ 

is an isomorphism. Analogously, we say a functor

$$S:\mathfrak{Mod} extsf{-}\mathcal{O}_\mathcal{X}G o\mathcal{A}b$$

commutes with exact limits, denoted here by  $\lim_{\leftarrow}$ , if, for every inverse system  $(N_{\lambda})_{\lambda \in \Lambda}$  of  $\mathcal{O}_{\mathcal{X}}G$ -comodules, the natural map

$$S(\lim N_{\lambda}) \to \lim S(N_{\lambda})$$

is an isomorphism.

We say an  $\mathcal{O}_{\mathcal{X}}G$ -module M(-) is finitely generated if there is a finitely generated free module mapping onto it. In particular, there is a *G*-finite *G*-set  $\Delta$  such that  $\mathbb{Z}[-, \Delta] \twoheadrightarrow M(-)$  (here we are extending the notation  $\mathbb{Z}[-, \Delta]$  to non transitive sets in the obvious way).

**Lemma 5.1.** Let M be an  $\mathcal{O}_{\mathcal{X}}G$ -module. Then M is the direct colimit of its finitely generated submodules.

 $\Box$ 

*Proof.* This follows from  $[13, \S 9.19]$ .

The notions of type Bredon-FP, Bredon-FP<sub>n</sub> and Bredon-FP<sub> $\infty$ </sub> are defined in terms of projective resolutions over  $\mathcal{O}_{\mathcal{X}}G$  analogously to the classical notions of type FP, FP<sub>n</sub> and FP<sub> $\infty$ </sub>.

**Proposition 5.2.** Let A be an  $\mathcal{O}_{\mathcal{X}}G$ -module of type Bredon-FP<sub>n</sub>,  $0 \leq n \leq \infty$ . Then

(i) For every exact limit, the natural homomorphism

$$\operatorname{Tor}_{k}^{\mathfrak{X}}(\lim N_{*}, A) \to \lim \operatorname{Tor}_{k}^{\mathfrak{X}}(N_{*}, A)$$

is an isomorphism for all  $k \leq n-1$  and an epimorphism for k = n. (ii) For every exact colimit, the natural homomorphism

$$\varinjlim \operatorname{Ext}^k_{\mathfrak{X}}(A, M_*) \to \operatorname{Ext}^k_{\mathfrak{X}}(A, \varinjlim M_*)$$

is an isomorphism for all  $k \leq n-1$  and a monomorphism for k = n.

*Proof.* The proof goes completely analogously to that of Bieri [1, Proposition 1.2]. It relies on the Yoneda isomorphisms, i.e that  $N \otimes_{\mathfrak{X}} \mathbb{Z}[-, G/K] \cong N(G/K)$  and  $\operatorname{Hom}_{\mathfrak{X}}(\mathbb{Z}[-, G/K], M) \cong M(G/K)$ , the fact that  $\lim_{K \to \infty} \mathbb{Z}[-, G/K] \cong \mathbb{Z}[-, M]$  commute with finite direct sums and that  $\lim_{K \to \infty} \mathbb{Z}[-, G/K] \cong \mathbb{Z}[-, M]$  exact and hence commute with the homology functor.

Bieri's argument can be carried through completely for Bredon-Ext and Bredon-Tor functors.

## **Theorem 5.3.** Let A be an $\mathcal{O}_{\mathcal{X}}G$ -module. Then the following are equivalent:

- (i) A is of type  $Bredon-FP_n$ .
- (ii) For every exact colimit, the natual homomorphism

 $\lim_{\mathfrak{X}} \operatorname{Ext}_{\mathfrak{X}}^{k}(A, M_{*}) \to \operatorname{Ext}_{\mathfrak{X}}^{k}(A, \lim_{\mathfrak{X}} M_{*})$ 

is an isomorphism for all  $k \leq n-1$  and a monomorphism for k = n.

(iii) For the direct limit of any directed system of  $\mathcal{O}_{\mathcal{X}}G$ -modules  $M_*$  with  $\varinjlim M_* = 0$ , one has  $\varinjlim \operatorname{Ext}^k_{\mathfrak{X}}(A, M_*) = 0$ , for all  $k \leq n$ .

