CENTRALISERS OF FINITE SUBGROUPS IN SOLUBLE GROUPS OF TYPE FP_n

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Abstract. We show that for soluble groups of type FP_n, centralisers of finite subgroups need not be of type FP_n.

1. Introduction

We study stabilizers of finite groups acting on soluble groups of type FP_n. Our interest in this problem derives from the study of groups G of type Bredon-FP_d with respect to the family $\mathcal{F}$ of all finite subgroups of G. For the family $\mathcal{F}$ of finite subgroups, the classifying space for proper actions, denoted by $\underline{E}G$, has been widely studied. In particular, Lück has shown [16] that G admits a cocompact model for $\underline{E}G$ if and only if G has finitely many conjugacy classes of finite subgroups and the stabilizer in G of every finite subgroup is finitely presented and of type FP_∞. Bredon cohomology with respect to the family of all finite subgroups can be viewed as the algebraic mirror to classifying spaces for proper actions. Its properties are similar to those of ordinary cohomology, the mirror for Eilenberg MacLane spaces. The following result is an algebraic version of Lück's result, generalised to arbitrary d, and serves as the main motivation for this paper.

Theorem 1.1. [14, Lemma 3.1] *A group is of type Bredon-*FP_n *if and only if it has finitely many conjugacy classes of finite subgroups and centralisers of finite subgroups are of type* FP_n.

Recently it was proved that every virtually soluble group of type FP_∞ is of type Bredon-FP_∞ [17]. Here we show that the equivalent statement does not hold for type FP_n using methods from Σ-theory developed by R. Bieri, J. Groves, R. Strebel and others. Even virtually metabelian groups of type FP_n are not necessarily of type Bredon-FP_n. In sections 4.1 and 4.2 we present two types of examples for which we calculate the homological type of the centralizers of finite actions. In these examples G is an extension of A by Q where A and Q are abelian groups with Q of rank n, G is of type FP_n but not FP_{n+1} and there is a finite group H of order n acting on G. Furthermore this A has Krull dimension 1 as a $\mathbb{Z}Q$-module. We show that if H_0 is a subgroup of index d in H then $C_G(H_0)$ is of type FP_d but not FP_{d+1}. In the case when A is of prime exponent, the finite group H of our examples is cyclic. But in the case when A is torsion-free, H can be any

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finite group which is realisable as a Galois group of a finite extension over $\mathbb{Q}$. In particular H can be any symmetric group.

In addition to these examples we prove some positive results describing the finiteness conditions of centralisers of finite subgroups. In particular, we show in Theorem 3.5 that a metabelian-by-finite group G of type FP_n with finite Prüfer rank is of type Bredon FP_d for $d = \lfloor \frac{n}{s} \rfloor$, where s denotes the upper bound on the orders of the finite subgroups of G (see 3.4). Some partial results concerning Bredon homological type FP_k for soluble-by-finite groups of finite Prüfer rank are included in section 5.

2. PRELIMINARIES ON THE BIERI-STREBEL SIGMA INVARIANT

Let G be a finitely generated group. The character sphere $S(G)$ of G is defined by

$$S(G) = (Hom_{\mathbb{Z}}(G, \mathbb{R}) \smallsetminus \{0\}) / \sim,$$

where $\mathbb{R}$ is considered as a group via addition and $\sim$ is the equivalence relation where $\chi_1 \sim \chi_2$ if there is a positive real number r such that $r\chi_1 = \chi_2$. We write $[\chi]$ for the class of χ in $S(G)$ and denote

$$G_\chi = \{g \in G | \chi(g) \geq 0\}.$$

Note that G_χ is a submonoid of G.

Let Q be a finitely generated abelian group and A be a $\mathbb{Z}Q$-module. The Bieri-Strebel invariant [5] is defined as

$$\Sigma_A(Q) = \{[\chi] \in S(Q) | A \text{ is finitely generated over } \mathbb{Z}Q_\chi\}.$$

Complements in the character sphere are denoted as follows:

$$\Sigma_A^c(Q) = S(Q) \smallsetminus \Sigma_A(Q).$$

We say that A is m-tame as a $\mathbb{Z}Q$-module if whenever $[\chi_1], \ldots, [\chi_m] \in \Sigma_A^c(Q)$ we have $\chi_1 + \ldots + \chi_m \neq 0$. The following conjecture was suggested in [2] after R. Bieri and R. Strebel had already resolved the case $m = 2$ [5].

The FP_m-Conjecture. *Let* $1 \to A \to G \to Q \to 1$ *be a short exact sequence of groups with* G *finitely generated and* A *and* Q *abelian. Then* G *is of type* FP_m *if and only if* A *is* m*-tame as a* $\mathbb{Z}Q$*-module.*

Though the FP_m-Conjecture is still open in general, it was shown to hold for metabelian groups of finite Prüfer rank [1] (recall that a group is said to be of finite Prüfer rank if there is an upper bound on the number of generators of the finitely generated subgroups).

3. METABELIAN GROUPS OF FINITE PRÜFER RANK

Let Q be a finitely generated abelian group acting on an abelian group A which is finitely generated as a $\mathbb{Z}Q$-module. If a group H acts on both Q and A we say that the actions are compatible if

$$(a^q)^h = (a^h)^{q^h}$$

for any $a \in A$, $q \in Q$, $h \in H$. Note that there is an induced action of H on the valuation sphere such that for $h \in H$, $[v^h]$ is given by

$$v^h(q) := v(q^{h^{-1}}).$$

If the actions of H and Q on A are compatible then $\Sigma^c_A(Q)$ is H-invariant (see [17, 3.4]).

