COHOMOLOGY RELATIVE TO A G-SET AND FINITENESS CONDITIONS

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ABSTRACT. We shall consider a cohomology theory relative to group actions on sets and develop a completion analogous to Mislin, Benson and Carlson. Later we restrict ourselves to cohomology relative to all finite subgroups. There we shall study relative analogues to finiteness conditions on kG-modules such as finite relative projective dimension and being of relative type FP_∞.

1. Introduction

Relative Cohomology has been studied extensively in various contexts. Cohomology relative to one subgroup has a vast literature and has many applications, for example in the representation theory of finite groups. We shall consider the more general set-up of cohomology relative to group actions on a set. This however is a special case of cohomology set up via proper exact sequences as in [10, Chapters IX and XII].

Unless otherwise specified k will denote a commutative ring of coefficients and G will denote an arbitrary group.

In Section 2 we will introduce the general set-up of cohomology relative to a G-set Δ and give an overview of the main definitions and facts.

In Section 3 we shall develop a completion of proper cohomology [10, §XII.4] analogous to Mislin's [11], which vanishes on proper projectives and is defined and can be non-zero in all dimensions. This is obtained via the generalized Satellites introduced in [10, §XII.7].

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In Sections 4 and 5 we shall develop alternative but equivalent approaches to complete cohomology relative to G-sets Δ, one analogous to Benson and Carlson [2] and the other one via complete Δ-projective resolutions analogous to [6]. Kropholler's definition of $\mathbf{H}\mathfrak{F}$-groups [8], a large class of groups described by their actions on finite dimensional contractible cell complexes, suggested looking at a cohomology theory relative to G-sets $\mathcal{F}$ with finite isotropy. The set

$$\mathcal{F} = \bigsqcup_{H\,finite} H\backslash G$$

would be a natural choice. In this context we shall discuss $\mathcal{F}$-split short exact sequences and $\mathcal{F}$-projective kG-modules. Consequently we have the notions $\mathcal{F}$-projective resolutions and $\mathcal{F}$-projective dimension of a kG-module. We also denote the relative cohomology functors by $\mathcal{F}\mathrm{Ext}^*_{kG}(-,-)$ and $\mathcal{F}\widehat{\mathrm{Ext}}^*_{kG}(-,-)$ respectively. In Section 6 we introduce such G-sets $\mathcal{F}$ and show that they give rise to the same cohomology. We also prove the following crucial proposition, which implies the main theorems of Section 7.

Proposition 6.3. *Let k be a commutative Noetherian ring of coefficients and G be an arbitrary group. Then every finitely generated $\mathcal{F}$-projective kG-module is of type FP_∞.*

In Section 7 we proceed to define two relative analogues to the property FP_∞. Unfortunately there is no obvious link between the two, as in the ordinary case. But for commutative Noetherian rings of coefficient k we can show the following two facts:

(1) (Theorem 7.1) For every kG-module M admitting a finite type $\mathcal{F}$-projective resolution the relative Ext-groups $\mathcal{F}\mathrm{Ext}^*_{kG}(M,-)$ commute with direct limits.

(2) (Theorem 7.2) Every kG module admitting an $\mathcal{F}$-projective resolution of finite type is of type FP_∞.

Actually, Section 6 and the main theorems of Section 7 are independent of Sections 3 to 5.

In Section 8 we consider finiteness conditions for kG-modules, where G belongs to $\mathbf{H}\mathfrak{F}$. It turns out (Theorem 8.3) that kG-modules satisfying either of the two relative analogues to FP_∞, have finite $\mathcal{F}$-projective dimension.

2. Cohomology relative to a G-set

Let G be an arbitrary group and k be a commutative ring of coefficients. Denote by Δ an arbitrary G-set. $k\Delta$ is the free k-module on Δ. It is given a G-module structure by the G-action on Δ.

Definition 2.1. *Let G be a group and k be a commutative ring of coefficients. A short exact sequence $A \rightarrowtail B \twoheadrightarrow C$ of G-modules is called Δ-split if and only if*

$$A \otimes k\Delta \rightarrowtail B \otimes k\Delta \twoheadrightarrow C \otimes k\Delta$$

is split.

If not specified otherwise we always tensor over the coefficient ring k.
Remark: The surjection $\sigma : k\Delta \otimes k\Delta \twoheadrightarrow k\Delta$ given by $\sigma(\delta_1 \otimes \delta_2) = \delta_2$ is split. Define the splitting $s : k\Delta \to k\Delta \otimes k\Delta$ by $s(\delta) = \delta \otimes \delta$. In particular, for every G-module A, the surjection $\sigma_A : A \otimes k\Delta \twoheadrightarrow A$, defined by $\sigma_A(a \otimes \delta) = a$, is Δ-split.

Furthermore, every homomorphism $\varphi : A \to B$ gives rise to a map: $\varphi_\Delta = \varphi \otimes id_{k\Delta} : A \otimes k\Delta \to B \otimes k\Delta$.

Definition 2.2. *A kG-module P is called Δ-projective if and only if it is a direct summand of a module of the form $N \otimes k\Delta$ where N is a kG-module.*

Δ-projectives satisfy all analogous properties to ordinary projectives. In particular, arbitrary direct sums of Δ-projectives are Δ-projective.

Lemma 2.3. *Let Δ_1 and Δ_2 be two G-sets and let $\phi : \Delta_1 \to \Delta_2$ be a G-map. Every Δ_1-projective kG-module is Δ_2-projective.*

Proof: Let P be Δ_1-projective. Therefore the surjection

$$\sigma_{1P} : P \otimes k\Delta_1 \twoheadrightarrow P$$

splits. Consider the following diagram:

$$\begin{tikzcd}
P \otimes k\Delta_1 \arrow[r, two heads, bend left, "\sigma_{1P}"] \arrow[d, "1\otimes\phi"'] & P. \arrow[l, bend left, "s"] \\
P \otimes k\Delta_2 \arrow[ur, two heads, "\sigma_{2P}"'] &
\end{tikzcd}$$

Since $\sigma_{2P}(1 \otimes \phi) = \sigma_{1P}$, the composite $(1 \otimes \phi)s : P \to P \otimes k\Delta_2$ splits σ_{2P}. Thus, P is Δ_2-projective. $\square$

Denote by Δ_0 a set of orbit representatives for Δ and by G_δ the stabilizer of $\delta \in \Delta_0$, then

$$\Delta = \bigsqcup_{\delta \in \Delta_0} \delta G \cong \bigsqcup_{\delta \in \Delta_0} G_\delta \backslash G.$$

Corollary 2.4. *For each $\delta \in \Delta$ the induced module $M{\uparrow}_{G_\delta}^G$ is Δ-projective.*

Proof: There is an obvious G-map $G_\delta \backslash G \to \Delta$. $\square$

Lemma 2.5. *Let Δ_1 and Δ_2 be two G-sets and let $\Phi : \Delta_1 \to \Delta_2$ be a G-map. Every Δ_2-split short exact sequence $A \rightarrowtail B \twoheadrightarrow C$ of kG-modules is Δ_1-split.*