*Proof.* The implications  $(i) \Rightarrow (ii) \Rightarrow (iii)$  are either obvious or follow from Proposition 5.2. Every  $\mathcal{O}_{\mathcal{X}}G$ -module is the directed colimit of finitely generated submodules, Lemma 5.1, and hence  $(iii) \Rightarrow (i)$  is proved completely analogously to [1, Theorem 1.3  $(iiib) \Rightarrow (i)$ ].

**Theorem 5.4.** Let A be an  $\mathcal{O}_{\mathcal{X}}G$ -module. Then the following are equivalent:

- (i) A is of type Bredon-FP<sub>n</sub>.
- (ii) For every exact limit, the natural homomorphism

$$\operatorname{Tor}_{k}^{\mathfrak{X}}(\varprojlim N_{*}, A) \to \varprojlim \operatorname{Tor}_{k}^{\mathfrak{X}}(N_{*}, A)$$

is an isomorphism for all  $k \leq n-1$  and an epimorphism for k = n.

(iii) For any  $K \in \mathfrak{X}$  consider any arbitrary direct product  $\prod_{\Lambda_K} \mathbb{Z}[G/K, -]$ . Then the natural map

$$\operatorname{Tor}_{k}^{\mathfrak{X}}(\prod_{K\in\mathfrak{X}}\prod_{\Lambda_{K}}\mathbb{Z}[G/K,-],A)\to\prod_{K\in\mathfrak{X}}\prod_{\Lambda_{K}}\operatorname{Tor}_{k}^{\mathfrak{X}}(\mathbb{Z}[G/K,-],A)$$

is an isomorphism for all  $k \leq n-1$  and an epimorphism for k = n.

*Proof.* The implications  $(i) \Rightarrow (ii) \Rightarrow (iii)$  are again either obvious or consequence of Proposition 5.2.

 $(iii) \Rightarrow (i)$ : The proof is in the same spirit as Bieri's proof. We begin by letting n = 0 and claim that A is finitely generated as an  $\mathcal{O}_{\mathcal{X}}G$ -module. As an index set we take  $\prod_{K \in \mathfrak{X}} A(G/K)$  and consider  $\prod_{K \in \mathfrak{X}} \prod_{a \in A(G/K)} \mathbb{Z}[G/K, -]$ . By (iii), the natural map

$$\mu: (\prod_{K \in \mathfrak{X}} \prod_{A(G/K)} \mathbb{Z}[G/K, -]) \otimes_{\mathfrak{X}} A(-) \to \prod_{A(G/K)} A(G/K)$$

is an epimorphism. Let c be the element with  $\mu(c) = \prod_{K \in \mathfrak{X}} \prod_{a \in A(G/K)} a$ . Then c is of the form

$$c = \sum_{i=1}^{\iota} (\prod_{K \in \mathfrak{X}} \prod_{A(G/K)} f_i^{a,K}) \otimes b_i,$$

for certain subgroups  $H_1, \ldots, H_l \in \mathfrak{X}$  and elements  $b_i \in A(G/H_i)$ . Here,  $f_i^{a,K} \in \mathbb{Z}[G/K, G/H_i]$ . Now we claim that there is an epimorphism

$$\tau: \bigoplus_{i=1}^{\iota} \mathbb{Z}[-, G/H_i] \twoheadrightarrow A$$

given by  $\tau(f) := f^*(b_i) \in A(G/K)$  whenever  $f \in \mathbb{Z}[G/K, G/H_i]$ . Observe that this is well defined. In particular it is functorial. To prove the claim, take any  $K \in \mathfrak{X}$  and any  $a \in A(G/K)$ . Note that

$$\mu(c) = \sum_{i=1}^{l} \prod_{K \in \mathfrak{X}} \prod_{a \in A(G/K)} (f_i^{a,K})^*(b_i) = \prod_{K \in \mathfrak{X}} \prod_{a \in A(G/K)} \sum_{i=1}^{l} (f_i^{a,K})^*(b_i)$$

so the fact that c maps onto the diagonal means that

$$a = \sum_{i=1}^{l} (f_i^{a,K})^*(b_i) = \tau(\sum_{i=1}^{l} f_i^{a,K}).$$

The case  $n \ge 1$  is now done analogously to [1, Theorem 1.3] using a diagram chase.