Lemma 3.1. *Let $1 \to A \to G \to Q \to 1$ be a short exact sequence of groups with G finitely generated and A and Q abelian. Let H be a finite group acting on G such that A is H-invariant. Suppose A is n-tame as a $\mathbb{Z}Q$-module, where Q acts on A by conjugation and d is a positive integer such that*

$$d|H| \leq n.$$

Then $A_0 = C_A(H)$ is a finitely generated and d-tame $\mathbb{Z}C_Q(H)$-module.

Proof. Let $Q_0 = C_Q(H)$. The finite group H acts compatibly on A and Q and we can apply [17, Lemma 3.5] to see that there is a subgroup Q_1 of Q such that $\widetilde{Q} = Q_0 \times Q_1$ is a subgroup of finite index in Q and

$$\sum_{t \in H} t \text{ acts trivially on } Q_1. \tag{1}$$

To prove that A_0 is a finitely generated d-tame $\mathbb{Z}Q_0$-module it suffices to show that A is a finitely generated d-tame $\mathbb{Z}Q_0$-module, as $\mathbb{Z}Q_0$ is Noetherian and furthermore d-tameness is preserved by submodules [5, Lemma 1.1]. We show first that A is finitely generated as a $\mathbb{Z}Q_0$-module and by Corollary [6, Cor. 4.5] this is equivalent to the following: for every non-zero character $\chi : \widetilde{Q} \to \mathbb{R}$ such that $\chi(Q_0) = 0$ we have that $[\chi] \in \Sigma_A(\widetilde{Q})$. Assume now that for some χ as above we have that $[\chi] \notin \Sigma_A(\widetilde{Q})$. Then using (1)

$$\sum_{t \in H} [\chi^t] = 0 \text{ and } [\chi^t] \in \Sigma^c_A(\widetilde{Q}),$$

so A is not $|H|$-tame, contradicting the hypothesis that A is n-tame as a $\mathbb{Z}Q$-module and therefore is also n-tame as a $\mathbb{Z}\widetilde{Q}$-module.
Suppose that A is not d-tame as a $\mathbb{Z}Q_0$-module. Then there are elements $[\chi_1], \ldots, [\chi_d] \in \Sigma^c_{A_0}(Q_0)$ such that $\sum_{1 \leq i \leq d} \chi_i = 0$. By [17, Lemma 3.3] there are homomorphisms $\mu_i : Q_1 \to \mathbb{R}$ such that $[\chi_i + \mu_i] \in \Sigma^c_A(\widetilde{Q})$. Using (1) we obtain that

$$\sum_{t \in H} \mu_i^t = 0 \text{ for every } 1 \leq i \leq d.$$

Hence

$$\sum_{1 \leq i \leq d} \sum_{t \in H} (\chi_i^t + \mu_i^t) = \sum_{1 \leq i \leq d} \sum_{t \in H} \chi_i^t = (\sum_i \chi_i)|H| = 0.$$

Thus we have $d|H|$ characters $\{\chi_i^t + \mu_i^t\}_{1 \leq i \leq d, t \in H}$ summing to 0 and $[\chi_i^t + \mu_i^t] \in \Sigma^c_A(\widetilde{Q})$, contradicting the fact that A is n-tame as a $\mathbb{Z}\widetilde{Q}$-module and that $d|H| \leq n$. Hence A is d-tame as a $\mathbb{Z}Q_0$-module, proving the claim. □

Let $1 \to A \to G \to Q \to 1$ be a short exact sequence of groups with G finitely generated and A and Q abelian. Consider the following two conditions:
(i) G has finite Prüfer rank and $A = G'$;
(ii) $G = Q \ltimes A$.

Proposition 3.2. *Assume G satisfies either i) or ii) above and that A is n-tame as a $\mathbb{Z}Q$-module. Let H be a finite group acting on G such that in case ii) A and Q are H-invariant and let d be a positive integer with*

$$d|H| \leq n.$$

Then $C_A(H)$ is a finitely generated d-tame $\mathbb{Z}[C_G(H)/C_A(H)]$-module.

Proof. By Lemma 3.1, $A_0 = C_A(H)$ is d-tame as a $\mathbb{Z}C_Q(H)$-module. In the finite Prüfer rank case it suffices to take into account that by [17, Lemma 3.10] the index $|C_Q(H) : AC_G(H)|$ is finite, so A_0 is also d-tame as a $\mathbb{Z}[C_G(H)/C_A(H)]$-module.

Assume now that $G = Q \ltimes A$. Then $C_G(H) = C_Q(H) \ltimes A_0$ so the result also follows. □

For a real number r denote by $\lfloor r \rfloor$ the integral part of r. This is the unique integer $\lfloor r \rfloor$ such that $r - \lfloor r \rfloor \in [0, 1)$.

Corollary 3.3. *Let G be a metabelian group of type* FP_n *with finite Prüfer rank. Let H be a finite group acting on G. Then $C_G(H)$ is of type* FP_d *for* $d = \lfloor \frac{n}{|H|} \rfloor$.