Proof: It follows from the definition of Δ_2-split and the fact that $k\Delta_1$ is k-free, that the top sequence in the following diagram is split.

$$\begin{tikzcd}
A \otimes k\Delta_2 \otimes k\Delta_1 \arrow[r, tail] \arrow[d] & B \otimes k\Delta_2 \otimes k\Delta_1 \arrow[r, two heads] \arrow[d] & C \otimes k\Delta_2 \otimes k\Delta_1 \arrow[d] \\
A \otimes k\Delta_1 \arrow[r, tail] & B \otimes k\Delta_1 \arrow[r, two heads] & C \otimes k\Delta_1
\end{tikzcd}$$

By Lemma 2.3 all the modules in the bottom row are Δ_2-projective. Hence the vertical maps are split. Thus we can conclude that the bottom row is split as required. $\square$

Lemma 2.6. *A short exact sequence of G-modules $A \rightarrowtail B \twoheadrightarrow C$ is Δ-split if and only if it is split restricted to each stabilizer G_δ.*

Proof: Since, by the above, $k\Delta \cong \bigoplus_{\delta \in \Delta_0} k[G_\delta \backslash G]$, the if-direction follows immediately from the fact that induction is exact.

Assume now that $B \otimes k\Delta \twoheadrightarrow C \otimes k\Delta$ is split over kG. For each $\delta \in \Delta$ there is an obvious G-map $G_\delta \backslash G \to \Delta$. Application of Lemma 2.5 now gives the result. $\square$

One can now proceed to define the Δ-analogues to projective resolution and projective dimension and from there define the cohomological functors $\Delta\mathrm{Ext}^*_{kG}(-,-)$. This is very straightforward and a crucial ingredient is the following generalized Schanuel's Lemma, which is probably well known. Nevertheless we shall include its proof, which I first saw in a lecture course of Peter Kropholler.

Lemma 2.7. *Let R be a ring and*

$$\begin{tikzcd}
K \arrow[r, tail, "\iota"] \arrow[d, shift right, "\psi'"'] & P \arrow[r, two heads, "\pi"] \arrow[d, shift right, "\psi"'] & M \arrow[d, equal] \\
L \arrow[r, tail, "\tau"] \arrow[u, shift right, "\phi'"'] & Q \arrow[r, two heads, "\rho"] \arrow[u, shift right, "\phi"'] & M
\end{tikzcd}$$

be a diagram of R-modules with exact rows and maps $\psi : P \to Q$ and $\phi : Q \to P$ such that $\phi\pi = \rho$ and $\psi\rho = \pi$. Then

(1) $K \oplus Q \cong P \oplus L$.

(2) Suppose there is only a map $\psi : P \to Q$ such that $\psi\rho = \pi$.Then there is a short exact sequence of R-modules

$$K \overset{\alpha}{\rightarrowtail} L \oplus P \overset{\beta}{\twoheadrightarrow} Q.$$

Proof: The proof of (2) is a repetition of the proof of one way of verifying Schanuel's Lemma. We will now show (1). To simplify we will identify K with its image in P and L with its image in Q. We define maps

$$\begin{array}{rccc} \Theta : & Q \oplus K & \to & P \oplus L \\ & \begin{pmatrix} q \\ k \end{pmatrix} & \mapsto & \begin{pmatrix} \phi & 1 \\ \psi\phi - 1 & \psi \end{pmatrix} \begin{pmatrix} q \\ k \end{pmatrix} \end{array}$$

and

$$\begin{array}{rccc} \Theta' : & P \oplus L & \to & Q \oplus K \\ & \begin{pmatrix} p \\ l \end{pmatrix} & \mapsto & \begin{pmatrix} \psi & -1 \\ 1 - \phi\psi & \phi \end{pmatrix} \begin{pmatrix} p \\ l \end{pmatrix}. \end{array}$$

Then Θ and Θ' are inverse to each other. $\square$

Remark: If the rows in the above Lemma are Δ-split then the short exact sequence in (2) is Δ-split as well.

We shall go into more detail in the next section where we set up complete relative cohomology in a more general context.

3. Complete Relative Cohomology

To later be able to develop a completion analogous to Mislin's [11], we shall now outline a more general approach to relative cohomology using proper classes of short exact sequences. The reader is referred to [10, Chapters IX, XII] for a detailed account.

We shall consider an abelian category $\mathcal{A}$ and a proper class of short exact sequences $\mathcal{P}$, cf. [10, p 367].

For example, if $\mathcal{P}$ is the class of all short exact sequences in an abelian category, then this is proper. And, as expected, the class of Δ-split short exact sequences as in Definition 2.1 is proper.

Lemma 3.1. *Let G be a group, k a commutative ring and $\mathcal{A} = \mathcal{M}od_{kG}$. For every G-set Δ the class $\mathcal{P}$ of Δ-split exact sequences is proper.* $\square$

There are obvious notions, analogous to the ordinary case, for proper projective modules and proper projective resolutions, cf [10, pp 376–377].

Since any two proper projective resolutions of an object A are chain homotopy equivalent [10, (IX,4.3)], it makes sense to define the following:

Definition 3.2. *Let $\mathcal{P}$ be a class of proper short exact sequences in an abelian category $\mathcal{A}$, and $\mathbf{Q} \twoheadrightarrow C$ a proper projective resolution of an object C. Then we define, for an object A,*

$$\mathrm{Ext}^n_{\mathcal{P}}(C, A) = H^n(\mathrm{Hom}(\mathbf{Q}, A)).$$

These cohomology groups satisfy all analogous properties to ordinary cohomology such as, for example long exact sequences with natural connecting homomorphisms in both variables.

Denote by $\mathrm{Ext}^*_{\mathcal{A}}(C, A)$ the ordinary cohomology groups obtained by considering all short exact sequences in $\mathcal{A}$. Then there is a natural transformation: $\mathrm{Ext}^n_{\mathcal{P}}(C, A) \to \mathrm{Ext}^n_{\mathcal{A}}(C, A)$ of bifunctors, which in dimension one is a monomorphism, cf. [10, XII, Section 4].

There is also the obvious notion of proper projective dimension. We say that an object M in $\mathcal{A}$ has finite proper projective dimension, if it has a proper projective resolution of finite length. Such objects satisfy all analogous equivalences to the ordinary case, see eg. [10, (I.6.3)]. These are proved in the same fashion using the Generalized Schanuel's Lemma 2.7.

Notation: When considering cohomology relative to a G-set Δ as set up above we will denote $\mathrm{Ext}^*_{\mathcal{P}}(C, A)$ by $\Delta\mathrm{Ext}^*_{kG}(C, A)$. Similarly we shall say a kG-module M has finite Δ-projective dimension ($\Delta pd_{kG} M < \infty$), if it has a Δ-projective resolution of finite length.

We shall now proceed to develop a completion of proper cohomological functors analogous to Mislin's. In this case we use generalized satellites as set up in Mac Lane's book. The main reference for this section is [10, §XII.7, §XII.8] and for the actual completion [11].
One can define $\mathcal{P}$-completion in the much more general context of abelian categories and proper short exact sequences, a set-up which will include relative as well as ordinary cohomology.