Remark 5.5. For  $n \geq 1$ , condition (iii) is equivalent to the following, which in ordinary homology is often referred to as the Bieri-Eckmann criterion for  $\operatorname{FP}_n$ : For every subgroup  $K \in \mathfrak{X}$  consider an arbitrary direct product  $\prod_{\Lambda_K} \mathbb{Z}[G/K, -]$ . Then the natural map

$$\prod_{K \in \mathfrak{X}} \prod_{\Lambda_K} \mathbb{Z}[G/K, -] \otimes_{\mathfrak{X}} A(-) \to \prod_{K \in \mathfrak{X}} \prod_{\Lambda_K} A(G/K)$$

is an isomorphism and  $\operatorname{Tor}_{k}^{\mathfrak{X}}(\prod_{K \in \mathfrak{X}} \prod_{\Lambda_{K}} \mathbb{Z}[G/K, -], A) = 0$ , for all  $1 \leq k \leq n-1$ . We call this condition the global Bieri-Eckmann criterion for Bredon homology.

We say a group satisfies the local Bieri-Eckmann criterion for Bredon cohomology if, for any K and direct product as before, the natural map

$$\prod_{\Lambda_K} \mathbb{Z}[G/K, -] \otimes_{\mathfrak{X}} A(-) \to \prod_{\Lambda_K} A(G/K)$$

is an isomorphism and  $\operatorname{Tor}_{k}^{\mathfrak{X}}(\prod_{\Lambda_{K}}\mathbb{Z}[G/K, -], A) = 0$  for all  $1 \leq k \leq n-1$ .

#### 6. Classifying spaces with finite isotropy

In this section we shall restrict ourselves to the family  $\mathcal{F}$  of all the finite subgroups of G.

To stay in line with notation previously used, we say a module is of type  $\underline{FP}_{\infty}$  if it is of type Bredon- $FP_{\infty}$  with respect to  $\mathcal{F}$ . The notions of  $\underline{FP}_n$  and  $\underline{FP}$  are defined analogously. For Bredon cohomology with respect to  $\mathcal{F}$  there is a good algebraic description for modules of type  $\underline{FP}_n$ . For the original approach via classifying spaces, see [14].

**Theorem 6.1.** [11] Let G be a group having finitely many conjugacy classes of finite subgroups. Then an  $\mathcal{O}_{\mathcal{F}}G$ -module M(-) is of type  $\underline{\mathrm{FP}}_n$  if and only if M(G/K) is of type  $\mathrm{FP}_n$  as a  $\mathbb{Z}(WK)$ -module for each finite subgroup K of G.

It was also shown, [11], that a group G is of type <u>FP</u><sub>0</sub> if and only if G has finitely many conjugacy classes of finite subgroups. Hence we have the following corollary:

**Corollary 6.2.** [11] A group G is of type  $\underline{FP}_n$  if and only if G has finitely many conjugacy classes of finite subgroups and  $C_G(K)$  is of type  $FP_n$  for every finite subgroup K of G.

Recall that we say a group G is of Bredon-type  $\operatorname{FP}_n$  if the trivial module  $\mathbb{Z}(-)$  is of type  $\operatorname{FP}_n$  as an  $\mathcal{O}_{\mathcal{X}}G$ -module. We can, of course rephrase Theorems 5.3 and 5.4 in terms of Bredon-cohomology and Bredon-homology replacing the module A(-) with  $\mathbb{Z}(-)$ ,  $\operatorname{Ext}^*_{\mathfrak{X}}(A,-)$  with  $\operatorname{H}^*_{\mathfrak{X}}(G,-)$  and  $\operatorname{Tor}^{\mathfrak{X}}_{\ast}(-,A)$  with  $\operatorname{H}^*_{\ast}(G,-)$ .