Proof. This follows from Proposition 3.2 and the fact that the FP_m-Conjecture holds for metabelian groups of finite Prüfer rank [1]. □

Proposition 3.4. *Let G be a soluble-by-finite group of finite Prüfer rank. Then there is a bound on the orders of the finite subgroups of G. Furthermore, G has finitely many conjugacy classes of finite subgroups.*

Proof. Let G be a soluble group of finite Prüfer rank. Then it is minimax; see for example [21, Exercise 14.1.4]. The proof of [20, Theorem 10.33] implies that G has a bound on the orders of its finite subgroups. Hence the argument of [17, Theorem 2.4] can be applied implying that G has finitely many conjugacy classes of finite subgroups. □

The above results give a lower bound for the Bredon-type of metabelian groups. This bound turns out to be the best possible as the examples in the next sections show.

Theorem 3.5. *Let G be a metabelian-by-finite group of finite Prüfer rank of type* FP_n. *Then G is of type Bredon* FP_d *for* $d = \lfloor \frac{n}{s} \rfloor$, *where s denotes the upper bound on the orders of the finite subgroups of G.*

Proof. Apply Corollary 3.3, Proposition 3.4 and Theorem 1.1. □

4. Examples of virtually metabelian groups

4.1. **An example of prime characteristic.** In this section we construct examples of metabelian groups $G = A \rtimes Q$ where Q is free abelian of finite rank and A an infinite abelian p-group, which show that the bound in Corollary 3.3 can be sharp. More precisely, for a certain integer n, these groups are of type FP_n but admit an action of a finite cyclic group H_0 such that $C_G(H_0)$ is not of type FP_{d+1} for $d = \lfloor \frac{n}{|H_0|} \rfloor$.

Let $\mathbb{F}_p$ be the field with p elements and $\mathbb{F}_p \subset k$ be a finite field extension of degree m; k is a field with p^m elements. Then the Galois group $Gal(k \mid \mathbb{F}_p)$

is a cyclic group generated by the Möbius map σ sending an element to its pth power. By [18, Ch. VIII, Sec. 12, Thm. 20] there is an element $a \in k$ such that the set $\{\sigma^j(a)\}_{0 \leq j \leq m-1}$ is linearly independent over $\mathbb{F}_p$. Hence $k = \mathbb{F}_p(a)$. Note that

$$b = tr(a) = \sum_{0 \leq j \leq m-1} \sigma^j(a) \in \mathbb{F}_p \smallsetminus \{0\}. \tag{2}$$

By multiplying a with a non-zero element of $\mathbb{F}_p$ we can assume that $b = 1$. Define the set

$$\{a_1 = a, a_{i+1} = \sigma(a_i) + a\}_{i \in \mathbb{N}}.$$

Lemma 4.1. *The set $\{a_j\}_{1 \leq j \leq mp}$ contains mp different elements and $a_m = 1, a_{pm} = 0$.*

Proof. Note that $a_m = b = 1$ and so $a_{m+i} = a_i + 1$ for every i. Then $a_{pm} = a_m + (p-1) = 1 + (p-1) = 0$. Furthermore $a_1 = a, a_2, \dots, a_m = 1$ are linearly independent over $\mathbb{F}_p$ and thus

$$\{a_j\}_{1 \leq j \leq mp} = \cup_{1 \leq i \leq m} \{a_i + \mathbb{F}_p\}$$

contains mp different elements. ☐

Define A to be the localization $k[x_1, 1/(x_1 + a_j)]_{1 \leq j \leq mp}$ of the polynomial ring $k[x_1]$. Let H be a cyclic group of order pm with a generator μ acting on the field $k(x_1)$ in the following way : the restriction of μ to the field k is the Möbius map σ, and $\mu(x_1) = x_1 + a$. Thus $\mu^i(x_1) = x_1 + a_i$. In particular $\mu^m(x_1) = x_1 + 1$ and by Lemma 4.1 μ has order pm. Moreover we get an induced action of H on A.

Consider the split extension $G = A \rtimes Q$, where Q is a free abelian group with generators $q_1, \dots, q_{pm}$ and q_i acts on A by conjugation as multiplication with $x_1 + a_i$. The generator μ of the cyclic group H acts on Q by sending q_i to q_{i+1}, where $q_{pm+1} = q_1$. This, together with the above action of H on A induces an action of H on G.

Lemma 4.2. *The $\mathbb{Z}Q$-module A is (pm)-tame but not $(pm+1)$-tame. Hence G is of type FP_{pm} but not of type FP_{pm+1}.*

Proof. Note first that A is a cyclic $\mathbb{Z}Q$-module, where Q acts via conjugation. So A is a quotient ring of $\mathbb{Z}Q$ and the embedding of Q in $\mathbb{Z}Q$ induces an embedding of Q in A.

Let $\chi : Q \to \mathbb{R}$ be a non-zero character such that $[\chi] \in \Sigma_A^c(Q)$. Then by [3, Thm. 8.1] and the fact that A is a cyclic $\mathbb{Z}Q$-module there is a real valuation $v : A \to \mathbb{R} \cup \{\infty\}$ such that the restriction of v to Q extends χ. Since k is finite $v(k \smallsetminus 0) = 0$ and so if v has a positive value on one of the elements of the basis $Y = \{x_1 + a_j\}_{1 \leq j \leq pm}$ of Q as embedded in A, then v has zero value on all other elements of the same basis.