Lemma 3.3. *[10, (XII.7.4)] Let $\mathcal{A}$ be an abelian category which has enough proper projectives and $\mathcal{R}$ be a selective category. Then each covariant additive functor $T : \mathcal{A} \to \mathcal{R}$ has a left satellite.*

Let $K \rightarrowtail P \twoheadrightarrow C$ be a proper short exact sequence with P proper projective. We then define

$$S(C) = ker(T(K) \to T(P)).$$

This is independent of the choice of the proper projective P, hence we can indeed talk about the left satellite of T.
We now define the iterated satellites as follows:

$$S^0T(C) = T(C), \quad S^{-1}T(C) = S(C) \quad \text{and} \quad S^{-n}T(C) = S^{-1}S^{-n+1}T(C).$$

By construction, for each proper projective Q and for all $n \geq 1$

$$S^{-n}T(Q) = 0.$$

Lemma 3.4. *If T is a half proper-exact functor and $\mathcal{A}$ has enough proper projectives, then there is a long exact sequence*

$$\cdots S^{-n}T(A) \to S^{-n}T(B) \to S^{-n}T(C) \to S^{-n+1}T(A) \to \cdots$$

$$\cdots \to S^{-1}T(C) \to T(A) \to T(B) \to T(A)$$

Proof: The fact that the above sequence is a complex is an iterated version of [10, XII.7.(7.3)]. Exactness is proved analogously to [4, (III.3.1)]. □

Definition 3.5. *A $\mathcal{P}$-connected sequence of covariant functors is a sequence of functors $T^* = \{T^n, n \in \mathbb{Z}\} : \mathcal{A} \to \mathcal{R}$, which assigns to each proper short exact sequence $0 \to A \to B \to C \to 0$ of $\mathcal{A}$ a complex*

$$\cdots \to T^{-n-1}(C) \to T^{-n}(A) \to T^{-n}(B) \to T^{-n}(C) \to T^{-n+1}(A) \to \cdots.$$

We shall say that the non-positive part $T^{\leq 0} = \{T^{-n}, n \geq 0\}$ of a $\mathcal{P}$-connected sequence of covariant functors is of proper cohomological type, if for each proper short exact sequence $A \rightarrowtail B \twoheadrightarrow C$ we have a long exact sequence

$$\cdots \to T^{-n}(A) \to T^{-n}(B) \to T^{-n}(C) \to T^{-n+1}(A) \to \cdots \to T^0(C)$$

Proposition 3.6. *Let $V^{\leq 0}$ and $T^{\leq 0}$ be the non-positive parts of $\mathcal{P}$-connected sequences of additive functors, and let $f^0 : V^0 \to T^0$ be a natural transformation. If, in addition, $T^{\leq 0}$ is of proper cohomological type and $T^{-n}(P) = 0$ for all $n \geq 0$ and all proper projective P, then:*

i) f^0 extends uniquely to $f^{\leq 0} : V^{\leq 0} \to T^{\leq 0}$ and $f^{\leq 0}$ factors uniquely through the canonical transformation $V^{\leq 0} \to S^{\leq 0}V^0$

ii) If V^0 is half $\mathcal{P}$-exact and f^0 is an equivalence then the induced transformation $S^{\leq 0}V^0 \to T^{\leq 0}$ is an equivalence.

Proof: This is analogous to [11, (1.2)] and follows directly from [10, (XII.8.3)] and [10, (XII.8.4)]. □

A $\mathcal{P}$-connected sequence of covariant functors $T^* = \{T^n, n \in \mathbb{Z}\}$ is called a proper $(-\infty, \infty)$-cohomological functor, if for each proper short exact sequence $A \rightarrowtail B \twoheadrightarrow C$ we have a long exact sequence

$$\cdots \to T^n(A) \to T^n(B) \to T^n(C) \to T^{n+1}(A) \to \cdots$$

The following definition is an analogue to [11, (2.1)] and gives us the desired universal property for a proper complete functor.

Definition 3.7. *A proper $(-\infty, \infty)$-cohomological functor $T^* = \{T^n, n \in \mathbb{Z}\}$ is called proper complete, if $T^n(P) = 0$ for all n and every proper projective module P. A morphism $V^* \to T^*$ of proper $(-\infty, \infty)$-cohomological functors is called a proper completion if T^* is proper complete and every morphism $V^* \to W^*$ into a proper complete functor W^* factors uniquely through $V^* \to T^*$.*

Theorem 3.8. *Every proper $(-\infty, \infty)$-cohomological functor $T^* = \{T^n, n \in \mathbb{Z}\}$ admits a unique proper completion $\tau^* : T^* \to \widehat{T}^*$. This completion is obtained as follows:*

$$\widehat{T}^j(M) = \varinjlim_{k \geq 0} S^{-k}T^{j+k}(M)$$

Proof: This theorem can be proved in exactly the same way as Mislin's Theorem 2.2 [11], as its proof only relies on the exactness of T^* and a universal property as in Proposition 3.6. □

The following two lemmas are a direct consequence of Lemmas 2.4 and 2.5 in [11], as they only rely on the theorem and the long exact sequence property of T^* without specifying where it comes from.

Lemma 3.9. *Let T^* be a proper $(-\infty, \infty)$-cohomological functor and $n_0 \in \mathbb{Z}$ such that $T^n(P) = 0$ for all proper projective P and all $n \geq n_0$. Then $\tau^n(C) : T^n(C) \to \widehat{T}^n(C)$ is an isomorphism for all $n \geq n_0$.*

Lemma 3.10. *If $f^* : T^* \to V^*$ is a morphism of proper $(-\infty, \infty)$-cohomological functors where V^* is proper complete and if $f^n : T^n \to V^n$ is an equivalence for all $n \geq n_0$ then the induced morphism $\widehat{T}^* \to V^*$ is an equivalence.*

As mentioned before, the Mislin-completion of ordinary cohomology, where proper short exact sequences are just all short exact sequences, is proper complete.

Cohomology relative to a G-set Δ has a proper completion as follows. Let

$$\Omega_\Delta N = ker(N \otimes k\Delta \twoheadrightarrow N)$$

and inductively,

$$\Omega^i_\Delta N = \Omega_\Delta \Omega^{i-1}_\Delta N.$$

As the definition of the left satellite does not depend on the choice of proper projectives, we define now:

$$S^{-1}\Delta\mathrm{Ext}^n_{kG}(M, N) = ker(\Delta\mathrm{Ext}^n_{kG}(M, \Omega_\Delta N) \to \Delta\mathrm{Ext}^n_{kG}(M, N \otimes k\Delta)$$

and the proper, or Δ-completion is as follows:

$$\Delta\widehat{\mathrm{Ext}}^n_{kG}(M, N) = \varinjlim_{k \geq |n|} S^{-k}\Delta\mathrm{Ext}^{n+k}_{kG}(M, N)$$

.

4. Complete Relative Cohomology: An Intuitive Approach

Another, probably more intuitive way of setting up a complete theory is using and modifying Benson and Carlson's [2] way of defining ordinary complete cohomology. We shall do this in the context of group rings and Δ-split exact sequences.