We shall now weaken the hypothesis on the conjugacy classes of finite subgroups:

**Definition 6.3.** We say a group is of type quasi- $\underline{FP}_n$  if, for each finite subgroup K of G there are finitely many conjugacy classes of subgroups isomorphic to K and the Weyl-groups  $WK := N_G(K)/K$  are of type  $FP_n$ .

Note that a group of type quasi- $\underline{FP}_n$  with a bound on the orders of the finite subgroups is of type  $\underline{FP}_n$ .

Let k be a positive integer. We denote by  $\mathbb{Z}_k(-)$  the  $\mathcal{O}_{\mathcal{F}}G$ -module defined by

$$\mathbb{Z}_k(G/H) = \begin{cases} \mathbb{Z} \text{ if } |H| \le k \\ 0 \text{ otherwise,} \end{cases}$$

together with the obvious morphisms.

**Lemma 6.4.** A group G is of type quasi- $\underline{FP}_0$  if and only if, for each  $k \ge 1$ , the module  $\mathbb{Z}_k(-)$  is finitely generated. Moreover, in that case, the finite G-set  $\Delta_k$  with  $\mathbb{Z}[-, \Delta_k] \twoheadrightarrow \mathbb{Z}_k(-)$  can be chosen to have stabilisers of order bounded by k.

*Proof.* Suppose G is of type quasi- $\underline{FP}_0$ . Take

$$\Delta_k = \bigsqcup_{|H| \le k, \text{up to } G\text{-conj.}} G/H.$$

This is a *G*-finite *G*-set with stabilisers of order bounded by k and  $\mathbb{Z}[-, \Delta_k] \twoheadrightarrow \mathbb{Z}_k(-)$ . For the converse, we need to show that, for each finite subgroup K, there are only finitely many conjugacy classes of subgroups of order bounded by k = |K|. Let  $\Delta_k$  be the finite *G*-set with  $\mathbb{Z}[-, \Delta_k] \twoheadrightarrow \mathbb{Z}_k(-)$  and take any finite subgroup *H* of *G* with  $|H| \leq k$ . Hence  $\mathbb{Z}_k(G/H) \cong \mathbb{Z} \neq 0$ . Since the map  $\mathbb{Z}[G/H, \Delta] \twoheadrightarrow \mathbb{Z}_k(G/H)$  is onto, it follows that  $\mathbb{Z}[G/H, \Delta] \neq 0$  and hence *H* has to be subconjugated to one of the finitely many stabilisers of  $\Delta$ .

Note that finitely generated  $\mathcal{O}_{\mathcal{F}}G$ -modules are precisely those of type  $\underline{\mathrm{FP}}_0$ . Fix an integer  $k \geq 1$  and let  $M_k(-)$  be an  $\mathcal{O}_{\mathcal{F}}G$ -module such that  $M_k(G/L) = 0$  whenever |L| > k. Suppose  $M_k(-)$  is finitely generated. Then there exists a *G*-finite *G*-set  $\Delta$  with stabilisers of order  $\leq k$  and a short exact sequence of  $\mathcal{O}_{\mathcal{F}}G$ -modules

$$N_k(-) \rightarrow \mathbb{Z}[-,\Delta] \twoheadrightarrow M_k(-)$$

with the property that  $N_k(G/L) = 0$  for all finite subgroups L with |L| > k.

**Proposition 6.5.** A group G is of type quasi- $\underline{FP}_n$  if and only if, for each integer  $k \ge 1$ , the  $\mathcal{O}_{\mathcal{F}}G$ -module  $\mathbb{Z}_k(-)$  is of type  $\underline{FP}_n$ .