If v has a negative value on one of the elements of Y then the v-value on all elements of Y is the same. Thus $\Sigma_A^c(Q)$ has exactly $pm + 1$ points, A is (pm)-tame but is not $(pm + 1)$-tame as a $\mathbb{Z}Q$-module.

Finally note that the FP$_n$-Conjecture for metabelian groups holds for groups which are extensions of A by Q, where Q is a finitely generated abelian group and A is a finitely generated $\mathbb{Z}Q$-module of Krull dimension 1 and of finite exponent as an additive group [11, Cor. C]. ☐

Define H_0 to be the subgroup of H generated by μ^d for some positive divisor d of m. Thus $[H : H_0] = d$. We want to study the centralizer $C_G(H_0)$ of H_0 in G. Note that $C_G(H_0) = C_A(H_0) \rtimes C_Q(H_0)$. We write A_0 for $C_A(H_0)$ and Q_0 for $C_Q(H_0)$ and consider A_0 as a $\mathbb{Z}Q_0$-module via conjugation. Note that $A_0 \neq 0$ since $\prod_{h\in H} h(x_1)$ is an element of A_0 and a non-zero polynomial of degree pm. Since the generator μ of H cyclicly permutes the basis $q_1, \dots, q_{pm}$ of Q, Q_0 is a free abelian group of rank d.

Lemma 4.3. *The $\mathbb{Z}Q_0$-module A_0 is not $(d+1)$-tame.*

Proof. Let T_0 be the image of Q_0 under the embedding of Q in A. Then there is no bound on the degree, as polynomials in the variable x_1, of the elements of $T_0 \prod_{h\in H} h(x_1)$ and $T_0 \prod_{h\in H} h(x_1) \subseteq A_0$. In particular A_0 is not finite dimensional.
Suppose now that A_0 is $(d+1)$-tame as a $\mathbb{Z}Q_0$-module. Note that by Proposition 3.2 A_0 is finitely generated as a $\mathbb{Z}Q_0$-module. Since d is the rank of Q_0 we get that A_0 is ∞-tame as a $\mathbb{Z}Q_0$-module. Then the main result of [4] implies that for all s the tensor powers $\otimes^s A_0$ are finitely generated as $\mathbb{Z}Q_0$-modules with diagonal Q_0-action. Hence they are also finitely generated as $\mathbb{F}_p Q_0$-modules. In particular, the Krull dimension of $\otimes^s A_0$ is at most the Krull dimension of $\mathbb{F}_p Q_0$, which is the rank of Q_0. On the other hand the Krull dimension of $\otimes^s A_0$ is at least s times the Krull dimension of A_0 [8, Lemma 1] and the Krull dimension of A_0 is not zero since A_0 is infinite dimensional. Thus s cannot be arbitrarily large, giving a contradiction. □

Corollary 4.4. *The group $C_G(H_0)$ is of type* FP_d *but not of type* FP_{d+1}.

Proof. By Proposition 3.2 and Lemma 4.3 A_0 is d-tame but not $(d+1)$-tame as a $\mathbb{Z}Q_0$-module. The claim now follows from the fact that the FP_m-Conjecture for metabelian groups holds for groups which are extensions of A_0 by Q_0, where Q_0 is a finitely generated abelian group and A_0 is a finitely generated $\mathbb{Z}Q_0$-module of Krull dimension 1 and of finite exponent as an additive group [11, Cor. C]. □

4.2. **An example of finite Prüfer rank.**

The group A constructed in the previous section was of infinite Prüfer rank. We construct examples similar to the above, but now A will be a torsion free group of finite Prüfer rank. Moreover, the finite group H_0 acting on G will be a subgroup of the Galois group of a finite extension of $\mathbb{Q}$. Note also that any finite group is a subgroup of some symmetric group and any symmetric group is the Galois group of some extension of $\mathbb{Q}$.

Let $K : \mathbb{Q}$ be a Galois extension with Galois group H. Choose a primitive element ζ with minimal polynomial $f(x)$, note that ζ can chosen to be integral over $\mathbb{Z}$ so we may assume $f(x) \in \mathbb{Z}[x]$. Let O_K be the integral closure of $\mathbb{Z}$ in K and d_K its discriminant.

By [13, Lemma 4.1] there are infinitely many primes q for which there exists some integer k_q with $q|f(k_q)$ so we may take $q, k := k_q$ such that

$$q|f(k), q \nmid |O_K/\mathbb{Z}[\zeta]| \text{ and } q \nmid d_K.$$

By Dedekind's Theorem the last condition implies that q is not ramified in O_K. Moreover, the extension $K:\mathbb{Q}$ is Galois and therefore

$$qO_K = I_1 \dots I_{t_0}$$

for distinct prime ideals I_i which form a single H-orbit. In particular, O_K/qO_K is a product of fields. Put $I = I_1$. Then for any i

$$\delta = [O_K/I : \mathbb{F}_q] = [O_K/I_i : \mathbb{F}_q],$$

and $n = [K:\mathbb{Q}] = \delta t_0$.
The condition $q \nmid |O_K/\mathbb{Z}[\zeta]|$ implies

$$O_K/qO_K \cong \mathbb{Z}[\zeta]/q\mathbb{Z}[\zeta] = \mathbb{Z}[x]/(q, f(x)) = \mathbb{F}_q[x]/(\bar{f}(x)).$$

As $x - \bar{k} \mid \bar{f}(x)$ in $\mathbb{F}_q[x]$, we deduce that $\delta = 1$. Therefore $n = t_0$ and

$$qO_K = \prod_{t \in H} I^t$$

with $I \trianglelefteq O_K$ prime and the I^t are all distinct.
Let r be the ideal class number of the Dedekind domain O_K. Then

$$I^r = \alpha O_K$$

for some $\alpha \in O_K$.
Take any natural prime number $p \neq q$. Then

$$pO_K = J_1^s \dots J_t^s$$

where the J_i are prime ideals which form a single H-orbit.