Definition 4.1. *For an arbitrary group G and a G-set Δ, let $\Delta\mathrm{Hom}_{kG}(M, N)$ be the subgroup of $\mathrm{Hom}_{kG}(M, N)$ consisting of homomorphisms in $\mathrm{Hom}_{kG}(M, N)$ factoring through a Δ-projective. Define*

$$[M, N]_\Delta = \mathrm{Hom}_{kG}(M, N)/\Delta\mathrm{Hom}_{kG}(M, N)$$

Also denote by $\Delta\mathcal{M}od_{kG}$ the category having as objects the kG-modules and whose morphisms lie in $[-, -]_\Delta$.

There now follows a fairly crucial lemma, whose proof is a routine check and therefore we will not include it.

Lemma 4.2. *Let M be a kG-module and $A \rightarrowtail B \twoheadrightarrow C$ be a Δ-split short exact sequence of kG-modules. Then every $[\phi] \in [M, C]_\Delta$ induces a unique $[\Psi] \in [\Omega_\Delta M, A]_\Delta$. In particular, Ω_Δ is a functor from $\Delta\mathcal{M}od_{kG}$ to itself.*

In particular, the lemma implies, that for any two G-modules M and N we have a well defined sequence of maps:

$$[M, N]_\Delta \to [\Omega_\Delta M, \Omega_\Delta N]_\Delta \to [\Omega^2_\Delta M, \Omega^2_\Delta N]_\Delta \to \cdots$$

This now enables us to define the 0-th relative Benson-Carlson group.

Definition 4.3. *Let G be a group and k a commutative ring of coefficients. Then, for all G-modules M and N we define the 0-th relative Benson–Carlson group to be:*

$$\Delta\widehat{BC}^0_{kG}(M, N) = \varinjlim_{i \geq 0} [\Omega^i_\Delta M, \Omega^i_\Delta N]_\Delta.$$

The following facts illustrate how this actually defines a proper $(-\infty, \infty)$-cohomological functor, which is proper complete. We will start with two useful observations, which can be checked easily. Let $A \rightarrowtail B \twoheadrightarrow C$ be a Δ-split short exact sequence. Then

(1) $[M, A]_\Delta \xrightarrow{\iota_*} [M, B]_\Delta \xrightarrow{\pi_*} [M, C]_\Delta$ is exact at $[M, B]_\Delta$, and

(2) $\Omega_\Delta A \rightarrowtail \Omega_\Delta B \twoheadrightarrow \Omega_\Delta C$ is a Δ-split short exact sequence.

Lemma 4.4. *Every Δ-split short exact sequence induces a long exact sequence*

$$\cdots \to [M, \Omega_\Delta B]_\Delta \to [M, \Omega_\Delta C]_\Delta \xrightarrow{\Psi_*} [M, A]_\Delta \to [M, B]_\Delta \to [M, C]_\Delta,$$

where Ψ_* is induced from Ψ of Lemma 4.2.

Proof: Exactness at $[M, B]_\Delta$ and $[M, \Omega_\Delta B]_\Delta$ follows directly from the above observations. An application of the generalized Schanuel's Lemma 2.7 gives us a Δ-split short exact sequence

$$\Omega_\Delta C \rightarrowtail (C \otimes k\Delta) \oplus A \twoheadrightarrow B$$

and thus exactness at $[M, A]_\Delta$. Exactness at $[M, \Omega_\Delta C]_\Delta$ is proved by a similar argument involving the Δ-split sequence

$$\Omega_\Delta B \rightarrowtail \Omega_\Delta C \oplus (B \otimes k\Delta) \twoheadrightarrow A \oplus (C \otimes k\Delta).$$

$\square$

In fact, Lemma 4.4 implies that for each $i, j \geq 0$ we get a long exact sequence

$$\cdots \to [\Omega^i_\Delta M, \Omega^j_\Delta B]_\Delta \to [\Omega^i_\Delta M, \Omega^j_\Delta C]_\Delta \to [\Omega^i_\Delta M, \Omega^{j-1}_\Delta A]_\Delta \to [\Omega^i_\Delta M, \Omega^{j-1}_\Delta B]_\Delta \to$$

We now define for all $n \in \mathbb{Z}$

$$\Delta\widehat{BC}^n(M,N) = \Delta\widehat{BC}^0(M, \Omega^{-n}_\Delta N) = \Delta\widehat{BC}^0(\Omega^n_\Delta M, N).$$

Even if $\Omega^n_\Delta M$ or $\Omega^{-n}_\Delta N$ are originally not defined, the n-th relative Benson Carlson group is still defined since we can take the direct limit starting after enough steps so that everything involved is indeed defined.
Taking the direct limit of each column, we have proved the following crucial fact.

Proposition 4.5. *The relative Benson-Carlson groups $\Delta\widehat{BC}^*(M,-)$ define a relative $(-\infty,\infty)$-cohomological functor.* □

It now remains to show, that the functor $\Delta\widehat{BC}^*_{kG}(M,-)$ is actually the same as the functor $\Delta\widehat{\mathrm{Ext}}^*_{kG}(M,-)$ we have defined in the previous section.

Theorem 4.6. *For G a group, k a commutative ring of coefficients and M an arbitrary kG-module there is a natural equivalence of functors*

$$\Theta^n : \Delta\widehat{\mathrm{Ext}}^n_{kG}(M,-) \to \Delta\widehat{BC}^n_{kG}(M,-)$$

for all $n \in \mathbb{Z}$.

Proof: The proof is essentially an analogue to Mislin's proof [11, (4.4)] that the original Mislin-groups and Benson-Carlson groups are naturally equivalent.

Take the Δ-projective resolution $d : \mathbf{P}_* \twoheadrightarrow M$ of M, where $\Omega^i_\Delta M = ker(P_{i-1} \to P_{i-2})$ for all $i \geq 1$. For all $j \geq 1$ there is a natural surjection $\theta^n : \Delta\mathrm{Ext}^n(M,N) \twoheadrightarrow [\Omega^n M, N]_\Delta$. Passing onto direct limits we obtain the following natural surjection:

$$\varinjlim_{k\geq|n|} \Delta\mathrm{Ext}^{n+k}(M, \Omega^k_\Delta N) \twoheadrightarrow \varinjlim_{k\geq|n|} [\Omega^{n+k}_\Delta M, \Omega^k_\Delta N]_\Delta,$$

which is defined for all $n \in \mathbb{Z}$. Obviously, the right hand side of this surjection is just $\Delta\widehat{BC}^n(M,N)$. From here on we proceed in exactly the same way as Mislin [11, (4.4)]. □

From now on we will not distinguish between the two notations and use the more natural looking $\Delta\widehat{\mathrm{Ext}}^n_{kG}(-,-)$.