*Proof.* The "if"-direction follows from Lemma 6.4, Theorem 6.1 and the definition as  $\mathbb{Z}_{|K|}(G/H)$  is of type FP<sub>n</sub> as a WH-module for each  $|H| \leq |K|$ . Now suppose G is of type quasi-<u>FP</u><sub>n</sub>. For each  $k \geq 1$  we construct a projective resolution of  $\mathbb{Z}_k(-)$  which is finitely generated in dimensions up to n; note that we may assume n > 0. By Lemma 6.4 and the above remark we have a short exact sequence

$$C_0(-) \rightarrow \mathbb{Z}[-, \Delta_0] \twoheadrightarrow \mathbb{Z}_k(-)$$

with  $\Delta_0$  a *G*-finite *G*-set and  $C_0(G/L) = 0$  for all |L| > k. We claim that  $C_0(-)$  is a finitely generated  $\mathcal{O}_{\mathcal{F}}G$ -module.

We know that there are finitely many conjugacy classes of subgroups of order bounded by k. Let H be one of those. As  $\Delta_0$  is G-finite, the WHmodule  $\mathbb{Z}[G/H, \Delta_0]$  is of type  $\operatorname{FP}_{\infty}$ . This is a consequence of the fact that for any K,  $\mathbb{Z}[G/H, G/K]$  is a sum of exactly  $|\{x \in N_G(H) \setminus G/K \mid H^{x^{-1}} \leq K\}|$ WH-modules, which are of type  $\operatorname{FP}_{\infty}$ . As K is finite, this sum must also be finite. So evaluating the previous short exact sequence at G/H, we see that the WH-module  $C_0(G/H)$  is of type  $\operatorname{FP}_{n-1}$  and in particular, finitely generated. Fix a finite WH-generating set  $X_H$  for  $C_0(G/H)$ . Then the  $\mathcal{O}_{\mathcal{F}}G$ -set formed by the union of all those  $X_H$  where  $H \in \operatorname{Stab}\Delta_0$ , generates  $C_0$ .

We can now proceed to construct the desired resolution by using the remark before Proposition 6.5.  $\hfill \Box$ 

**Theorem 6.6.** Let G be of type quasi- $\underline{FP}_n$ , where  $n \ge 1$ . Then G satisfies the local Bieri-Eckmann criterion for Bredon homology.

*Proof.* It follows from the definition of the modules  $\mathbb{Z}_k(-)$  that

$$\mathbb{Z}(-) = \lim_{k \in \mathbb{N}} \mathbb{Z}_k(-).$$

In the category of  $\mathcal{O}_{\mathcal{F}}G$ -modules the construction of a free module mapping onto a given one is functorial. Hence, we can get a direct colimit of free resolutions  $\varinjlim_{k\in\mathbb{N}}(F_{*,k}(-) \twoheadrightarrow \mathbb{Z}_k(-)) = F_*(-) \twoheadrightarrow \mathbb{Z}(-)$ , which gives us a flat resolution of  $\mathbb{Z}(-)$ . For details the reader is referred to [20, Lemma 3.4]. Hence

$$\begin{split} \mathrm{H}_{k}^{\mathcal{F}}(G, \prod_{\Lambda} \mathbb{Z}[G/K, -]) &= \mathrm{H}_{*}(\prod_{\Lambda} \mathbb{Z}[G/K, -] \otimes_{\mathcal{F}} F_{*}(-)) \\ &= \mathrm{H}_{k}(\prod_{\Lambda} \mathbb{Z}[G/K, -] \otimes_{\mathcal{F}} \lim_{k \in \mathbb{N}} F_{*,k}(-)) \\ &= \lim_{k \in \mathbb{N}} \mathrm{H}_{k}(\prod_{\Lambda} \mathbb{Z}[G/K, -] \otimes_{\mathcal{F}} F_{*,k}(-)) \\ &= \lim_{k \in \mathbb{N}} \mathrm{Tor}_{k}(\prod_{\Lambda} \mathbb{Z}[G/K, -], \mathbb{Z}_{k}(-)) = 0 \quad , \end{split}$$

where the last line follows from Proposition 6.5 and Theorem 5.4. The first assertion follows by a similar argument.

For each  $k \geq 1$  we consider the family  $\mathcal{F}_k$  and the orbit category  $\mathcal{O}_{\mathcal{F}_k}G$ . For a given positive integer k the family  $\mathcal{F}_k$  consists of all subgroups H of G with  $|H| \leq k$ . By using the arguments of the proofs of Lemma 6.4 and Proposition 6.5 we can show:

**Proposition 6.7.** A group is of type quasi- $\underline{FP}_n$  if and only if it is of type Bredon- $FP_n$  over  $\mathcal{O}_{\mathcal{F}_k}G$  for each k.