Now, let $Q = \langle q_t \rangle_{t \in H}$ be free abelian of rank $n = |H|$ and let

$$A = O_K[\alpha^t/p, p/\alpha^t]_{t \in H}.$$

Let each q_t act on A by multiplication by α^t/p. The group H acts on Q by $q_t^{t_1} = q_{tt_1}$, where $t, t_1 \in H$. The group H also acts on K, and A is invariant under this action. Moreover the actions of Q and H on A are compatible so this gives an action of H on the group $G := Q \ltimes A$.

Proposition 4.5. *The group G is of type* FP_n *and not of type* FP_{n+1}.

Proof. Consider the homomorphism of groups

$$\tau : Q \to A$$

given by $\tau(q_h) = \alpha^h/p$. This map can be extended to an epimorphism

$$\tau : O_K Q \to A,$$

where $O_K Q$ is the group algebra of Q with coefficients in O_K. As A is a domain, $A \cong O_K Q/P$ for some prime ideal $P \triangleleft O_K Q$. Note also that $P \cap O_K = 0$. Thus, by [6, 2.4] with $R = O_K$, the set $\Sigma_A^c(Q)$ is finite and discrete. Therefore [6, 2.2 and 2.1] imply that

$$\Sigma_A^c(Q) = \{[\upsilon \circ \tau] : \upsilon \circ \tau \neq 0\}$$

where $\upsilon : K \to \mathbb{Z}_\infty$ is a discrete valuation of K. Note that any real valuation of $\mathbb{Z}$ is non-negative $\upsilon(O_K) \geq 0$. The discrete valuations of K, up to multiplication with a positive real number, are the P-adic valuations v_P where $P \trianglelefteq O_K$ is a prime ideal. The valuation v_P is given by

$$v_P(a) = m$$

such that

$$aO_K = P^m M$$

where M is a fractionary ideal such that P does not appear in its decomposition in primes. Note that for $x \in H$

$$(\alpha^x/p)O_K = (I^x)^r (pO_K)^{-1}$$

and therefore for $h, x \in H$

$$v_{I^h}(\alpha^x/p) = \begin{cases} r \text{ if } x = h \\ 0 \text{ otherwise.} \end{cases}$$

Also, for any prime ideal J of O_K with $J|pO_K$ there is some $s_0 > 0$ with

$$v_J(\alpha^x/p) = -s_0.$$

Therefore $\Sigma^c_A(Q)$ has exactly $(n+1)$ points and the $\mathbb{Z}Q$-module A is n-tame but not $n+1$-tame. By [1] the FP_m-Conjecture holds for metabelian groups of finite Prüfer rank, so the group G is of type FP_n but not FP_{n+1}. □

Now let $H_0 \leq H$ be a subgroup of index d. Note that H_0 acts on G and

$$d|H_0| = n.$$

Moreover $C_G(H_0) = Q_0 \ltimes A_0$ with $Q_0 = C_Q(H_0)$ and $A_0 = C_A(H_0)$. The group Q_0 is free abelian of rank d with basis

$$\{s_x := \prod_{t \in H_0} q_{xt} : x \in H/H_0\}.$$

Proposition 4.6. *The group* $C_G(H_0)$ *is of type* FP_d *but not of type* FP_{d+1}.

Proof. Note first that Lemma 3.2 and the fact that the FP_m-Conjecture for groups of finite Prüfer rank holds, see [1], imply that $C_G(H_0)$ is of type FP_d. In particular A_0 is a finitely generated $\mathbb{Z}Q_0$-module.

Suppose that $C_G(H_0)$ is of type FP_{d+1}. Since the FP_m-Conjecture holds for groups of finite Prüfer rank, we deduce that A_0 is $(d+1)$-tame as a $\mathbb{Z}Q_0$-module: every $d+1$ elements of $\Sigma^c_{A_0}(Q_0)$ lie in an open hemisphere. Since $d+1 > rk(Q_0)$ we get that A_0 is ∞-tame as a $\mathbb{Z}Q_0$-module.

