Representations of locally compact $p$-adic groups and Number Theory

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Notations

- $F$ a locally compact non-archimedean local field, with ring of integers $\mathfrak{o}_F$, maximal ideal $\mathfrak{p}_F$, uniformizer $\varpi_F$, residue field $k_F = \mathfrak{o}_F / \mathfrak{p}_F$ of characteristic $p$.

$$\mathbb{Q}_p \supset \mathbb{Z}_p \supset p\mathbb{Z}_p.$$  

- $G = G(F)$ the points of a connected reductive group.

$$G = GL_n(F), \quad G = Sp_{2n}(F).$$  

- For $H \subseteq G$ a closed subgroup, $\mathcal{R}(H)$ the category of smooth complex representations of $H$.

A complex representation $(\pi, \mathcal{V})$ of $H$ is smooth if

$$\text{Stab}_H(\nu)$$  

is open, for all $\nu \in \mathcal{V}$.  

- $\text{Irr}(H)$ the set of equivalence classes of irreducible representations in $\mathcal{R}(H)$.  

Let $K$ be a compact open subgroup of $G$. Any $\rho \in \text{Irr}(K)$ factors through a finite quotient and is finite-dimensional. Any $\rho \in \mathfrak{R}(K)$ is semisimple.

For $G$ the situation is somewhat different. Any $\pi \in \text{Irr}(G)$ which is finite-dimensional is in fact 1-dimensional. In general, representations in $\mathfrak{R}(G)$ are not semisimple. Schur’s Lemma does hold.
Restriction to compact open subgroups

**Theorem**

Let \( \pi \in \text{Irr}(G) \) and let \( K \) be a compact open subgroup of \( G \). The restriction \( \pi|_K \) decomposes as a direct sum of smooth irreducible representations of \( K \), each appearing with finite multiplicity.

**Remarks**

- Not all representations of compact open subgroups are “interesting” – eg. the trivial representation of \( \text{GL}_n(\mathcal{O}_F) \).
- We want those which are *typical* of representations of \( G \).
- Maximal compact subgroups are not in general all conjugate to \( G(\mathcal{O}_F) \).
- The *typical* representations “naturally” live on more complicated compact subgroups: *parahoric* subgroups.
Parahoric subgroups

Let $K$ be a maximal compact (open) subgroup of $G$, with pro-$p$ radical $K_{0+}$:

$$1 \to K_{0+} \to K \to M(k_F) \to 1,$$

for $M$ a reductive group.

A parahoric subgroup is the inverse image in $K$ of a parabolic subgroup of the connected component of $M$.

Parahoric subgroups $J$ come with left-continuous decreasing filtrations by normal subgroups $J_r$, for $r \geq 0$, with

$$J_0 = J, \quad [J_r, J_s] \subset J_{r+s}, \text{ for } r, s \geq 0,$$

and we put $J_{r+} = \bigcup_{s > r} J_s$.

The basic example is the Iwahori subgroup.
Given $\pi \in \text{Irr}(G)$, we can consider

$$\left\{ r \in \mathbb{R} : \text{there is a parahoric subgroup } J \text{ with } \pi^{Jr+} \neq 0 \right\}.$$  

It is non-empty (by smoothness) and attains its minimum, which is a rational number. This is the normalized level $n(\pi)$.

For $G = \text{GL}_n(F)$, the set of possible normalized levels is

$$\left\{ \frac{a}{b} : a, b \in \mathbb{Z}_{\geq 0}, 1 \leq b \leq n \right\}.$$  

For $G = \text{Sp}_{2n}(F)$, the set of possible normalized levels is

$$\left\{ \frac{a}{2b} : a, b \in \mathbb{Z}_{\geq 0}, 1 \leq b \leq n \right\}.$$
If $r = n(\pi) > 0$ and $\pi^{J_{r^+}} \neq 0$ then $\pi|_{J_r}$ contains a non-trivial (1-dimensional) character $\chi$ of $J_r/J_{r^+}$,

and these characters are parametrized by cosets in the (dual of the) Lie algebra of $G$.

The minimality of $r$ implies:

- this coset is *non-nilpotent* (mod $p$).

In fact, we can assume it is *semisimple* (mod $p$).

This character $\chi$ is often called an *unrefined minimal $K$-type*.
Let $K$ be a compact open subgroup of $G$ and $\rho \in \text{Irr}(K)$.

- $\mathcal{A}^\rho(G)$ the full subcategory whose objects are representations generated by their $\rho$-isotypic component.
- $\mathcal{A}_\rho(G)$ the full subcategory whose objects are representations with zero $\rho$-isotypic component.

**Definition**

$(K, \rho)$ is a type if $\mathcal{A}^\rho(G)$ closed under subquotients.

If $(K, \rho)$ is a type then $\mathcal{A}(G) = \mathcal{A}^\rho(G) \oplus \mathcal{A}_\rho(G)$.

The archetype is the pair

$$K = \text{Iwahori subgroup}, \rho = 1.$$ 

But $(GL_n(o_F), 1)$ is not a type.
Parabolic induction

If $M$ is a Levi subgroup of $G$ and $\tau \in \text{Irr}(M)$, we form the parabolically induced representation

$$\text{Ind}_{M,P}^G \tau,$$

a smooth representation of finite length.

An irreducible representation $\pi$ of $G$ is **cuspidal** if it is *not* a subrepresentation of any parabolically induced representation $\text{Ind}_{M,P}^G \tau$, for $M$ a proper Levi subgroup.