We can also rephrase Theorems and 5.3 and 5.4:

**Corollary 6.8.** Let G be a group. Then the following are equivalent:

- (i) G is of type quasi- $\underline{FP}_n$ .
- (ii) For every exact colimit and any k, the natual homomorphism

$$\varinjlim \mathrm{H}^{l}_{\mathfrak{F}_{k}}(G, M_{*}) \to \mathrm{H}^{l}_{\mathfrak{F}_{k}}(G, \varinjlim M_{*})$$

is an isomorphism for all  $l \leq n-1$ , and a monomorphism for l = n.

(iii) For any k and any  $K \in \mathfrak{F}_k$  consider an arbitrary direct product  $\prod_{\Lambda_K} \mathbb{Z}[G/K, -]$ . Then the natural map

$$\mathrm{H}_{l}^{\mathfrak{F}_{k}}(\prod_{K\in\mathfrak{F}_{k}}\prod_{\Lambda_{K}}\mathbb{Z}[G/K,-],A)\rightarrow\prod_{K\in\mathfrak{F}_{k}}\prod_{\Lambda_{K}}\mathrm{H}_{l}^{\mathfrak{F}_{k}}(\mathbb{Z}[G/K,-],A)$$

is an isomorphism for all  $l \leq n-1$  and an epimorphism for l = n.

One may also add the statements analogous to 5.3 ii) and 5.4 ii). Note also that for  $n \ge 1$  the above is equivalent to:

(iv) For any k, any  $K \in \mathfrak{F}_k$  and any arbitrary direct product  $\prod_{\Lambda_K} \mathbb{Z}[G/K, -]$ , the natural map

$$\mathbb{Z}_k(-) \otimes_{\mathfrak{F}_k} \prod_{K \in \mathfrak{F}_k} \prod_{\Lambda_K} \mathbb{Z}[G/H, -] \to \prod_{K \in \mathfrak{F}_k} \prod_{\Lambda_K} \mathbb{Z}_k$$

is an isomorphism and  $\operatorname{H}_{l}^{\mathfrak{F}_{k}}(G, \prod_{K \in \mathfrak{F}_{k}} \prod_{\Lambda_{K}} \mathbb{Z}[G/H, -]) = 0$  for all  $1 \leq l \leq n-1$ .

**Definition 6.9.** We say a group G is of type quasi- $\underline{F}_{\infty}$  if for all positive integers k, G admits a finite type model for  $E_{\mathfrak{F}_k}G$ .

Analogously to the algebraic case, any group of type quasi- $\underline{F}_{\infty}$ , which has a bound on the orders of the finite subgroups, is of type  $\underline{F}_{\infty}$ .

Lück's Theorem [14, Theorem 4.2] (Theorem 1.2) goes through for arbitrary families of finite subgroups. Hence combining Theorems 1.1 and 1.2 yields:

**Proposition 6.10.** A group G is of type quasi- $\underline{F}_{\infty}$  if and only if G is of type quasi- $\underline{FP}_{\infty}$  and G and all centralisers  $C_G(K)$  of finite subgroups are finitely presented.

We can now prove what is largely equivalent to Proposition 6.7:

**Theorem 6.11.** A group G is of type quasi- $\underline{F}_{\infty}$  if and only if it admits a model for  $\underline{E}G$ , which is the mapping telescope of finite type models for  $E_{\mathfrak{F}_k}G$  for each  $K \in \mathcal{F}$ .

*Proof.* The "if'-direction follows directly from the definition. Now suppose we have finite type models  $X_K$  for  $E_{\mathfrak{F}_k}G$  for all  $K \in \mathcal{F}$ . For each  $H \leq$ K the universal property for classifying spaces for a family yields G-maps  $\nu_H^K: X_H \to X_K$ . Now the mapping telescope yields a G-CW-complex X, for which  $X^K$  is contractible for all  $K \in \mathcal{F}$  and empty otherwise. 