Let $\chi : Q \to \mathbb{R}$ be a non-zero homomorphism with $\chi(Q_0) \neq 0$. We claim:

$$\text{If } [\chi] \in \Sigma^c_A(Q) \text{ then } [\chi\,|_{Q_0}] \in \Sigma^c_{A_0}(Q_0). \tag{3}$$

Indeed, A is a cyclic $O_K Q$-module. So $A \simeq O_K Q/I$ for $I = ann_{O_K Q}(A)$. Since $[\chi] \in \Sigma^c_A(Q)$ and O_K is finitely generated as a $\mathbb{Z}$-module one sees that A is not finitely generated as a $O_K Q_\chi$-module. Then by applying [3, Thm. 8.1], χ lifts to a real valuation $v : A \to \mathbb{R} \cup \infty$ such that $v(O_K) \geq 0$. The restriction

$$w = (v\,|_{A_0}) : A_0 \to \mathbb{R} \cup \infty$$

has the following property: $w(aq) = w(a) + \chi(q)$ for every $a \in A_0, q \in Q_0$. This shows that, provided $w(A_0) \neq \infty$, A_0 is not finitely generated as $O_K Q_\chi$-module and consequently $[\chi\,|_{Q_0}] \in \Sigma^c_{A_0}(Q_0)$. Note that $v^{-1}(\infty)$ is a prime ideal in A and that A has Krull dimension 1. Thus $v^{-1}(\infty) = 0$.

Finally, since A_0 is an ∞-tame $\mathbb{Z}Q_0$-module, (3) implies that $\Sigma^c_A(Q)$ lies in a closed half subsphere of $S(Q)$. This contradicts the description of $\Sigma^c_A(Q)$ in the proof of Lemma 4.5. □

5. Soluble groups of finite Prüfer rank

In this section we shall consider soluble groups of finite Prüfer rank of type FP$_n$ and cohomological finiteness conditions for centralisers of finite subgroups. Soluble groups G of finite Prüfer rank are nilpotent-by-abelian-by-finite, which follows from the proof of [20, Theorem 10.38]. Then G has a subgroup of finite index G_1 that is characteristic and is nilpotent-by-abelian. For such a group G there is a group extension:

$$N \rightarrowtail G \twoheadrightarrow Q \tag{4}$$

where N is the commutator of G_1, N is nilpotent and Q is abelian-by-finite.

For soluble groups of finite Prüfer rank there is as of now no result like Theorem 3.5. We have, however, some partial results which suggest a general conjecture for Bredon type FP$_m$:

Conjecture 5.1. *Let G be a soluble group of finite Prüfer rank and of type* FP$_n$*. Then G is of type Bredon-*FP$_m$ *for $m = \lfloor \frac{n}{sc} \rfloor$, where s is the bound on the orders of the finite subgroups of G and c is the nilpotency class of N in (4) above.*

Note that by Proposition 3.4, G has finitely many conjugacy classes of finite subgroups. Therefore Theorem 1.1 implies that G is of type Bredon-FP$_m$ if and only if the centralizer in G of any finite group of automorphisms of G is of type FP$_m$. The FP$_m$ property is invariant under finite group extensions and finite index subgroups. In particular, let $M_1 \leq M_2$ be groups such that M_1 has finite index in M_2. Then M_1 is of type FP$_m$ if and only if M_2 is of type FP$_m$. It therefore suffices to show that the centralizer in G_1 of any finite group H of automorphisms of G is of type FP$_m$. Thus without loss of generality we can assume that $G = G_1$. Hence from now on let Q be abelian, G be nilpotent-by-abelian and let H be a finite group of automorphisms of G. Note that we may also assume that $m \geq 1$ in the above conjecture. Otherwise there is nothing to prove. Thus we can suppose that

$$n \geq cs. \tag{5}$$

We denote by $A = N/N'$ the abelianization of N. We shall make use of the following result by Åberg:

Proposition 5.2. [1, IV.2.2] *Let G be a soluble group of finite Prüfer rank of type* FP$_n$ *. Then A is an n-tame $\mathbb{Z}Q$-module.*

Lemma 5.3. *Let G be a nilpotent-by-abelian group of finite Prüfer rank and of type* FP$_n$*. Let H be a finite group acting on G. Let N be as in (4) above. Then $A = N/N'$ is finitely generated and $\lfloor \frac{n}{|H|} \rfloor$-tame as a $C_G(H)/C_N(H)$-module.*

Proof. As G is of type FP_n and of finite Prüfer rank, A is n-tame as a Q-module by [1, Proposition IV 2.2] . Using a similar argument to Lemma 3.1, we deduce that A is finitely generated and $\lfloor \frac{n}{|H|} \rfloor$-tame as a C/N-module, where $C/N = C_Q(H)$. Moreover, the index $|C : C_G(H)N|$ is finite [17, Proposition 3.11] (note that although the groups considered in [17] are of type FP_∞, in this result the group only has to be finitely generated and of finite Prüfer rank). □

For a group M denote by $\gamma_i(M)$ the i-th term of the lower central series of M.

Theorem 5.4. *Let G be a nilpotent-by-abelian group of finite Prüfer rank, of type FP_n and let H be a finite group of order s acting on G such that (5) holds. Let N be as in (4) above. Then $D = \oplus_i \gamma_i(C_N(H))/\gamma_{i+1}(C_N(H))$ is finitely generated and $\lfloor \frac{n}{sc} \rfloor$-tame as a $C_G(H)/C_N(H)$-module, where c is the nilpotency class of N.*