- Given $\pi \in \text{Irr}(G)$, there are a Levi subgroup $M$ and a cuspidal representation $\tau$ of $M$ such that $\pi$ is a subrepresentation of $\text{Ind}_{M,P}^G \tau$. Moreover, $(M, \tau)$ is unique up to conjugacy. It is called the *cuspidal support* of $\pi$. 
Cuspidal types

For \( \pi \in \text{Irr}(G) \) cuspidal, a \( \pi \)-type is a type \((K, \rho)\) such that the irreducible representations in \( \mathcal{R}^\rho(G) \) are precisely

\[
\{ \pi \otimes \chi, \text{ for some unramified character } \chi \text{ of } G \}.
\]

All known cuspidal representations can be constructed as

\[
\pi \cong \text{Ind}^G_K \tilde{\rho},
\]

for \( \tilde{K} \) a compact-mod-center open subgroup of \( G \) and \( \tilde{\rho} \in \text{Irr}(\tilde{K}) \), such that, for \( K \) the unique maximal compact subgroup of \( \tilde{K} \), the restriction \( \rho = \tilde{\rho}|_K \) is irreducible.

In this situation, \((K, \rho)\) is a \( \pi \)-type.
Theorem

This is all cuspidal representations in the following cases:

- $GL_n(F)$;
- $SL_n(F)$;
- Level zero representations for arbitrary $G$;
- $G$ arbitrary but several conditions on $F$;
- $GL_m(D)$, $D$ a division algebra over $F$;
- $G$ a classical group, $p \neq 2$.

[Howe, Carayol, Bushnell–Kutzko, Morris, Moy–Prasad, Yu, Kim, Sécherre, S.]
Cuspidal types

When $p \nmid n$, Howe parametrized the irreducible cuspidals of $GL_n(F)$ by isomorphism classes of admissible pairs $(E, \xi)$, where:

- $E/F$ is a field extension of degree $n$;
- $\xi$ is a character of $E^\times$ such that:
  - $\xi$ does not factor through $N_{E/L}$, for any $E \supseteq L \supseteq F$;
  - if $\xi|_{U_E^1}$ factors through $N_{E/L}$, for some $E \supseteq L \supseteq F$, then $E/L$ is unramified.

From such a pair one constructs $\pi_\xi$, a cuspidal representation of $GL_n(F)$, giving a non-canonical bijection

$$
\left\{ \text{isomorphism classes of admissible pairs} \right\} \leftrightarrow \left\{ \text{equivalence classes of irreducible cuspidal representations of } GL_n(F) \right\}
$$
Local Class Field Theory

We have short exact sequences

\[
1 \rightarrow \mathcal{I}_F \rightarrow \mathcal{G}_F \rightarrow \hat{\mathbb{Z}} \rightarrow 1
\]

\[
1 \rightarrow \mathcal{I}_F \rightarrow \mathcal{W}_F \rightarrow \mathbb{Z} \rightarrow 1
\]

\(\mathcal{W}_F\) is the Weil group of \(F\).

There is a canonical bijection

\[
\mathcal{W}_F^{ab} \leftrightarrow F^\times.
\]
Put $\mathcal{W}_D = \mathcal{W}_F \times SL_2(\mathbb{C})$, the Weil-Deligne group of $F$.

**Local Langlands Correspondence**

There is a canonical bijection

$$\left\{ \text{continuous complex } \right. \atop \left. \text{n-dimensional representations of } \mathcal{W}_D \right\} \leftrightarrow \left\{ \text{irreducible smooth complex representations } \right. \atop \left. \text{of } GL_n(F) \right\}$$

[Laumon–Rapoport–Stuhler, Harris–Taylor, Henniart]

This can be reduced to a bijection

$$\left\{ \text{irreducible n-dimensional } \right. \atop \left. \text{representations of } \mathcal{W}_F \right\} \leftrightarrow \left\{ \text{irreducible cuspidal representations of } GL_n(F) \right\}$$

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Representations of $p$-adic groups
When \( p \nmid n \), one can parametrize the irreducible \( n \)-dimensional representations of \( \mathcal{W}_F \) in terms of *admissible pairs* \((E, \xi)\):

\[
\sigma_\xi = \text{Ind}_{\mathcal{W}_E}^{\mathcal{W}_F} \xi.
\]

[Bushnell–Henniart] The Local Langlands Correspondence is given by

\[
\sigma_\xi \leftrightarrow \pi_{\xi \mu},
\]

for \( \mu = \mu_\xi \) an explicit tamely ramified character of \( E^\times \) (which is non-trivial in general).
Local Langlands Conjectures

Let $G = G(F)$ be a connected reductive group, $\hat{G}$ the complex dual group and

$$L G = \hat{G} \rtimes \mathcal{W}_F,$$

the Langlands dual group.

The Langlands Conjectures predict a finite-one correspondence

$$\begin{align*}
\text{continuous complex representations } \mathcal{W}_D \to L G & \leftrightarrow \\
\text{irreducible smooth complex representations of } G
\end{align*}$$

The fibres of the map are called $L$-packets.

Langlands Functoriality predicts that if there is a continuous map $L G \to L H$ then there is a corresponding map from the set of $L$-packets for $G$ to the set of $L$-packets for $H$. 
For generic cuspidal representations $\pi$ of $G = \text{Sp}_{2n}(F)$, the corresponding representation of $\mathcal{W}_F$ should take the form

$$\sigma_1 \oplus \cdots \oplus \sigma_r \oplus \omega,$$

for $\sigma_i$ inequivalent irreducible representations $\mathcal{W}_F \to \text{SO}_{2n_i}(\mathbb{C})$, $\sum_i n_i = n$, and $\omega$ a character.

If $\pi_i$ is the representation of $\text{GL}_{2n_i}(F)$ corresponding to $\sigma_i$, then the pairs $(\text{GL}_{2n_i}(F) \times G, \pi_i \otimes \pi)$ in $\text{Sp}_{2(n+n_i)}(F)$ should be exactly those which give “interesting” reducibilities in parabolic induction.
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