Lemma 4.7. *Let G be a group and k a commutative ring of coefficients. Then the following is true for kG-modules M and N:*

i) If M or N have finite Δ-projective dimension then $\Delta\widehat{\mathrm{Ext}}^n_{kG}(M,N) = 0$ for all $n \in \mathbb{Z}$.

ii) $\Delta pd_{kG}M < \infty \iff \Delta\widehat{\mathrm{Ext}}^0_{kG}(M,M) = 0$

Proof: This can be shown analogously to Kropholler's proof for ordinary complete cohomology [8, (4.2)]. $\square$

5. Complete Δ-projective Resolutions

Definition and construction of complete Δ-projective resolutions are analogous to Cornick and Kropholler's [6] approach for ordinary complete resolutions and proofs can be more or less carried over. We shall, for the sake of completeness, give a short outline of the main facts about complete Δ-projective resolutions.

Definition 5.1. *Let G be a group, k a commutative ring of coefficients and M a kG-module. Then a complete Δ-projective resolution of M is a Δ-split exact sequence (P_*, δ), indexed by the integers such that*

(1) P_* coincides with a Δ-projective resolution of M in sufficiently high dimensions,

(2) $\mathrm{Hom}_{kG}(\mathbf{P}, Q)$ is exact for every Δ-projective kG-module Q.

Lemma 5.2. *Let kG and M be as above.*

(1) If $\mathbf{Q} \twoheadrightarrow M$ is a Δ-projective resolution of M with $Q_i = 0$ for all $i < 0$, and $\mathbf{P}$ is a complete Δ-projective resolution of M coinciding with $\mathbf{Q}$ in sufficiently high dimensions then there is a chain map $\phi : \mathbf{P} \to \mathbf{Q}$ unique up to chain homotopy.

(2) Any two complete Δ-projective resolutions of a kG-module M are chain homotopy equivalent.

Proof: The proof is essentially the same as the proof of Lemma 2.4 [6]. It is done by induction and depends basically on the exactness of $\mathrm{Hom}_{kG}(\mathbf{P}, Q)$ for Δ-projective modules Q. $\square$

Taking cohomology now obviously gives us a proper $(-\infty, \infty)$-cohomological functor $\mathrm{H}^*(\mathrm{Hom}_{kG}(\mathbf{P}, -))$, which is proper complete.

Theorem 5.3. *Let G be a group, k a commutative ring of coefficients and $\mathbf{P} = (P_*, \delta)$ a complete Δ-projective resolution of a kG-module M, then for every kG-module N we have the following equivalence:*

$$\Delta\widehat{\mathrm{Ext}}^*_{kG}(M,N) \cong \mathrm{H}^*(\mathrm{Hom}_{kG}(P_*, N)).$$

Proof: Suppose $\mathbf{Q} \twoheadrightarrow M$ is a Δ-projective resolution of M such that $\mathbf{P}$ coincides with $\mathbf{Q}$ above a sufficiently high dimension. Thus, by the previous lemma, there is a natural map

$$\Delta\mathrm{Ext}^*_{kG}(M,-) \to \mathrm{H}^*(\mathrm{Hom}_{kG}(\mathbf{P},-))$$

Application of Lemma 3.10 now gives the desired natural equivalence. $\square$

Corollary 5.4. *Let a kG-module M have a complete Δ-projective resolution. Then, for every Δ-projective module Q in sufficiently high dimensions, $\Delta\mathrm{Ext}^i_{kG}(M,Q) = 0$.*

We shall now proceed to give an account of how to construct complete Δ-projective resolutions and give conditions for their existence. Similarly to [6], it involves considering the ring of bounded functions from Δ to k.

Definition 5.5. *Let G be group, k a commutative ring of coefficients and Δ be a G-set. Then define*

(1) $\mathrm{B}(\Delta,\mathbb{Z})$ to be the set of bounded functions from Δ to $\mathbb{Z}$ and

(2) $\mathrm{B}(\Delta,k) = \mathrm{B}(\Delta,\mathbb{Z}) \otimes_{\mathbb{Z}} k$ to be the k-algebra of functions from Δ to k which only take finitely many values.

There is a well defined action of G on $\mathrm{B}(\Delta,k)$ defined as follows:

$${}^g\varphi(\delta) = \varphi(\delta g^{-1})$$

Lemma 5.6. *Let G, k and Δ be as above. Then*

i) $\mathrm{B}(\Delta,k)$ is free as a k-module,

ii) there is a k-split inclusion $k \rightarrowtail \mathrm{B}(\Delta,k)$ of kG-modules, and

iii) the inclusion $k \rightarrowtail \mathrm{B}(\Delta,k)$ is Δ-split.

Proof: i) is Lemma 4.4 of [5]; it can be deduced directly from Bergman's result [7, (97.2)].

ii) follows immediately from iii) and Lemma 2.6. So let us now prove iii): k can be regarded as the subset of $\mathrm{B}(\Delta,k)$ of constant functions and thus there is a G-map $k \rightarrowtail \mathrm{B}(\Delta,k)$. To prove this is Δ-split, is equivalent to showing that the following inclusion is G-split:

$$k\Delta \rightarrowtail \mathrm{B}(\Delta,k) \otimes k\Delta$$

Define the splitting s, which is indeed a G-map by $s(\varphi \otimes \delta) = \varphi(\delta)\delta$. This splits the inclusion sending $\delta \mapsto 1 \otimes \delta$, where $1 \in \mathrm{B}(\Delta,k)$ is the constant function sending everything to $1 \in k$. $\square$

Lemma 5.7. *Let Q be a Δ-projective kG-module. Then, for every k-projective kG-module L, the module $Q \otimes L$ is Δ-projective.*

Proof: Denote by $hom(L, N)$ the set of k-module homomorphisms from L to N, where G acts on $\phi \in hom(L, N)$ by $\phi^g(l) = \phi(lg)$. Then the lemma follows directly from the natural isomorphism

$$\mathrm{Hom}_{kG}(Q \otimes L, -) \cong \mathrm{Hom}_{kG}(Q, hom(L, -)).$$

$\square$

Lemma 5.8. *If Q is a Δ-projective kG-module then it is a direct summand of $hom_k(\mathrm{B}(\Delta, k), Q)$.*

Proof: $hom_k(\mathrm{B}(\Delta, k), Q)$ is a G-module where the action comes from the action of G on $\mathrm{B}(\Delta, k)$. Since, by Lemma 5.6 ii), the inclusion $\iota : k \rightarrowtail \mathrm{B}(\Delta, k)$ is k-split we obtain a surjective G-map:

$$\pi : hom_k(\mathrm{B}(\Delta, k), Q) \twoheadrightarrow Q.$$

From Lemma 5.6 iii) and Lemma 2.6 it follows, that ι, and therefore π, is actually split for all stabilizers G_δ. Hence π is a split G-map. $\square$

The next theorem, besides giving necessary conditions for a module to have a complete Δ-projective resolution provides a method to construct them as well.