Proof. Let $Q_0 = C_G(H)/C_N(H)$ and let $d = \lfloor \frac{n}{s} \rfloor$. By Lemma 5.3 $A = N/N'$ is d-tame as a Q_0-module. By (5) $d \geq c$ and hence for $i \leq c$ we have that $\gamma_i(N)/\gamma_{i+1}(N)$ is finitely generated as a $\mathbb{Z}Q_0$-module, as it is a quotient of the finitely generated $\mathbb{Z}Q_0$-module $\otimes^i A$.
Let B be any abelian subsection of N that is invariant under the action of Q_0 and let A_i be the i-th factor of the lower central series of N. Then B has a series of $\mathbb{Z}Q_0$-modules with factors $B_1, ..., B_c$ such that each B_i is a subsection of A_i and A_i is a subsection of $\otimes^i A$. Then all A_i and B_i are finitely generated as $\mathbb{Z}Q_0$-modules and so B is finitely generated as a $\mathbb{Z}Q_0$-module. By [6, Lemma 1.1]

$$\Sigma^c_B(Q_0) = \bigcup_{i=1}^{c} \Sigma^c_{B_i}(Q_0),$$

and for $i \leq c$

$$\Sigma^c_{B_i}(Q_0) \subseteq \Sigma^c_{\otimes^i A}(Q_0) \subseteq conv_{\leq i}\Sigma^c_A(Q_0) \subseteq conv_{\leq c}\Sigma^c_A(Q_0).$$

Therefore

$$\Sigma^c_B(Q_0) \subseteq conv_{\leq c}\Sigma^c_A(Q_0).$$

Now apply this for $B = \gamma_i(C_N(H))/\gamma_{i+1}(C_N(H))$. Then $\gamma_i(C_N(H))/\gamma_{i+1}(C_N(H))$ is finitely generated as a $\mathbb{Z}Q_0$-module and using [6, Lemma 1.1] again

$$\Sigma^c_D(Q_0) = \cup_i \Sigma^c_{\gamma_i(C_N(H))/\gamma_{i+1}(C_N(H))}(Q_0) \subseteq conv_{\leq c}\Sigma^c_A(Q_0).$$

Therefore if $0 \notin conv_{\leq d}\Sigma^c_A(Q_0)$ then also $0 \notin conv_{\leq \frac{d}{c}}\Sigma^c_D(Q_0)$ and D is $\lfloor \frac{d}{c} \rfloor$-tame as a $\mathbb{Z}Q_0$-module. □

Corollary 5.5. *Let G be a finitely generated nilpotent-by-abelian group of finite Prüfer rank. Let N be nilpotent of class c as in (4) and assume that G has type FP_n for some $n \geq c$. Then $\oplus_i \gamma_i(N)/\gamma_{i+1}(N)$ is finitely generated and $\lfloor \frac{n}{c} \rfloor$-tame as a $\mathbb{Z}Q$-module.*

Proof. Take $H = \{1\}$ in Theorem 5.4. □

The following result together with the remarks after Conjecture 5.1 show that Conjecture 5.1 holds for $m = 1$.

Corollary 5.6. *Let G be a finitely generated nilpotent-by-abelian group of finite Prüfer rank. Let N be nilpotent of class c as in (4) with Q abelian. Let H be a finite group of order s acting on G. Assume also that G is of type FP_n such that $\frac{n}{cs} \geq 1$. Then $C_G(H)$ is finitely generated.*

Proof. Since N is a nilpotent group it suffices to show that

$$B = (C_G(H) \cap N)/(C_G(H) \cap N)'$$

is finitely generated as a Q_0-module, where $Q_0 = C_G(H)/C_N(H)$. This follows directly from the fact that D from Theorem 5.4 is finitely generated as a $\mathbb{Z}Q_0$-module.

□

In contrast to the metabelian case there is no criterion for finite presentability of nilpotent-by-abelian groups. There are, however, results giving sufficient conditions implying finite presentability and hence type FP_2. The following results are some of these sufficient conditions.

Proposition 5.7. [10, Cor. C] *Let G be a finitely generated nilpotent-by-abelian group with normal nilpotent subgroup N of nilpotency length c with G/N abelian. Suppose N/N' is $(c+1)$-tame as a G/N-module. Then G is finitely presented.*

Proposition 5.8. [10, Thm. B] *Let G be a finitely generated nilpotent-by-abelian group with normal nilpotent subgroup N of nilpotency length c with $Q = G/N$ abelian. Suppose that $\gamma_i(N)/\gamma_{i+1}(N)$ is a finitely generated $\mathbb{Z}Q$-module and that $\Sigma^c_{N/N'}(Q) \cap -\Sigma^c_{\gamma_i(N)/\gamma_{i+1}(N)}(Q) = \varnothing$ for every $1 \leq i \leq c$. Then G is finitely presented.*

The following result together with the remarks after Conjecture 5.1 shows that Conjecture 5.1 holds for $m = 2$.

Corollary 5.9. *Let G be a finitely generated nilpotent-by-abelian group of finite Prüfer rank. Let N be nilpotent of class c as in (4) with Q abelian. Let H be a finite group of order s acting on G. Assume also that G is of type FP_n such that $\frac{n}{cs} \geq 2$. Then $C_G(H)$ is finitely presented. In particular, $C_G(H)$ is of type FP_2.*

Proof. By Corollary 5.6 $C_G(H)$ is finitely generated. By Theorem 5.4 $D = \oplus_i \gamma_i(C_N(H))/\gamma_{i+1}(C_N(H))$ is $\lfloor \frac{n}{sc} \rfloor$-tame as a $C_G(H)/C_N(H)$-module, in particular is 2-tame as a $C_G(H)/C_N(H)$-module. Then by Proposition 5.8 $C_G(H)$ is finitely presented. □

Theorem 5.4, its corollaries and Groves' results now lead us to make the following conjecture:

Conjecture 5.10. *Let G be a finitely generated nilpotent-by-abelian group. Let N be nilpotent as in (4) with Q abelian and suppose that the direct sum of all factors in the lower central series of N is finitely generated and m-tame as a G/N-modules. Then G is of type* FP_m .