#### 7. Bredon cohomological finiteness properties for generalised THOMPSON-HIGMAN GROUPS

We can now prove one of our main results.

**Theorem 7.1.** The following conditions are equivalent:

- i)  $G_r(\Sigma)$  is quasi-<u>FP</u><sub> $\infty$ </sub> for any  $1 \le r \le d$ .
- ii)  $G_r(\Sigma)$  is quasi- $\underline{\underline{FP}}_{\infty}$  for some  $1 \leq r \leq d$ .
- iii)  $G_r(\Sigma)$  is of (ordinary) type  $\operatorname{FP}_{\infty}$  for any  $1 \leq r \leq d$ .

*Proof.* Assume that ii) holds. For any  $1 \le r_2 \le d$  choose some  $0 \ne r_1$  with  $r_1 + 2r_2 \equiv r \mod d$ . Then there is some admissible subset Y of cardinality  $r_1 + 2r_2$  and we may define an action of  $C_2$  on Y with exactly  $r_1$  fixed points. This yields a subgroup  $C_2 \cong Q \leq G_r(\Sigma)$  and by 4.5, for some finite K we have

$$C_{G_r(\Sigma)}(Q)/K \cong G_{r_1}(\Sigma) \times G_{r_2}(\Sigma).$$

So we deduce that  $C_{G_r(\Sigma)}(Q)$  and thus  $G_{r_2}(\Sigma)$  is of type  $FP_{\infty}$ . So we have iii).

Assume now iii) and choose any  $1 \leq r \leq d$ . Then Theorem 4.3 and Theorem 4.5 imply that  $G_r(\Sigma)$  is quasi-<u>FP</u><sub> $\infty$ </sub>

The analogue to Theorem 7.1 also holds for classifying spaces for proper actions. The following corollary is a direct consequence of Proposition 6.10, Theorem 4.5 and Theorem 7.1.

Corollary 7.2. The following conditions are equivalent:

- i)  $G_r(\Sigma)$  is quasi- $\underline{\mathbf{F}}_{\infty}$  for any  $1 \leq r \leq d$ . ii)  $G_r(\Sigma)$  is quasi- $\underline{\mathbf{F}}_{\infty}$  for some  $1 \leq r \leq d$ .
- iii)  $G_r(\Sigma)$  is of (ordinary) type  $F_{\infty}$  for any  $1 \leq r \leq d$ .

**Corollary 7.3.** Let  $G = nV_{2,r}$  be a Brin-Thompson-Higman group of arity 2. Then G is quasi- $\underline{FP}_{\infty}$  or quasi- $\underline{F}_{\infty}$  if and only if it is of ordinary type  $FP_{\infty}$  or  $F_{\infty}$  respectively. In particular, for n = 2, 3 G is quasi- $\underline{F}_{\infty}$ .

*Proof.* Note that d = 1. For the last assertion, use the main result of [10].

To consider finiteness conditions for the groups  $T_r(\Sigma)$  we need to consider Cantor-Algebras  $U_r(\Sigma)$ , where  $\Sigma$  preserves the induced ordering. Any finite subgroup  $Q \leq T_r(\Sigma)$  is cyclic. In particular, following the argument of Proposition 4.2, we get that  $\pi(Q) = \{0, ..., 0, r_i, 0, ...0\}$  with  $r_i w_i \equiv r \mod d$ . Furthermore, there is a  $Y \in \mathfrak{A}_r(\Sigma)^Q$  with  $|Y| = r_i w_i$  such that any other element in  $\mathfrak{A}_r(\Sigma)^Q$  can be obtained from Y by a finite sequence of simple Q-expansions and Q-contractions. Applying Lemma 4.4 hence proves:

**Lemma 7.4.** Let  $U_r(\Sigma)$  be a Cantor-Algebra with order preserving  $\Sigma$ . Let  $Q \leq T_r(\Sigma)$  be a finite subgroup. Then, for a certain  $0 < l \leq d$ , there is a poset isomorphism  $\mathfrak{A}_r(\Sigma)^Q \cong \mathfrak{A}_l(\Sigma)$ .