Theorem 5.9. *Let G be a group, k a commutative ring of coefficients and M a kG-module such that $\Delta pd_{kG}(M \otimes \mathrm{B}(\Delta, k)) < \infty$. Then M has a complete Δ-projective resolution which splits under tensoring with $\mathrm{B}(\Delta, k)$.*

Proof: This can be shown analogously to [6, (3.5)]. Throughout this proof let $B = \mathrm{B}(\Delta, k)$. We can replace M by an appropriate kernel in a Δ-projective resolution and from now on assume that $M \otimes B$ is Δ-projective. The short exact sequence

$$k \rightarrowtail B \twoheadrightarrow \bar{B}$$

is Δ-split (Lemma 5.6) implying that $\bar{B}$ is k-projective. For all $i \geq 0$, let $\bar{B}^{i+1} = \bar{B}^i \otimes \bar{B}$, where $\bar{B}^0 = k$. Thus we obtain, for all $i \geq 0$, the following Δ-split short exact sequence:

$$\bar{B}^i \rightarrowtail \bar{B}^i \otimes B \twoheadrightarrow \bar{B}^{i+1}.$$

Splicing all these together and then tensoring with M now gives us the following "backwards" Δ-projective resolution:

$$M \rightarrowtail M \otimes B \to M \otimes \bar{B} \otimes B \to \cdots \to M \otimes \bar{B}^i \otimes B \to M \otimes \bar{B}^{i+1} \otimes B \to \cdots .$$

Since $M \otimes \bar{B}^i \otimes B$ is Δ-projective (Lemma 5.7) this is split under tensoring with B. Splice the "backwards" relative projective resolution together with a "forwards" relative projective resolution $\mathbf{Q}_*$ of M to obtain our desired complete relative projective resolution $\mathbf{P}_*$ of M the following way:

$$P_i = Q_i \quad \forall i \geq 0$$
$$P_i = M \otimes \bar{B}^{-i-1} \otimes B \quad \forall i < 0$$

We still need to check that for all relative projectives L, $\mathrm{Hom}_{kG}(\mathbf{P}_*, L)$ is exact, which follows from an application of Lemma 5.8 and the adjoint isomorphism. $\square$

6. A special G-set Δ

We shall now restrict ourselves to considering a special G-set Δ, which will result in a cohomology theory relative to all finite subgroups as mentioned in the introduction. For this special case it is possible to prove slightly stronger results about finitely generated Δ-projectives and we obtain a link to ordinary FP_∞-modules.
Let H be a subgroup of G and let Δ^H denote the set of all fixed points of the G-set Δ under the action of H. From now on we shall consider the G-set Δ satisfying the following property:

$$\Delta^H \neq \emptyset \iff H \text{ finite.}$$

An example of a G-set Δ satisfying the above condition is the set of all cosets Hg where H is a finite subgroup and $g \in G$.

Lemma 6.1. *Let Δ_1 and Δ_2 be two G-sets satisfying the above condition. Then there are G-maps $\Delta_1 \to \Delta_2 \to \Delta_1$.* $\square$

This lemma suggests that it does indeed make sense to consider G-sets having the above property since they all give rise to the same relative cohomology, cf. Lemmas 2.3 and 2.5. To distinguish it from the general case we shall from now on replace Δ by $\mathcal{F}$ as indicated in the Introduction.

In particular, Corollary 2.4 implies that, for a finite subgroup H, the induced module $M{\uparrow}_H^G$ is $\mathcal{F}$-projective. Also, Lemma 2.6 gives rise to the following Corollary.

Corollary 6.2. *A short exact sequence of G-modules $A \rightarrowtail B \twoheadrightarrow C$ is $\mathcal{F}$-split if and only if it splits restricted to each finite subgroup of G.*

The following, main result of this section, besides being interesting in its own right has proved to be of great importance for later results on relative finiteness conditions.

Proposition 6.3. *Let k be a commutative Noetherian ring of coefficients and G an arbitrary group. Then every finitely generated $\mathcal{F}$-projective kG-module is of type FP_∞.*

Proof: Every finitely generated $\mathcal{F}$-projective P is a direct summand of $P \otimes k\Delta = \bigoplus_{\delta \in \Delta_0} P{\uparrow}_{G_\delta}^G$, where Δ_0 is a set of orbit representatives and G_δ are finite. Hence there is a splitting:

$$\bigoplus_{\delta \in \Delta_0} P{\uparrow}_{G_\delta}^G \underset{\iota}{\overset{\pi}{\rightleftarrows}} P.$$

Since P is finitely generated, ι factors through a finite subsum $\bigoplus_{\delta \in \bar{\Delta}_0} P{\uparrow}_{G_\delta}^G$, where $\bar{\Delta}_0$ is a finite subset of Δ_0.

P regarded as a G_δ-module is not necessarily finitely generated but is the direct limit of its finitely generated submodules P_{λ_δ}. Hence, for each $\delta \in \bar{\Delta}_0$:

$$P{\uparrow}_{G_\delta}^G \cong \varinjlim_{\lambda_\delta \in \Lambda_\delta} P_{\lambda_\delta}{\uparrow}_{G_\delta}^G,$$

which is, in fact, the directed union.

Let $\{p_1, ..., p_n\}$ be a set of generators of the kG-module P. Then, for each i, the image ιp_i lies in some $P_{\lambda_{i_\delta}}{\uparrow}_{G_\delta}^G$. Since, for each δ, Λ_δ is a directed set and there are only finitely many p_i, there exist, for all δ, $\lambda_{0_\delta} \in \Lambda_\delta$ such that

$$\bigcup_{i=1,\dots,n} P_{\lambda_{i_\delta}} \subseteq P_{\lambda_{0_\delta}}.$$

Hence ι factors through $\bigoplus_{\delta \in \bar{\Delta}_0} P_{\lambda_{0_\delta}}{\uparrow}_{G_\delta}^G$. Since k is Noetherian and the G_δ are finite, each $P_{\lambda_{0_\delta}}$ is a kG_δ module of type FP_∞. Thus, $\bigoplus_{\delta \in \bar{\Delta}_0} P_{\lambda_{0_\delta}}{\uparrow}_{G_\delta}^G$ is a kG-module of type FP_∞. Since P is a direct summand of it, it is also of type FP_∞ as required. $\square$

Remark. For non-Noetherian commutative rings of coefficients the above statement does not remain true. If k is not Noetherian, for example, $f[X_0, X_1, ...]$ the polynomial ring of infinitely many indeterminants over a field f, then there exists an infinitely generated ideal J. In our example we can take J to be the ideal generated by $(X_0, X_1, ...)$. Denote by M the cokernel of the inclusion $J \rightarrowtail k$. This is a finitely generated k-module, which is not finitely presented. Let G be a torsion free group. Then the induced module $M\!\uparrow_1^G$ is a finitely generated $\mathcal{F}$-projective kG-module, which is not finitely presented.

7. Modules of type $\mathcal{F}\mathbf{FP}_\infty$

Let k be a commutative ring, G be an arbitrary group and M be a kG-module. As mentioned in the introduction there are two possibilities to define conditions on M analogously to the property FP_∞, which are as follows:

(A) $\mathcal{F}\mathrm{Ext}^i_{kG}(M,-)$ commutes with direct limits for all $i \geq 0$. We also say that $\mathcal{F}\mathrm{Ext}^*_{kG}(M,-)$ is continuous.