This conjecture obviously implies Conjecture 5.1. Just combine Proposition 3.4, Theorem 5.4 and Corollary 5.6 with Theorem 1.1.

Also note that Conjecture 5.10 is also true if the Hirsch length $h(G/N) \leq m-1$. For in this case a result by H. Meinert [19] implies that G is of type FP_∞. Furthermore, by [17, Theorem 3.13] centralisers of finite subgroups in soluble groups of type FP_∞ are also of type FP_∞.

The sufficient conditions for finite presentability of G given in Proposition 5.7 show that even in the case when $m = 2$ and N is nilpotent of class 2 the converse of Conjecture 5.10 is not true. Indeed if $A = N/N'$ is 3-tame then G is finitely presented but if $N' = A \wedge A$ then N' does not need be 2-tame. Actually if A is 4-tame then N' is 2-tame but in general this does not hold if A is only 3-tame.

It is a well known fact, see [2, Proposition 5.3], that for nilpotent-by-abelian groups of type FP_m the group $H_t(N, \mathbb{Z})$ is finitely generated for all $1 \leq t \leq m$. Here we show that this condition holds under the assumptions of Conjecture 5.10.

Proposition 5.11. *Suppose that G is a nilpotent-by-abelian group with derived subgroup N of nilpotency class c, $Q = G/N$ abelian and assume that $D = \oplus_i \gamma_i(N)/\gamma_{i+1}(N)$ is finitely generated and n-tame as a $\mathbb{Z}Q$-module and $n \geq c$. Then $H_t(N, \mathbb{Z})$ is finitely generated and $\lfloor \frac{n}{t} \rfloor$-tame as a $\mathbb{Z}Q$-module for $1 \leq t \leq n$.*

Proof. Denote by A_i the i-th factor in the lower central series of N. The proof of the proposition depends on the structure of $H_j(A_i, \mathbb{Z})$. For any torsion-free abelian group A there is a natural isomorphism $H_j(A, \mathbb{Z}) \simeq \wedge^j A$ [9, Ch. V, Thm. 6.4]. The situation turns a bit more complicated when A has torsion. By the proof of [12, Thm. C] if Q is a finitely generated abelian group and A is a finitely generated $\mathbb{Z}Q$-module then $H_i(A, \mathbb{Z})$ has a finite filtration with factors isomorphic to $\mathbb{Z}Q$-subsections of, possibly different, tensor powers of A, where the action of Q is the diagonal one. Inspection of the proof of [12, Thm. C] shows that the tensor powers are of type $\otimes^j A$ for $j \leq i$.

By repeating a Lyndon-Hochschild-Serre spectral sequence argument one proves that $H_t(N, \mathbb{Z})$ has a series with factors which are subsections of modules of the form

$$H_{i_1}(A_1, \mathbb{Z}) \otimes \ldots \otimes H_{i_c}(A_c, \mathbb{Z})$$

such that $i_1 + i_2 + \ldots + i_c = t$. Note that every $H_{i_j}(A_j, \mathbb{Z})$ has a filtration with quotients that are subsections of $\otimes^s A_j$ for some $s \leq i_j$. Then $H_{i_1}(A_1, \mathbb{Z}) \otimes \ldots \otimes H_{i_c}(A_c, \mathbb{Z})$ has a filtration with quotients that are subsections of $(\otimes^{s_1} A_1) \otimes (\otimes^{s_2} A_2) \otimes \ldots \otimes (\otimes^{s_c} A_c)$ for $s_j \leq i_j$, hence these quotients are subsections of $\otimes^{s_1+s_2+\ldots+s_c} D$ for $s_1 + s_2 + \ldots + s_c \leq i_1 + i_2 + \ldots + i_c = t$. Since $H_t(N, \mathbb{Z})$ has a filtration with quotients that are subsections of $\otimes^m D$ for $m \leq t \leq n$ and $\otimes^i D$ is finitely generated as a $\mathbb{Z}Q$-module for $i \leq n$, we deduce that $H_t(N, \mathbb{Z})$ is finitely generated as a $\mathbb{Z}Q$-module for $t \leq n$.

Finally [6, Lemma 1.1. and Theorem 1.3.] yield

$$\Sigma^c_{H_t(N,\mathbb{Z})}(Q) \subseteq \bigcup_{s \leq t} \Sigma^c_{\otimes^s D}(Q) \subseteq \bigcup_{s \leq t} \mathrm{conv}_{\leq s} \Sigma^c_D(Q) \subseteq \mathrm{conv}_{\leq t} \Sigma^c_D(Q).$$

As $0 \notin \mathrm{conv}_{\leq n}\Sigma^c_D(Q)$, we deduce

$$0 \notin \mathrm{conv}_{\leq \frac{n}{t}}\Sigma^c_{H_t(N,\mathbb{Z})}(Q).$$

$\square$

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