**Theorem 7.5.** Let  $U_r(\Sigma)$  be a Cantor-Algebra with order preserving  $\Sigma$  and let  $Q \leq T_r(\Sigma)$  be a finite subgroup. Then, for a certain  $0 < l \leq d$ , depending on Q there is a central extension

$$K \rightarrow C_{T_r(\Sigma)}(Q) \twoheadrightarrow T_l(\Sigma)$$

with K a cyclic group of finite order.

*Proof.* The proof is essentially the same as the proof of Theorem 4.5.  $\Box$ 

*Remark* 7.6. For  $U_r(\Sigma) = V_{2,1}$ , the Higman algebra and  $T_r(\Sigma) = T$ , the original Thompson-group T this reproves [17, Theorem 7.1.5].

**Lemma 7.7.** For every finite subgroup  $Q \leq T_r(\Sigma)$  there are only finitely many conjugacy classes in  $T_r(\Sigma)$  of subgroups isomorphic to Q.

Proof. Again, the proof goes through exactly as in the case for  $G_r(\Sigma)$ , see Theorem 4.3. In particular, if  $Q_1 \cong Q_2$  and  $\pi(Q_1) = \pi(Q_2)$  then there are admissible subsets  $V_1$  and  $V_2$  with the same structure as  $Q_i$ -sets, then there is a cyclic order preserving bijection  $\psi: V_1 \to V_2$ . The rest of the proof goes through as in 4.3.

**Theorem 7.8.** Let  $U_r(\Sigma)$  be a Cantor-Algebra with order preserving  $\Sigma$ . Then the following conditions are equivalent:

- i)  $T_r(\Sigma)$  is quasi- $\underline{F}_{\infty}$  for any  $1 \leq r \leq d$ .
- ii)  $T_r(\Sigma)$  is quasi- $\underline{\mathbf{F}}_{\infty}$  for some  $1 \leq r \leq d$ .
- iii)  $T_r(\Sigma)$  is of (ordinary) type  $F_{\infty}$  for any  $1 \le r \le d$ .

*Proof.* This follows from Theorem 7.5 and Lemma 7.7.

We also have the analogous result of Corollary 7.2.

**Corollary 7.9.** Let  $U_r(\Sigma)$  be a Higman algebra. Then  $G_{n,r} = G_r(\Sigma)$  and  $T_{n,r} = T_r(\Sigma)$  are quasi- $\underline{F}_{\infty}$ .

*Proof.* This follows directly from [5] and Theorems 7.1 and 7.8.

Remark 7.10. By [5], Proposition 4.1,  $F_{n,r} \cong F_{n,s}$ , for any r, s. However, this is false for the groups G, in fact  $G_{n,r} \cong G_{n,s}$  implies (n-1,r) = (n-1,s) ([8] Theorem 6.4). Recently, Pardo has proven that the converse also holds true ([21]).

**Corollary 7.11.** Let  $U_r(\Sigma)$  be a Brown-Stein algebra. Then  $G = G_r(\Sigma)$ and  $T = T_r(\Sigma)$  are quasi- $\underline{F}_{\infty}$ .

*Proof.* This is a consequence of Theorems 7.1, 7.8 and [22, Theorem 2.5] where it is proven that  $F_r(\Sigma)$  is finitely presented and of type  $FP_{\infty}$  for any r. Stein's argument carries over to G and T, [22].

**Corollary 7.12.** Brin groups 2V and 3V are quasi- $\underline{F}_{\infty}$ .

*Proof.* This follows from [10] and Corollary 7.2.

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Conchita Martínez-Pérez, Departamento de Matemáticas, Universidad de Zaragoza, 50009 Zaragoza, Spain *E-mail address*: conmar@unizar.es

BRITA E. A. NUCINKIS, SCHOOL OF MATHEMATICS, UNIVERSITY OF SOUTHAMPTON, Southampton, SO17 1BJ, United Kingdom

 $E\text{-}mail\ address: \texttt{bean@soton.ac.uk}$