(B) M admits an $\mathcal{F}$-projective resolution $\mathbf{Q}_* \twoheadrightarrow M$ of finite type, i.e. each Q_i, $i \geq 0$, is finitely generated. We shall say that such M are of type $\mathcal{F}FP_\infty$.

We will give the main result linking the two and also give a link to ordinary FP_∞. Both theorems are an application of Proposition 6.3.

Theorem 7.1. *Let G be a group and k a commutative Noetherian ring of coefficients. Let M be a kG-module of type $\mathcal{F}FP_\infty$. Then $\mathcal{F}\mathrm{Ext}^*_{kG}(M,-)$ is continuous.*

Proof: This proof, besides following from Proposition 6.3, relies on the repeated application of the 5-Lemma. We have to prove, that $\mathcal{F}\mathrm{Ext}^n_{kG}(M,-)$ is continuous for all $n \geq 0$. Consider an $\mathcal{F}$-projective resolution of M of finite type:

$$\cdots \to P_{i+1} \to P_i \to \cdots \to P_0 \twoheadrightarrow M$$

and let $K_i = ker(P_i \to P_{i-1})$ for all $i \geq 0$, where we denote M by K_{-1}. Obviously, the K_i are finitely generated. Thus, for all $i \geq -1$ we have the following inclusion:

$$\varinjlim_{\lambda\in\Lambda} \mathrm{Hom}(K_i, N_\lambda) \overset{\tau_i}{\rightarrowtail} \mathrm{Hom}(K_i, \varinjlim_{\lambda\in\Lambda} N_\lambda)$$

for each directed system of kG-modules N_λ. Since, by Proposition 6.3, the P_i are in particular finitely presented there is a natural isomorphism

$$\varinjlim_{\lambda\in\Lambda} \mathrm{Hom}_{kG}(P_i, N_\lambda) \cong \mathrm{Hom}_{kG}(P_i, \varinjlim_{\lambda\in\Lambda} N_\lambda).$$

An application of the 5-Lemma to the two long exact sequences obtained from

$$\mathrm{Hom}_{kG}(K_{i-1},-) \rightarrowtail \mathrm{Hom}_{kG}(P_i,-) \to \mathrm{Hom}_{kG}(K_i,-)$$

gives that τ_{i-1} is an isomorphism for all $i \geq 0$. Hence all the K_i are finitely presented. A dimension shift in the first variable of the long exact sequence of $\mathcal{F}\mathrm{Ext}^*_{kG}(-,-)$ yields, for all $i \geq 1$, the following natural isomorphism:

$$\mathcal{F}\mathrm{Ext}^i(M,-) \cong \mathcal{F}\mathrm{Ext}^1(K_{i-2},-)$$

Thus, by another application of the 5-Lemma, the natural morphism

$$\varinjlim_{\lambda \in \Lambda} \mathcal{F}\mathrm{Ext}^i_{kG}(M, N_\lambda) \xrightarrow{\phi_i} \mathcal{F}\mathrm{Ext}^i_{kG}(M, \varinjlim_{\lambda \in \Lambda} N_\lambda)$$

is an isomorphism for all $i \geq 1$, which proves the theorem. □

Theorem 7.2. *Let k be a commutative Noetherian ring of coefficients, and let M be a kG-module of type $\mathcal{F}FP_\infty$. Then M is of type FP_∞.*

Proof: Let

$$\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

be a finite type $\mathcal{F}$-projective resolution. Then by Proposition 6.3 all the P_i are of type FP_∞. Choose an integer $n \geq 0$ arbitrarily. Then consider the exact sequence

$$0 \to K_n \to P_n \to \cdots \to P_0 \to M \to 0.$$

P_{n+1} is of type FP_∞, which implies that K_n is finitely generated. Apply [3, (1.4.a)] to the short exact sequence $K_n \rightarrowtail P_n \twoheadrightarrow K_{n-1}$ to obtain that K_{n-1} is finitely presented. Proceed with applying [3, (1.4.a)] to show that M is of type FP_{n+1}. Since this is true for all $n \geq 0$, we can conclude that M is of type FP_∞ as required. □

Even though we don't know if the converse of Theorem 7.1 holds, modules satisfying either of the conditions **A** or **B** share a few properties.

Lemma 7.3. *Let k be an arbitrary ring of coefficients and G be a group. Let M be a kG-module satisfying* **(A)**. *Then $\mathcal{F}\widehat{\mathrm{Ext}}^i_{kG}(M,-)$ is continuous for all $i \in \mathbb{Z}$.*

Proof: This is an analogue to Kropholler's result for ordinary complete cohomology [8, (4.1)(ii)]. □

Lemma 7.4. *Let k and G be as in the previous lemma. For every kG-module M of type $\mathcal{F}FP_\infty$, $\mathcal{F}\widehat{\mathrm{Ext}}^*_{kG}(M,-)$ commutes with arbitrary direct sums.*

Proof: For this proof we will use the Benson-Carlson definition of Section 4. Let K_s be the s-th kernel of a finite type $\mathcal{F}$-projective resolution of M, which is obviously finitely generated. By the generalized Schanuel's Lemma (2.7) we have the following isomorphism:

$$[K_s, -]_{\mathcal{F}} \cong [\Omega^s_{\mathcal{F}} M, -]_{\mathcal{F}}.$$

It can be shown by a standard argument analogously to the well known fact that $\mathrm{Hom}_{kG}(L, -)$ commutes with direct sums for finitely generated kG-modules L, that $[K_s, -]_{\mathcal{F}}$ commutes with direct sums for all $s \geq 0$. $\square$

8. Modules of finite $\mathcal{F}$-projective dimension over $\mathbf{H}\mathfrak{F}$-groups

This final section will consist of a collection of results about finiteness conditions for $\mathbf{H}\mathfrak{F}$-groups. The first preliminary result is an analogue to [8, (3.1)] and can be proved using the same arguments. It could also be proved for any relative cohomological functor.

Lemma 8.1. *Let k be a commutative ring, G a group and let*

$$0 \to M_s \to \cdots \to M_0 \to L \to 0$$

be an $\mathcal{F}$-split long exact sequence of kG-modules. If for some kG-module N and some $i \in \mathbb{Z}$, $\mathcal{F}\widehat{\mathrm{Ext}}^i_{kG}(N, L) \neq 0$, then there is a $0 \leq j \leq s$ such that $\mathcal{F}\widehat{\mathrm{Ext}}^{i+j}_{kG}(N, M_j) \neq 0$. $\square$

Lemma 8.2. *Let k and G be as in the previous lemma and let H be a subgroup of G. Suppose $A \rightarrowtail B \twoheadrightarrow C$ is a short exact sequence of kH-modules which is $\mathcal{F}$-split. Then*

$$A{\uparrow}^G_H \rightarrowtail B{\uparrow}^G_H \twoheadrightarrow C{\uparrow}^G_H$$

is an $\mathcal{F}$-split short exact sequence of kG-modules.

Proof: Let Δ be a G-set satisfying the property $\Delta^K \neq \emptyset$ if and only if K is a finite subgroup of G. Then Δ regarded as an H-set satisfies this property as well. Hence

$$A \otimes k\Delta \rightarrowtail B \otimes k\Delta \twoheadrightarrow C \otimes k\Delta$$

is a split sequence of kH-modules, see Definition 2.1. Since induction is an exact functor, the sequence

$$(A \otimes k\Delta){\uparrow}^G_H \rightarrowtail (B \otimes k\Delta){\uparrow}^G_H \twoheadrightarrow (C \otimes k\Delta){\uparrow}^G_H$$

is a split sequence of kG-modules. Thus, it remains to show that for each kH-module A there is an isomorphism of kG-modules

$$(A \otimes k\Delta)\uparrow_H^G \cong A\uparrow_H^G \otimes k\Delta,$$

which is a result of the Frobenius Reciprocity. $\square$

Now we can prove the following main theorem, which in particular holds for modules satisfying conditions **(A)** and **(B)** of Section 7, cf. (7.3) and (7.4).

Theorem 8.3. *Let k be a commutative ring of coefficients and G an $\mathbf{H}\mathfrak{F}$-group. Let M be a kG-module such that $\mathcal{F}\widehat{Ext}^*_{kG}(M,-)$ commutes with arbitrary direct sums. Then M has finite $\mathcal{F}$-projective dimension.*

Proof: We will aim for a contradiction and assume that $\mathcal{F}\widehat{\mathrm{Ext}}^0_{kG}(M,M) \neq 0$, (cf. 4.7). We will show that this would imply that there is a finite group H such that $\mathcal{F}\widehat{\mathrm{Ext}}^i_{kG}(M, M\uparrow_H^G) \neq 0$, for some $i \geq 0$, which will give us the desired contradiction since $M\uparrow_H^G$ is $\mathcal{F}$-projective.

We will do this using a transfinite induction. Consider the collection $\mathcal{O}$ of ordinals β such that there is a $j \geq 0$ and a subgroup H of G lying in $\mathrm{H}_\beta\mathfrak{F}$ satisfying $\mathcal{F}\widehat{\mathrm{Ext}}^j_{kG}(M, M\uparrow_H^G) \neq 0$.

Step 1: Prove that $\mathcal{O}$ is not empty:

As $G \in \mathbf{H}\mathfrak{F}$ it actually lies in some $\mathbf{H}_\alpha\mathfrak{F}$ for some ordinal α, and by our assumption $\mathcal{F}\widehat{\mathrm{Ext}}^0_{kG}(M,M) \neq 0$. Hence $\alpha \in \mathcal{O}$.

Step 2: We will have to show that if there is a $0 \neq \beta \in \mathcal{O}$ satisfying the properties above, then there exists a subgroup K of G lying in $\mathbf{H}_\gamma\mathfrak{F}$, for some $\gamma < \beta$, and there is an $l \geq j$ such that $\mathcal{F}\widehat{\mathrm{Ext}}^l_{kG}(M, M\uparrow_K^G) \neq 0$.

As $H \in \mathrm{H}_\beta\mathfrak{F}$, it acts cellularly on a finite dimensional contractible space such that all the isotropy groups belong to $\mathbf{H}_\gamma\mathfrak{F}$, some $\gamma < \beta$.

Consider the following cellular chain complex, which is an exact sequence of $\mathbb{Z}H$-modules:

$$0 \to C_s \to \cdots \to C_0 \to \mathbb{Z} \to 0$$

It follows from a theorem of Kropholler and Wall [9] that this is split for all finite subgroups of H. Applying $M \otimes_{\mathbb{Z}} -$ to the above we obtain an exact sequence of kH-modules, which is split for all finite subgroups of H:

$$0 \to M \otimes_{\mathbb{Z}} C_s \to \cdots \to M \otimes_{\mathbb{Z}} C_0 \to M \to 0.$$

By inducing the above sequence to G, we obtain, by Corollary 6.2 and Lemma 8.2 an $\mathcal{F}$-split exact sequence of kG-modules:

$$0 \to (M \otimes_{\mathbb{Z}} C_s)\uparrow_H^G \to \cdots \to (M \otimes_{\mathbb{Z}} C_0)\uparrow_H^G \to M\uparrow_H^G \to 0.$$

As by assumption $\mathcal{F}\widehat{\mathrm{Ext}}^j_{kG}(M, M\uparrow_H^G) \neq 0$, we can apply Lemma 8.1. Hence there exists a $0 \leq i \leq s$ such that $\mathcal{F}\widehat{\mathrm{Ext}}^{j+i}_{kG}(M, (M \otimes_{\mathbb{Z}} C_i)\uparrow_H^G) \neq 0$.

Each of the terms in the above sequence is of the following form:

$$\begin{aligned}(M \otimes C_i)\uparrow_H^G &= \bigoplus_{\sigma\in\Sigma}(M \otimes_{\mathbb{Z}} \mathbb{Z}\uparrow_{H_\sigma}^H)\uparrow_H^G \\ &= \bigoplus_{\sigma\in\Sigma} M\uparrow_{H_\sigma}^G\end{aligned}$$

where Σ is the set of orbit representatives for the i-dimensional cells and H_σ denote the isotropy groups.
As $\mathcal{F}\widehat{\mathrm{Ext}}^{j+i}_{kG}(M, -)$ commutes with arbitrary direct sums there exists a $H_\sigma \in \mathbf{H}_\gamma\mathfrak{F}$, where $\gamma < \beta$, such that

$$\mathcal{F}\widehat{\mathrm{Ext}}^{j+i}_{kG}(M, M\uparrow_{H_\sigma}^G) \neq 0,$$

which proves the theorem. $\square$

Our next result will bring us back to modules satisfying the criterion of Theorem 5.9 to have a complete $\mathcal{F}$-projective resolution. We will start with a preliminary lemma.

Lemma 8.4. *Let G be any group and k a commutative ring of coefficients. Let M be a G-module such that $M \otimes \mathrm{B}(\Delta, k)$ is $\mathcal{F}$-projective and $\mathcal{F}pd_{kG}M < \infty$. Then M is $\mathcal{F}$-projective.*

Proof: This is a relative analogue to [1, (5.4)]. $\square$

Theorem 8.5. *Let k be a commutative ring and G an $\mathbf{H}\mathfrak{F}$-group. Then for every G-module M, $\mathcal{F}pd_{kG}M < \infty$ if and only if $\mathcal{F}pd_{kG}(M \otimes \mathrm{B}(\Delta, k)) < \infty$.*

Proof: The "only if"–direction follows from Lemma 5.7.
The "if"–direction can be proved by a transfinite induction analogously to a result of Cornick and Kropholler, an account of which can be found in [1, (5.7)], and an application of Lemma 8.4. $\square$

The most obvious question to ask is whether the converse of Theorem 7.1 is true: For commutative Noetherian rings of coefficients, is every kG-module satisfying condition **(A)** of type $\mathcal{F}FP_\infty$? A slightly weaker question to ask is whether every such module is of type FP_∞. Theorem 8.3 suggests that these conjectures might be true for $\mathbf{H}\mathfrak{F}$-groups.

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